Exploring the behavior of solitons on a desktop personal computer

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In recent years, there has been a growing interest in studying and applying nonlinear wave equations and their soliton solutions. In this paper we discuss at undergraduate level a simple finite–difference numerical method for solving nonlinear wave equations. This method is applied for studying the striking behavior of the optical solitons. The procedures presented can be reproduced by enthusiastic students and instructor with a minimum of programming experience. We provide a set of interesting problems that could be taken as starting point to numerically explore the solutions of nonlinear wave equations.

Keywords: Soliton; nonlinear differential equations; numerical methods.

El interés en el estudio de las ecuaciones de onda no lineales y sus soluciones solitónicas se ha incrementado recientemente. En este artículo estudiamos un método númerico muy simple basado en diferencias finitas para solucionar ecuaciones de onda no lineales. Este método es aplicado en el estudio del comportamiento de los solitones ópticos. Los procedimientos expuestos en este trabajo pueden reproducirse por estudiantes e instructores con un mínimo de experiencia en programación. Adicionalmente, incluimos una lista de problemas que pueden servir como punto de inicio para explorar las interesantes soluciones de las ecuaciones no lineales.

Descriptores: Solitones; ecuaciones diferenciales no lineales; métodos numéricos.

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1. Introduction

The term *soliton* was derived from the name *solitary wave*, which was first observed by Scott Russell in 1834 [1–3]. In his studies, he saw a smooth, rounded, well-defined "heap" of water that traveled without changing its shape and speed for many miles along the canal linking Edinburgh with Glasgow. This initial observation was followed by extensive theoretical and experimental research that established the existence of solitons as one of the most striking aspects of nonlinear wave phenomena. Solitons have been observed and studied in various fields ranging from optics and fluids [3–6] to solid-state and chemical systems [7, 8].

Exact analytical solutions to nonlinear wave equations do not exist in most cases, and thus there is a need for numerical techniques. Despite the existence of many applications, these numerical techniques are often avoided in undergraduate textbooks and curricula, and other texts present only limited discussions [9–13]. When facing a nonlinear wave problem that requires numerical methods, we can consult specialized books [4, 14] and classic articles [15–20]. However, this experience can be rather painful if one is trying to learn this subject for the first time. In most cases we are confronted with a large variety of sophisticated methods or references with a high density of equations. We believe that this lack of literature is due to the belief that nonlinear physics is difficult and cannot be taught to undergraduates.

The purpose of this article is to provide an introduction to the soliton phenomenon and discuss some basic soliton properties using elementary numerical methods based on linear algebra. This discussion is intended for students, teachers, and researchers who are unfamiliar with numerical methods for studying soliton propagation. We shall include only the essential formulas needed to explain the numerical method for propagation. The procedures and results presented in this paper can be reproduced by students with a minimum of programming experience on a desktop personal computer. To this end we propose some interesting numerical problems. Because the paper is relatively self-contained, we believe that undergraduate students and instructors can take our problems as a starting point for exploring the numerical solutions of nonlinear wave equations in various fields of physics.

1.1. Solitary waves and solitons

A solitary wave is a stable isolated (localized) traveling solution of a nonlinear wave equation, for instance a solitary wave propagating in positive z direction with velocity v could be $\exp[-(z-vt)^2]$. For a long time, the solitary wave was considered only as an unimportant curiosity in nonlinear wave theory. It seemed clear that if two solitary waves were initially launched into a collision course, the interaction upon collision would destroy their original identity. But it was a great surprise when a special kind of solitary wave was discovered that maintains its velocity and shape after collision with other solitary waves [21]. These solitary waves that behave like "particles" were named *solitons* in 1965 by Zabusky and Kruskal [2]. The stability of solitons stems from the delicate balance of "nonlinearity" and "dispersion" in the model equations. Nonlinearity drives a solitary wave to concentrate further; dispersion is the effect of spreading such a localized wave.

At the present time, solitons are studied in many fields of Physics. The following three nonlinear differential equations are well known examples of equations that have soliton solutions.

- Korteweg-de Vries equation: The KdV equation [22] is given by

$$\frac{\partial \Psi}{\partial t} + k\Psi \frac{\partial \Psi}{\partial x} + \frac{\partial^3 \Psi}{\partial x^3} = 0, \tag{1}$$

where $\Psi(x,t)$ depends on both position x and time t, and k is a constant. The KdV equation was developed by the mathematicians Korteweg and de Vries, and it is typically used to describe the lossless evolution of shallow water waves. In addition, the KdV equation describes longitudinal dispersive waves in elastic rods, magnetohydrodynamic waves in plasma, Ion-acoustic waves in plasma, the anharmonic lattice and thermally excited phonon packets in low-temperature nonlinear crystal [21].

- The sine-Gordon equation: This nonlinear equation [23]

$$\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial t^2} = \sin(\Psi), \tag{2}$$

appears in differential geometry and relativistic field theory. It has been used also to describe the propagation of light through a crystal dislocation and lipid membrane, propagation of magnetic flux on a Josephson line, Bloch wall motion of magnetic crystals and a unitary theory for elementary particles [21].

- The Nonlinear Schrödinger Equation: The NLSE [24]

$$\frac{\partial^2 \Psi}{\partial x^2} + i\frac{\partial \Psi}{\partial t} + k|\Psi|^2\Psi = 0, \tag{3}$$

is widely used in nonlinear optics. This equation has been used to describe one-dimensional self-modulation of monochromatic waves, the self-trapping phenomena of nonlinear optics, Langmuir waves in plasmas and is also related to the Ginzburg-Landau equation of superconductivity and to the propagation of a heat pulse in a solid [21].

There are several other well known nonlinear wave equations that exhibit solitons. For instance: the Klein-Gordon equation, the Boussinesq systems, the Landau-Lifshitz system, and the Burgers equation that arises in fluid mechanics to describe one-dimensional turbulence [3, 19, 25].

2. Nonlinear Schrödinger equation and solitons

In addition to the applications of NLSE mentioned above, the NLSE also describes soliton propagation in nonlinear dispersive optical fibers [4, 5, 18]. Because the application of solitons in fiber optic communication systems is a rapidly growing field, we have chosen the NLSE for discussing the soliton properties and basic numerical methods for solving nonlinear wave equations. We stress that the discussion given in this paper can be applied straightforwardly to other nonlinear wave equations.

If $u(z,t)$ denotes the complex amplitude of a wave pulse travelling along an optical fiber, the spatiotemporal evolution of the pulse is governed by the NLSE with the form:

$$\frac{\partial u}{\partial z} = -i\frac{\beta}{2}\frac{\partial^2 u}{\partial t^2} + i\gamma|u|^2 u, \tag{4}$$

where z is the distance along the direction of propagation in the optic fiber, t is the time a travelling frame of reference, β is the group-velocity dispersion parameter (this parameter quantifies how much the index of refraction changes with respect to the frequency of the wave) and γ is a constant that quantifies the nonlinear phenomena of the media where the wave pulse is propagated [4, 5]. An important condition for ensuring the existence of solitons is $\beta < 0$ [26]. Equation (4) is often written in a simpler normalized form by taking $\beta = -1$ and $\gamma = 1$. Here we have kept the parameters β and γ in order to control the strength of the dispersive and nonlinear effects.

The NLSE does not generally lend itself to analytic solutions except for some special cases in which some special methods for solving nonlinear partial differential equations can be used, (*i.e.* the inverse scattering method, the Hirota method and the Bäcklund transform) [15, 16, 20, 25] Equation (4) allows exact N-order temporal soliton solutions for the special case when the initial condition $U(t)$ is given by

$$u(z=0,t) = U(t) = NE_0\,\mathrm{sech}(t/t_0), \tag{5}$$

where the soliton order N is an integer, E_0 is the pulse amplitude, and t_0 is the time width parameter. The amplitude and the width of a soliton are not independent, but satisfy the inverse relation

$$E_0^2 t_0^2 = \frac{|\beta|}{\gamma}. \tag{6}$$

This relation is a crucial property of the optical solitons of the NLSE. For reference purposes we include in Appendix A the complete expressions of the analytic solutions for the first three order solitons ($N = 1$, 2, and 3).

An interesting property of the N–soliton is that the intensity $|u\,(z,t)|^2$ is periodic in z with the period

$$L = \frac{\pi}{2}\frac{t_0^2}{|\beta|} = \frac{\pi}{2}\frac{1}{\gamma E_0^2}, \tag{7}$$

and this periodicity occurs for all high–order solitons.

3. Propagating solitons

If the initial condition $U(t)$ does not correspond to a N-soliton, then a numerical approach is necessary for a full understanding of the propagation. Roughly speaking, numerical methods for obtaining solutions to initial value problems of nonlinear wave equations fall into two categories: finite difference methods and spectral methods. In general, spectral methods are faster for achieving the same accuracy. For historical reasons the literature on spectral methods has been disjoint from the literature on classical numerical analysis. The existence of highly efficient algorithms such as the Fast Fourier Transform (FFT) has made spectral methods common in current numerical research [4, 9, 10]. However, finite difference methods are more intuitive, more easily understood, and relatively easy to program in a personal computer. For this reason, we have chosen a direct explicit finite difference method to simulate the propagation of solitons in this paper. For the interested reader, a review of the split-step Fourier spectral method is included in Appendix B.

3.1. Description of the direct explicit method

The feature of a finite difference method is the approximation of the spatial and temporal derivatives in the governing equation by finite differences that relate the values of the unknown function at a set of neighboring grid points at various times and positions. The goal is to calculate the values of the unknown function at the nodes of a grid that covers the domain of the solution. The finite difference grid may be defined in rectangular coordinates (see for example Fig. 1) or other orthogonal coordinates depending on the boundary conditions.

The *direct explicit method* is perhaps the simplest algorithm based on finite–differences to solve the NLSE. Assume that the initial conditions of the pulse launched is given by the temporal shape $u(z=0,t)=U(t)$, where the time extends in the range $t_i \leq t \leq t_f$, and the function $u(z,t)$ to be determined is the spatiotemporal evolution of the pulse. We discretize time and space and introduce the shorthand notation

$$u_k^j = u(z_j, t_k), \tag{8}$$

where $z_j = jh$ and $t_k = t_i + (k-1)\tau$. The finite difference grid consists of an array of perpendicular lines that run parallel to the z and t axes. This (z,t) space-time plane is represented by the grid matrix shown in Fig. 1, where $j = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots, K$ denote the spatial and temporal indices or levels of a given grid point respectively. The space and time increments are denoted by h and τ, respectively. Note that the boundary points are u_1^j and u_K^j, so the grid spacing is $\tau = (t_f - t_i)/(K-1)$.

Our objective is to compute the values of the function u_k^j at the grid points t_k, at a sequence of z successive grid points z^j beginning from the initial condition at $z^0 = 0$, and subject to the boundary conditions $u_1^j = 0$ and $u_K^j = 0$. We approximate the spatial derivative in Eq. (4) by a first-order two-point difference and the time derivative by a second-order centered difference, [9–12] and obtain

$$\frac{u_k^{j+1} - u_k^{j-1}}{2h} = -i\frac{\beta}{2}\left(\frac{u_{k+1}^j - 2u_k^j + u_{k-1}^j}{\tau^2}\right) + i\gamma \left|u_k^j\right|^2 u_k^j. \tag{9}$$

Solving for u_k^{j+1}, we obtain

$$u_k^{j+1} = -\alpha\left[u_{k+1}^j - 2u_k^j + u_{k-1}^j\right] + i2h\gamma \left|u_k^j\right|^2 u_k^j + u_k^{j-1}, \tag{10}$$

where $\alpha \equiv i\beta h/\tau^2$. The direct explicit method provides us with a straightforward algorithm to compute u at level $j+1$ in terms of the values of u at levels j and $j-1$. Because everything that depends on j and $j-1$ is on the right-hand side, while only the next value of u is on the left, this method is an example of an explicit method.

Equation (10) can be applied to the internal grid points $k = 2, \ldots, K-1$, but not at the boundary points $k=1$ and $k=K$. If we express the values of u_k^j as a column vector $\mathbf{u}^j = [u_2^j, \ldots, u_{K-1}^j]^T$, we can conveniently rewrite Eq. (10) in matrix form as

$$\mathbf{u}^{j+1} = \mathbf{A}\mathbf{u}^j + \mathbf{v}^j + \mathbf{u}^{j-1}, \tag{11}$$

where $\mathbf{A}$ is a tridiagonal square matrix of size $(K-2)\times(K-2)$ given by

$$\mathbf{A} = \begin{pmatrix} 2\alpha & -\alpha & 0 & \cdots & \cdots \\ -\alpha & 2\alpha & -\alpha & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \cdots & 0 & -\alpha & 2\alpha & -\alpha \\ \cdots & \cdots & 0 & -\alpha & 2\alpha \end{pmatrix}, \tag{12}$$

and $\mathbf{v}^j$ is the column vector

$$\mathbf{v}^j = i2\gamma h\left[\left|u_2^j\right|^2 u_2^j, \ldots, \left|u_{K-1}^j\right|^2 u_{K-1}^j\right]^T. \tag{13}$$

Equation (11) is the basic propagating equation for solving the NLSE numerically with the direct explicit method and then for simulating temporal solitons. Note that the direct explicit method in Eqs. (10) and (11) is a three space-level method, that is, the method uses the levels $\mathbf{u}^{j-1}$, $\mathbf{u}^j$ to determine the next one, $\mathbf{u}^{j+1}$. Because only an initial condition at level $j=0$ is specified, the scheme is not self-starting. To get it started, u^1 can be determined from u^0 by applying an auxiliary first-order forward difference method to approximate the spatial derivative, that is,

$$\frac{u_k^1 - u_k^0}{h} = -i\frac{\beta}{2}\left(\frac{u_{k+1}^0 - 2u_k^0 + u_{k-1}^0}{\tau^2}\right) + i\gamma\left|u_k^0\right|^2 u_k^0. \tag{14}$$

Solving for u_k^1, we obtain

$$u_k^1 = -\frac{\alpha}{2}\left[u_{k+1}^0 - 2u_k^0 + u_{k-1}^0\right] + ih\gamma|u_k^0|^2 u_k^0 + u_k^0, \tag{15}$$

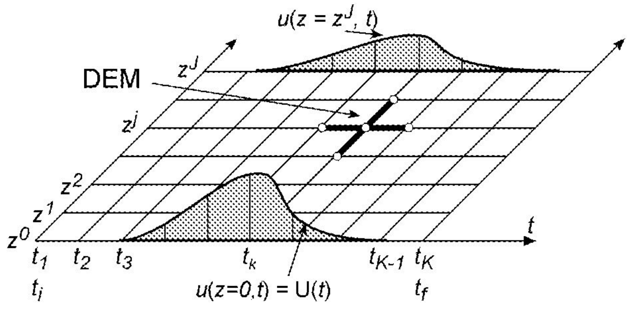

FIGURE 1. Grid matrix showing the (z,t) space-time plane.

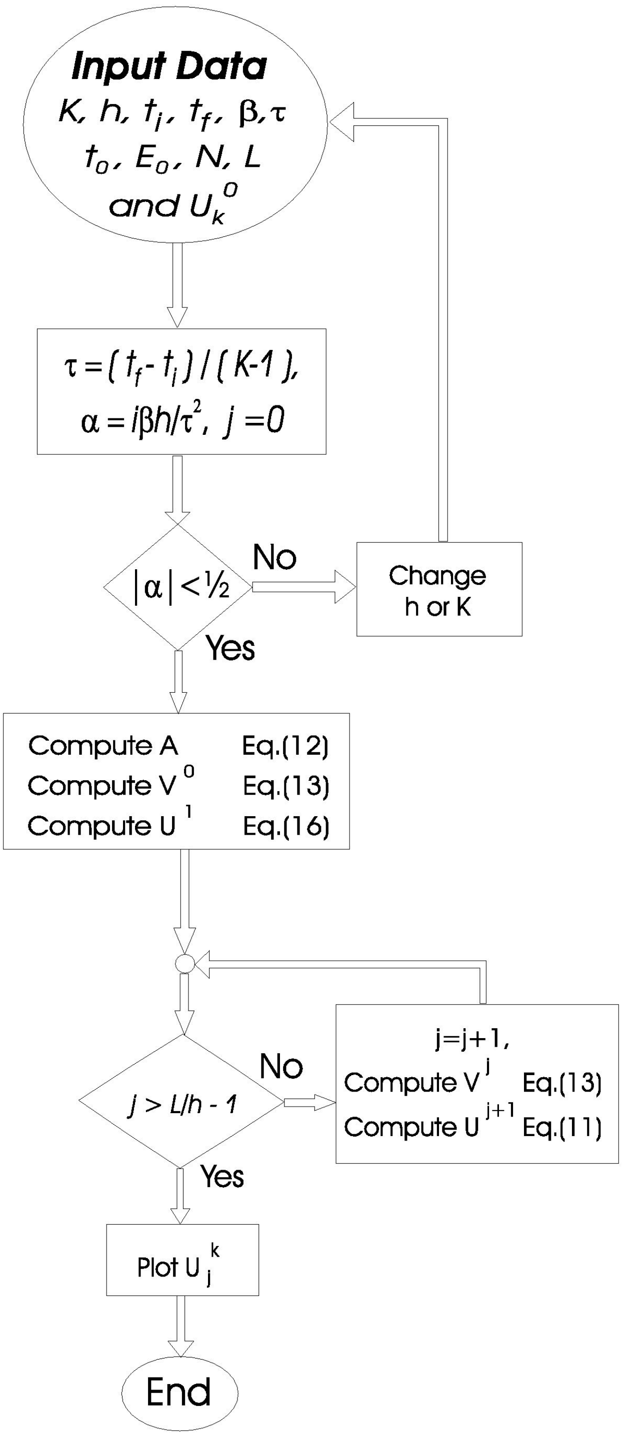

FIGURE 2. Flowchart for the direct explicit method.

or in matrix form

$$\mathbf{u}^1 = \frac{\mathbf{A}}{2}\mathbf{u}^0 + \frac{\mathbf{v}^0}{2} + \mathbf{u}^0. \tag{16}$$

Once $\mathbf{u}^1$ has been determined, Eq. (11) is used to produce the initial condition. The flowchart of the direct explicit algorithm for solving the NLSE is shown is Fig. 2. The stability of simple finite difference methods for partial differential equations may be assessed by several methods including the Von Neumann stability method, the projection matrix method, and the discrete perturbation method [12]. Application of the Von Neumann stability analysis reveals that the direct explicit method is numerically stable if $|\alpha| < 0.5$. [18]

3.2. Numerical problems

In order that students and instructors may gain a numerical insight into propagating solitons, we propose a set of simple numerical problems. The exercises are chosen in such a way as to show interesting physical phenomena.

Problem 1: Fundamental soliton

The evolution of the fundamental soliton (first order soliton, $N = 1$) is given by $u(z,t) = E_0 \mathrm{sech}(t/t_0)\exp(i\gamma E_0^2 z/2)$. Verfy by direct substitution that this expression is a solution of Eq. (4) and that Eq. (6) is a required condition.

Problem 2: Propagation of the fundamental soliton

Write a program that uses the direct explicit method to solve the NLSE. First we will use the direct explicit method to propagate a fundamental soliton ($N = 1$). Launch an initial profile given by Eq. (5) with $N = 1$. Assume the typical physical constants $\beta = -1.5\,\mathrm{ps}^2\,\mathrm{km}^{-1}$ and γ =3 W^{-1} km^{-1}. Choose the numerical parameters $K = 100$, $t \in [-5t_0, 5t_0]$, $t_0 = 2\,\mathrm{ps}$ and propagation step size $h = L/1000$ (where $L = \pi t_0^2/2|\beta|$). Use a mesh plot to graph the soliton evolution versus time and propagation distance. Remember to plot the intensity $|u(z,t)|^2$ (the squared absolute value of the field) and not the field itself. Observe how the form of this pulse remains constant because there is an equilibrium between the dispersion and nonlinear phenomena. Try a variety of values for τ and h and verify that the method is conditionally stable. You also could run the same simulation but now using $\beta = +1.5\,\mathrm{ps}^2\,\mathrm{km}^{-1}$ and observe how the soliton is not formed.

Important points to keep in mind: Remember that the amplitude and the width of the initial shape of the soliton are not independent [see Eq. (6)]. Be careful with the units of the physical constants and verify that the input data and your algorithm are dimensionally consistent. Keep in mind that the propagated field $u(z,t)$ is a complex function, consequently it carries information of both amplitude and phase of the field. The physical quantity to observe is the intensity, *i.e.* $|u|^2 = u^*u$, where the asterisk denotes complex conjugate. We have suggested some values for the numerical step sizes and physical constants in the simulation, however we encourage you to run your simulation several times adjusting the step sizes and changing the physical constants until you obtain nice, acceptable propagations.

In Fig. 3a we show the evolution of the fundamental soliton obtained with the direct explicit method using the values of the parameters from Problem 2. All numerical calculations done in this paper were completed within less than 1 min on a 1 GHz personal computer with 512 Mbyte RAM. Matlab was used, mainly because of its wealth of built-in matrix functions.

Problem 3: Propagating the second and third order solitons

In the fundamental soliton propagation shown in Fig. 3a, we can appreciate how nonlinearity and dispersion are in perfect balance along the propagation distance. Unlike fundamental soliton, higher–order solitons exhibit a periodic balance, *i.e.* the propagation is governed by a balance between broadening (due to dispersion) and narrowing (due to nonlinearity) such that the pulse recovers its original shape after a period L given by Eq. (7).

Test your routine made in Problem 2 by propagating the second and third order solitons. Launch an initial profile given by Eq. (5) using $N = 2$ and $N = 3$ respectively, and satisfying the condition (6). Use the same physical constants that were used in Problem 2. For visualizing purposes, the evolution of the second and third order solitons along a period L are depicted in Figs. 3b and 3c. These plots were computed with the direct explicit method using the propagation step size $h = L/5000$ and $t \in [-5t_0, 5t_0]$ for the second order soliton, and $h = L/15000$ and $t \in [-4t_0, 4t_0]$ for the third order soliton. For higher-order solitons smaller propagation step sizes are needed because the phase of the field $u(z,t)$ exhibits increasing longitudinal variations as the order increases. You could try a variety of values for h and τ to show that the accuracy of the propagation decreases with increasing h. Remember that the intensity profile is recovered after a distance L, so this can be useful to monitor the accuracy of your simulations.

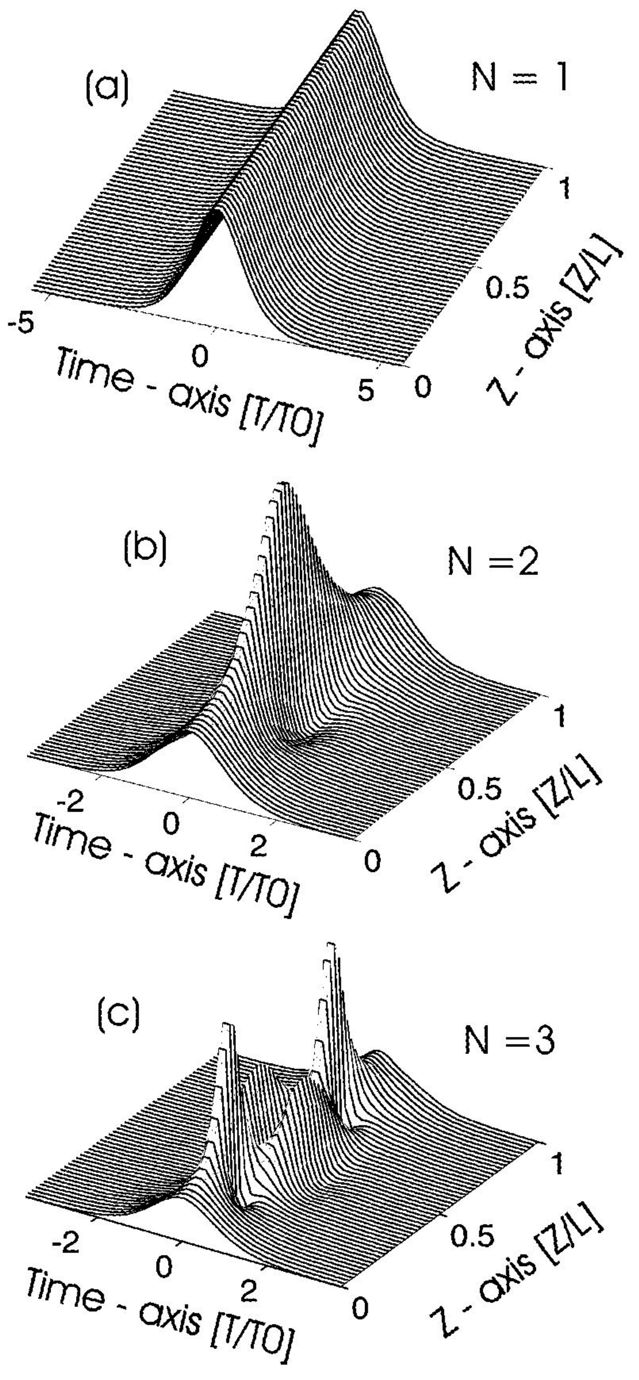

FIGURE 3. Space-time evolution over one soliton period for the first three order solitons. Plots correspond to the intensity $|u(z,t)|^2$. Numerical values of the parameters are included in Problems 2 and 3.

Problem 4: Soliton stability I: propagation of a perturbed fundamental soliton

From a practical point of view, one may ask how the soliton is affected if the initial pulse shape or the peak amplitude is not matched to that required by Eq. (5). In this problem, we numerically explore the case for which N in Eq. (5) is not an integer. To this end, we perform a variety of simulations with an initial pulse shape of the form $U(t) = (N+\delta)E_0 \operatorname{sech}(t/t_0)$, where $|\delta| < 1/2$. Choose $K = 100$, $t \in [-5t_0, 5t_0]$, and $h = L/5000$. The propagation of this perturbed pulse is depicted in Fig. 4. Verify the following known results: Fig. 4a The pulse broadens as it propagates along the fiber if $\delta < 0$ and narrows if $\delta > 0$. Fig. 4b The pulse adjusts its shape and width as it propagates and evolves into a soliton whose order is an integer closest to the value of $N + \delta$. Hint: You need to propagate for several soliton periods (typically 5) to see this asymptotic behavior. The soliton stability is a very important characteristic because it permits initial conditions for forming a soliton to be easily reproduced (you will need to launch an initial pulse like a hyperbolic secant profile and not an exact hyperbolic secant form). This condition plus soliton robustness against collisions with other pulses, have made the soliton a strong candidatefor the next bit representation in optical telecommunications.

Problem 5: Soliton stability II: propagation of Gaussian pulses

Let us now investigate the case when the initial pulse shape does not correspond to a sech shape. Assume an initial super-

Gaussian pulse of the form

$$U = E_0 \exp\left(-\frac{t^{2m}}{3mt_0^{2m}}\right), \qquad (17)$$

with $t_0 = 2\,\text{ps}$, and $m = 1, 2, 3, \ldots$ The case $m = 1$ reduces to a simple Gaussian pulse. Use the same physical parameters as in Problem 2, and $K = 100$, and $h = L/5000$, $t \in [-8t_0, 8t_0]$, and $m = 1$. Show that this pulse adjusts its width and evolves asymptotically into a fundamental soliton after a distance of about three soliton periods. The propagation of the super–Gaussian pulse is shown in Figs. 5a and 5b for $m = 1$ and $m = 2$, respectively. Try a variety of input shapes such as triangular and square pulses, and verify that the qualitative behavior remains the same. Keep the energy of the perturbed pulses approximately the same as for the fundamental soliton. The reason why the initial perturbed pulse tends to the soliton shape is that the soliton solution is a nonlinear mode of the NLSE. In this reshaping process, the energy dispersed is known as continuous radiation [4].

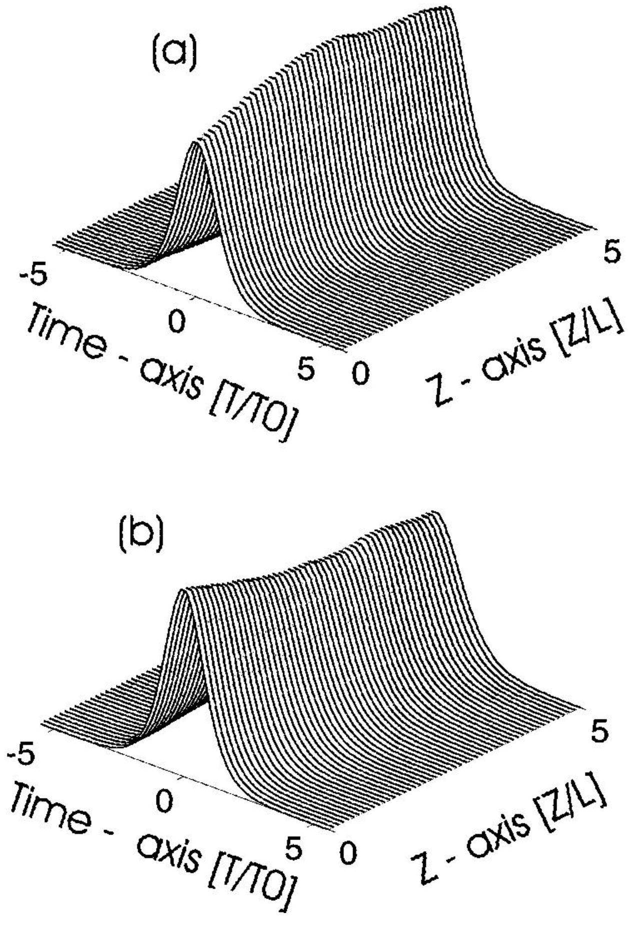

FIGURE 4. (a) Fundalmental soliton with an initial positive perturbation ($\delta = +0.075$). (b) Fundamental soliton with an initial negative perturbation ($\delta = -0.075$).

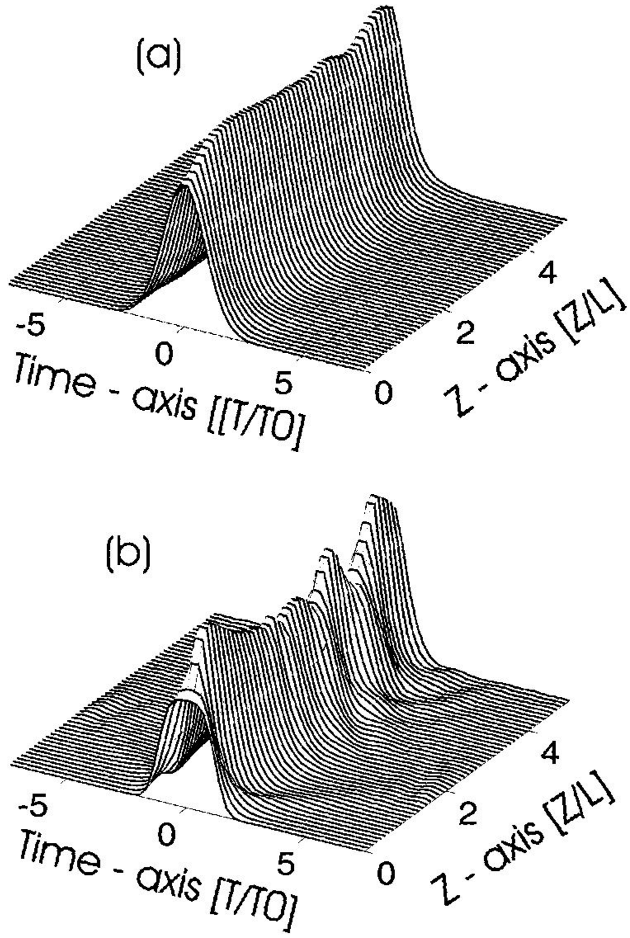

FIGURE 5. Time evolution of a superGaussian input pulse with (a) $m = 1$, and (b) $m = 2$.

You could study also the dispersion and nonlinear phenomena selecting either β or γ equal to zero and observing the pulse intensity and phase propagation. If there is no nonlinearity, Eq. (4) reduces to $\partial u/\partial z = -i(\beta/2)\partial^2 u/\partial t^2$, which is a special case of the diffusion equation [10]. If there is no dispersion, Eq. (4) reduces to $\partial u/\partial z = i\gamma|u|^2 u$, where in your phase propagation simulation, you can appreciate how the nonlinear phenomenon has a great impact on the phase components of the field.

Problem 6: Interacting solitons

The interaction between two solitons launched into a fiber is important from a practical point of view [4, 5] and also illustrates the particle-like behavior of the solitons. Propagate two fundamental solitons initially separated by a time $2q$ from each other. Consider the following initial pulse waveform:

$$U(t) = E_0 \text{sech}\left(\frac{t-q}{t_0}\right) + QE_0 \text{sech}\left(\frac{Q(t+q)}{t_0}\right) \times \exp(i\phi), \quad (18)$$

where Q is the relative amplitude and ϕ is the initial phase difference. First study the case of equal amplitude solitons

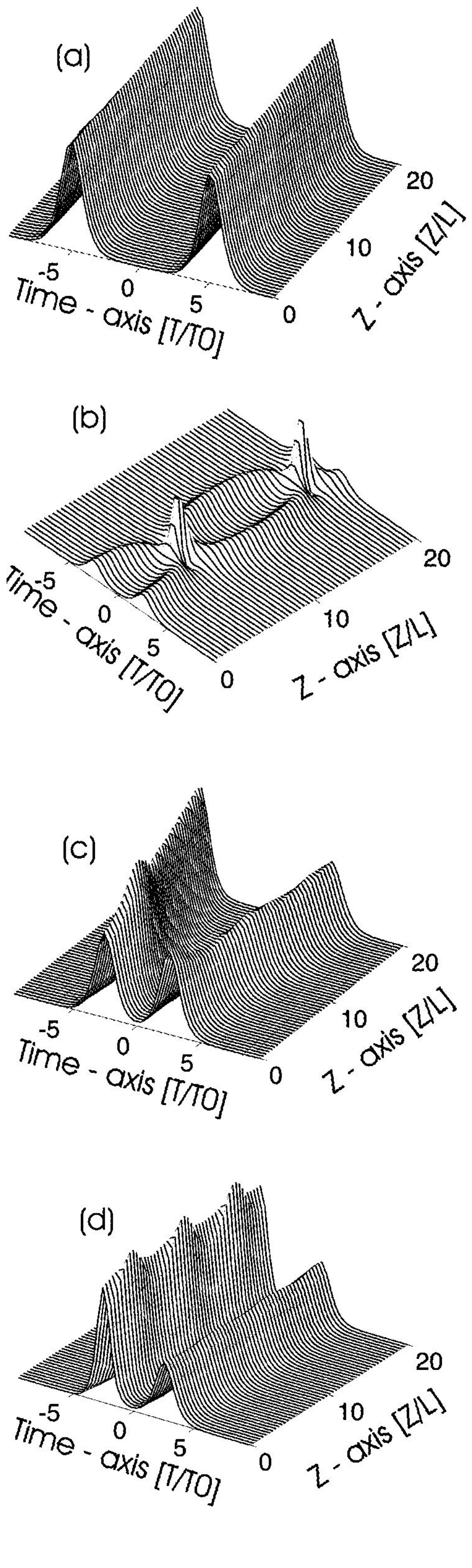

FIGURE 6. Time evolution of the interaction of two solitons over 20 soliton periods. (a) No interaction, (b) attraction effect, (c) repulsion effect, and (d) interaction of solitons with different intensities.

($Q = 1$). In the following exercises, take a time range $t \in [-10t_0, 10t_0]$ and propagate until at least 20 soliton periods.

(a) Show that a time separation equal to $q \approx 5t_0$ is large enough for avoiding the interaction between the solitons, see Fig. 6a.

(b) Choose an initial separation $T = 2.5t_0$, and propagate about 20 soliton periods. Show that the two solitons attract each other and collide periodically along the fiber in the in-phase case ($\phi = 0$). Set $K = 150$, $t \in [-5t_0, 5t_0]$, and $h = L/15000$. The propagation is shown in Fig. 6b. A very small step size h must be used for propagating the attractive interaction of solitons because the phase of the attractive interaction oscillates very quickly along the propagation distance. Plot the evolution of the phase in order to see this behavior. By performing several simulations, you could find the optimum distance of separation between solitons where there is no interaction between them. This distance would be translated into the optimum bandwidth capacity of the fiber using solitons (*i.e.* the capacity of transmitting the fastest and the largest amount of information).

(c) Using the same parameters used in exercise (b), show that the solitons repel each other and their spacing increases monotonically with the distance for $\phi = \pi/2$ [see Fig. 6c].

(d) Study the case of slightly different amplitudes by choosing $Q = 1.4$ and $\phi = 0$. Show that the two solitons oscillate periodically but never collide or move far away from each other. From this simulation you can see that you can increase your bandwidth capacity of your optical system, thus sending soliton with different amplitudes. The results are shown in Fig. 6d.

4. Conclusions

We have assessed a simple numerical method to solve nonlinear wave equations and have used it to study the physical behavior of temporal optical solitons in fiber optics. Our main goal was to encourage familiarity with soliton phenomena among the undergraduate community. We have proposed a set of challenging numerical problems in order to encourage the students and instructors to explore the striking behavior of solitons and nonlinear wave phenomena in various fields of physics.

Acknowledgments

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Appendix A: Explicit expressions for soliton solutions

For reference purposes, in this appendix we include the evolution expressions for the N-soliton solutions of the NLSE for $N = 1, 2, 3$. In the following, $T = t/t_0$ is the normalized time and $Z = z/L_D$ is the normalized distance, where $L_D = 1/(\gamma E_0^2) = t_0^2/|\beta|$ is the dispersion length.

For $N = 1$:

$$u(Z,T) = E_0 \exp(iZ/2) \operatorname{sech}(T), \tag{19}$$

For $N = 2$:

$$u(Z,T) = 4E_0 \exp(iZ/2) \frac{G_2(Z,T)}{H_2(Z,T)}, \tag{20}$$

where

$$G_2(Z,T) = \cosh(3T) + 3\cosh(T)\exp(i4Z),$$

$$H_2(Z,T) = \cosh(4T) + 4\cosh(2T) + 3\cos(4Z).$$

For $N = 3$:

$$u(Z,T) = 3E_0 \exp(iZ/2) \frac{G_3(Z,T)}{H_3(Z,T)}, \tag{21}$$

where

$$\begin{aligned} G_3(Z,T) = {} & 2\cosh(8T) + 32\cosh(2T) \\ & + \exp(i4Z)\left[36\cosh(4T) + 16\cosh(6T)\right] \\ & + \exp(i12Z)\left[20\cosh(4T) + 80\cosh(2T)\right] \\ & + 5\exp(-i8Z) + 45\exp(i8Z) + 20\exp(i16Z), \end{aligned}$$

and

$$\begin{aligned} H_3(Z,T) = {} & \cosh(9T) + 9\cosh(7T) \\ & + 64\cosh(3T) + 36\cosh(T) \\ & + 36\cosh(5T)\cos(4Z) \\ & + 20\cosh(3T)\cos(12Z) \\ & + 90\cosh(T)\cos(8Z). \end{aligned}$$

Appendix B: Split-step Fourier method

In this Appendix we briefly discuss one of the most popular algorithms for solving nonlinear differential equations, the split-step Fourier method (SSFM). This method relies on ideas associated with the Fourier transform and is usually placed under the general heading of *spectral* methods. The existence of highly efficient algorithms such as the Fast Fourier Transform (FFT) has made spectral methods common in current numerical research. The split-step Fourier method has been used extensively to solve the NLSE in the context of optical fibers. [4].

To understand the SSFM, we start by writing the NLSE in the operator form

$$\frac{\partial u}{\partial z} = \left[\widehat{D} + \widehat{S}\right] u, \tag{22}$$

where $\widehat{D} \equiv -i(\beta/2)\,\partial^2/\partial t^2$ and $\widehat{S} \equiv i\gamma|u|^2$ are the differential operators that account for dispersion and nonlinear effects in the medium. The method assumes that when propagating $u(z,t)$ from z to $z+h$, the dispersive and nonlinear effects act independently. In the first step, only dispersion acts, and $\widehat{S} = 0$. The effect of the nonlinearity is now taken into account in the midplane of the segment.

By integrating Eq. (22) with respect to z we have

$$\int_z^{z+h} \frac{\partial u}{u} = \int_z^{z+h} \widehat{D}dz + \int_z^{z+h} \widehat{S}dz. \tag{23}$$

If the step size h is small enough, then the operators $\widehat{D}$ and $\widehat{S}$ are approximately constant and we obtain $u(z+h,t) \simeq \exp(h\widehat{S})\left[\exp(h\widehat{D})\, u(z,t)\right]$. Notice here that the exponential operator $\exp(h\widehat{D})$ acts over the argument $u(z,t)$, while the operator $\exp(h\widehat{S})$ operates over the argument $\exp(h\widehat{D})\, u(z,t)$.

The application of the operator $\exp(h\widehat{D})$ over $u(z,t)$ is easily performed in Fourier space, namely

$$\exp\!\left(h\widehat{D}\right) u(z,t) = \Im^{-1}\!\left\{\exp\left(h\widehat{D}(i\omega)\right)\Im\{u(z,t)\}\right\}, \tag{24}$$

where $\Im$ stands for the one dimensional Fourier transform over the time variable, ω is the angular frequency variable, and $\widehat{D}(i\omega)$ is the operator $\widehat{D}$ evaluated at $\partial/\partial T \to i\omega$ in such a way that $\widehat{D}(i\omega) = i(\beta/2)\omega^2$.

The implementation of the SSFM is relatively straightforward. The field $u(z,t)$ is first propagated for a distance h with dispersion only. After this, the field is multiplied by a complex term that accounts for the change phase suffered by the field due to the nonlinearity over the whole segment. This prescription is written mathematically as follows

$$u' = \Im^{-1}\left\{\exp\left(i\frac{\beta\omega^2}{2}h\right)\Im\{u(z,t)\}\right\} \tag{25}$$

$$u(z+h) = \exp\left(ih\gamma|u'|^2\right)\, u'. \tag{26}$$

The use of the SSFM has become widespread in recent years because of its fast execution compared with most finite–difference methods. The Fourier transform and its inverse can be performed efficiently with the fast Fourier transform algorithm [9,10] that is widely available in commercial mathematical software.

A comparison between the finite differences methods (FDM) and the SSFM is summarized as follows.

- SSFM has an error of second order in the step size, whereas for the FDMs it is adjustable depending on the specific method
- SSFM is up to one order of magnitude faster than FDM.
- Applicability of the SSFM is narrower than FDMs.
- FDMs permit boundaryless propagation.
- FDMs are easier to program than SSFM.

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26. For the case $\beta > 0$, the NLSE also admits soliton solutions. The intensity shape of these solitons exhibits a dip in a uniform background, and consequently they are referred to as *dark* solitons, see Ref. 5.