Some integrals involving a class of filtering functions†

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We discuss some properties of the function $\sin \pi x/\pi x$ which is (sometimes) indicated by the symbol sinc x. This function is associated with problems involving filtering or interpolating functions. Several integrals are presented and a general rule is discussed.

Keywords: Residue theorem; filtering function; interpolating function.

Algunas propiedades de la función $\sin \pi x/\pi x$ comunmente llamada sinc x, son discutidas. Tal función surge en problemas que involucran funciones de filtro o de interpolación. Algunas integrales son presentadas y una regla general és discutida.

Descriptores: Teorema de los residuos; función filtro; función interpolación.

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1. Introduction

Several real integrals can be performed by means of integrals on the complex plane. The problems which appear in such calculations are associated with two questions: (a) What is the function that must be considered?, and (b) What is the best (or most convenient) contour, of integration? After choosing the function and determining the contour we proceed to the calculation using the residue theorem and Jordan lemma.

The accumulation of chance effects and the Gaussian frequency distribution is discussed by Silberstein [1] and Grimsey [2], respectively. Some operations involving "white" noise, for example, the intermodulation distortion, are presented by Medhurst and Roberts [3].

Here we discuss a methodology for evaluating integrals related to the filtering functions which appear in several problems, for example, in the theory of probability and the Fourier transform technique.

This paper is organized as follows: in Sec. 2, we introduce a filtering function known also as an interpolating function; in Sec. 3, we obtain the integral explicitly, using the function $\text{sinc}^3\, x$, and we present some other integrals involving powers of sinc x.

2. Filtering function

The filtering function^i, also called the interpolating function, is defined by the quotient

$$\frac{\sin \pi x}{\pi x} \equiv \text{sinc}x$$

and obeys the following properties:

$$\begin{aligned} \text{sinc}\, 0 &= 1 \\ \text{sinc}\, k &= 0 \qquad k = \text{nonzero integer} \\ \int_{-\infty}^{\infty} \text{sinc}\, x\, dx &= 1, \end{aligned}$$

where the integral is interpreted as a normalization, *i.e.* the central ordinate is unity and the total area under the curve is also unity.

Another frequently needed function is the square of $\text{sinc}\, x$, *i.e.*

$$\text{sinc}^2\, x = \left(\frac{\sin \pi x}{\pi x}\right)^2,$$

which represents the pattern of radiation power of a uniformly excited antenna, or the intensity of light in the Fraunhofer diffraction pattern in a slit.

The properties associated with $\text{sinc}\, x$ are valid for the square of $\text{sinc}\, x$. More about these functions can be seen in Ref. 4, where the Fourier transform is considered, with its pictorial representation. Several applications can also be seen in Ref. 5 where information theory is discussed; applications to radar are also presented.

3. Integrals of filtering functions

Here, we introduced the sinc x function as a function normalized to unity. The same is true for the square of sinc x, *i.e.* its integral is normalized to unity. One might ask if for all powers of sinc x we have the same result, that is if their integrals are also normalized to unity, *i.e.* if the following results are valid:

$$\int_{-\infty}^{\infty} \text{sinc}^3\, x\, dx \stackrel{?}{=} 1; \qquad \int_{-\infty}^{\infty} \text{sinc}^4\, x\, dx \stackrel{?}{=} 1,$$

and so on. The answer to this question is no.

In this paper we discuss how to calculate a class of integrals involving a power of $\text{sinc}\, x$, *i.e.* integrals of the form

$$\int_{-\infty}^{\infty} \text{sinc}^k\, x\, dx$$

where $k = 1, 2, 3, \ldots$ We have already seen that in the cases $k = 1$ and $k = 2$ the integrals or the areas under the curves are unitary.

For simplicity, and for pedagogical reasons, we discuss the integral involving the function $\mathrm{sinc}^3 x$ only, but the methodology presented is the same for the other cases, with $k = 4, 5, \ldots$ Then, to calculate the integral

$$\int_{-\infty}^{\infty} \mathrm{sinc}^3\, x\, dx = \int_{-\infty}^{\infty} \left(\frac{\sin \pi x}{\pi x}\right)^3 dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^3 dx,$$

we firstly consider another convenient integral in the complex plane, *i.e.*, the following integral

$$\int_C \frac{A_1\, \mathrm{e}^{3iz} + A_2\, \mathrm{e}^{iz} + A_3}{z^3} dz \qquad (1)$$

where A_1, A_2 and A_3 are constants which will be determined in a convenient way.

C is a contour in the complex plane composed of two straight line segments, $-R < x < -\epsilon$ and $\epsilon < x < R$, where $x = \Re(z)$, and enclosed by two semicircles, C_1 and C_2 centered at $z = 0$ with radii ϵ and R, respectively. The contour is oriented in the positive direction, *i.e.* counterclockwise.

We note that the singularity in $z = 0$ is outside of the contour. We can also note that $z = 0$ can be a pole or a removable singularity, depending on the constants A_1, A_2 and A_3.

Then, to evaluate the integral, we require that the singularity be a removable singularity. Using the residue theorem [6] with the contour defined above, we can write

$$\int_C \frac{f(z)}{z^3} dz = 0$$

where $f(z) = A_1\, \mathrm{e}^{3iz} + A_2\, \mathrm{e}^{iz} + A_3$. As a result we get

$$\int_{-\infty}^{\infty} \frac{f(x)}{x^3} dx = -\lim_{\epsilon\to 0} \int_{C_\epsilon} \frac{f(z)}{z^3} dz, \qquad (2)$$

where $x = \Re(z)$, and we have used the Jordan lemma. In this expression, C_ϵ denotes the semicircle centered at $z = 0$ with radius ϵ.

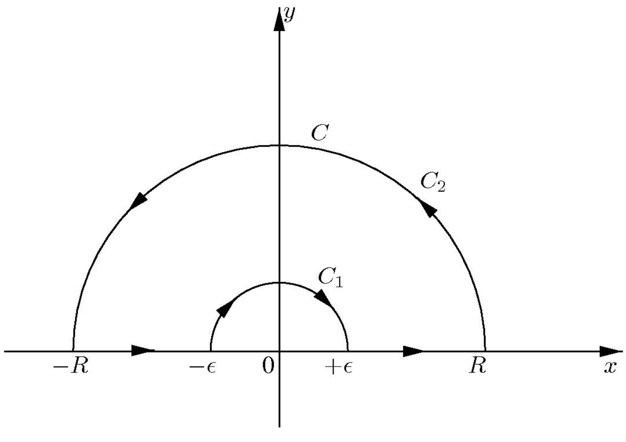

FIGURE 1. Contour for integration of Eq. (1).

We parameterize the semicircle as follows:

$$z = \epsilon\, \mathrm{e}^{i\theta}$$

with $0 < \theta < \pi$ and $\epsilon > 0$, and substituting in Eq.(2), we obtain

$$\int_{-\infty}^{\infty} \frac{f(x)}{x^3} dx = i \int_0^{\pi} \left\{ \lim_{\epsilon\to 0} \frac{f(\epsilon\, \mathrm{e}^{i\theta})}{\epsilon^2 \mathrm{e}^{2i\theta}} \right\} d\theta.$$

Using l'Hôpital's theorem, we can write[ii]

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{3ix} - 3\,\mathrm{e}^{ix} + 2}{x^3} dx = -3i\pi,$$

and by means of Euler's relation we obtain two integrals involving trigonometric functions, $\cos x$ and $\sin x$,

$$\int_{-\infty}^{\infty} \frac{\cos 3x - 3\cos x + 2}{x^3} dx = 0,$$

which is a well known result because the function under the integral is an even function integrated in a symmetric interval, and

$$\int_{-\infty}^{\infty} \frac{\sin 3x - 3\sin x}{x^3} dx = -3\pi.$$

Finally, to obtain our integral of the sinc x function, we use a relation involving the trigonometric functions of the triple angle written in terms of $\mathrm{sinc}^3 x$ and $\mathrm{sinc}\, x$, *i.e.*

$$\frac{1}{x^3}\sin 3x = \frac{3}{x^2}\mathrm{sinc}\, x - 4\,\mathrm{sinc}^3\, x,$$

and then

$$\int_0^{\infty} \mathrm{sinc}^3\, x\, dx = \frac{1}{\pi}\int_0^{\infty} \frac{\sin^3 x}{x^3} dx = \frac{3}{8}.$$

In the same way, we can show the following results:

$$\int_0^{\infty} \mathrm{sinc}^4\, x\, dx = \frac{1}{3};$$

$$\int_0^{\infty} \mathrm{sinc}^5\, x\, dx = \frac{115}{384};$$

$$\int_0^{\infty} \mathrm{sinc}^6\, x\, dx = \frac{11}{40}.$$

which are the same results that appear in Ref. 7. As a by-product, we can obtain a result for other filtering functions,

$$\int_0^{\infty} \mathrm{sinc}^7\, x\, dx = \frac{7\cdot 29^2}{2^9\cdot 3^2\cdot 5}$$

and

$$\int_0^\infty \operatorname{sinc}^8 x\, dx = \frac{151}{2 \cdot 3^2 \cdot 5 \cdot 7}.$$

4. Conclusion

In this paper we have pointed out a general methodology for evaluating some integrals involving a class of filtering functions, by means of a convenient integration in the complex plane. Another way to evaluate this type of integrals is discussed and presented by Sofo [8].

For pedagogical reasons, we have calculated explicitly only the integral involving $\operatorname{sinc}^3 x$, but the methodology can be extended to all integer powers of $\operatorname{sinc}^k x$, with $k = 1, 2, 3 \ldots$.

Unfortunately it was impossible to write a closed expression for these calculations.

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Appendix A: Delta function as a filtering function

We consider a function $D_T(x - \xi)$ defined as

$$D_T(x-\xi) = \frac{1}{2\pi} \int_{-T}^{T} \mathrm{e}^{i\mu(x-\xi)} d\mu$$

with $T > 0$ and, finding the integral over μ, we have

$$D_T(x-\xi) = \frac{1}{\pi} \left[\frac{\sin T(x-\xi)}{x-\xi} \right].$$

If we plot the graph of $D_T(x - \xi)$ as a function of x, maintaining $\xi =$ constant, for example, equal to zero, we note that the width of the curve decreases as T increases. Calculating the area under the curve, we have

$$\int_{-\infty}^{\infty} D_T(x-\xi) dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin T(x-\xi)}{x-\xi} dx = \frac{1}{\pi}$$

and, most important, the area is independent of width T.

Therefore, when T increases, the function $D_T(x - \xi)$ becomes like the Dirac delta functions, *i.e.*

$$\delta(x-\xi) = \lim_{T\to\infty} \frac{1}{\pi} \left[\frac{\sin T(x-\xi)}{x-\xi} \right]$$

and, remembering the property of the Dirac delta function,

$$\int_{-\infty}^{\infty} f(\xi)\delta(x-\xi) d\xi$$

$$= \lim_{T\to\infty} \int_{-\infty}^{\infty} f(\xi) \frac{1}{\pi} \left[\frac{\sin T(x-\xi)}{x-\xi} \right] d\xi = f(x),$$

we associate the function $D_T(x - \xi)$ as a filtering function.

Appendix B: Determining the constants A_1, A_2 and A_3

To calculate constants A_1, A_2 and A_3, we first set $f(0) = 0$ and then we get

$$A_1 + A_2 + A_3 = 0.$$

Now, deriving the function $f(z)$ in relation to z and taking $z = 0$ (singularity), we obtain

$$3A_1 + A_2 = 0.$$

Then, we have a system for three constants, A_1, A_2 and A_3, but only two equations. This system is an indeterminate system, *i.e.* it has infinite solutions. For example, taking $A_1 = 1$, $A_2 = -3$, and $A_3 = 2$ (we can take any one of the infinite solutions of the indeterminated system), we obtain a solution of this system and then our function can be written as follows:

$$f(z) = \mathrm{e}^{3iz} - 3\,\mathrm{e}^{iz} + 2.$$

For example, for the case $k = 7$, we must consider the following function:

$$f(z) = \mathrm{e}^{7iz} - 7\mathrm{e}^{5iz} + 21\mathrm{e}^{3iz} - 35\mathrm{e}^{iz} + \frac{77}{6} z^4 + 14z^2 + 20.$$

In order to get this result, we must use l'Hôpital's rule five times.

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i. See bellow the Appendix A.

ii. See the Appendix B.

1. L. Silberstein, *Philos. Mag.* **35** (1944) 395.
2. A.H.R. Grimsey, *Philos. Mag.* **36** (1945) 294.
3. R.G. Medhurst and J.H. Roberts, *Math. Comp.* **19** (1965) 113.
4. R.N. Bracewell, *The Fourier transform and its applications*, Second Edition, Revised (WCB McGraw-Hill, New York, 1986).
5. P.M. Woodward, *Probability and information theory with applications to radar* (McGraw-Hill Book Company, New York, 1953).
6. E. Capelas de Oliveira, *Int. J. Math. Educ. Sci. Technol.* **32** (2001) 156.
7. I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series and products* (Academic Press, New York, 1980).
8. A. Sofo, *Int. J. Math. Educ. Sci. Technol.* **29** (1998) 914.