Electrostatic and magnetostatic fields for Bessel-Fourier distributed sources on infinite planes

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The fields produced by Bessel-Fourier distributions of static electric charges on an infinite plane are constructed by integrating the differential equations for the electrostatic potential and the electric intensity field, subject to the boundary conditions at the source plane. Similarly, the fields produced by Bessel-Fourier distributions of stationary electric currents on an infinite plane are constructed via the integration of the differential equations for the potential and the magnetic induction field, subject to the corresponding boundary conditions.

Keywords: Bessel-Fourier distributed sources; static electric charges and intensity fields; stationary electric currents and magnetic induction fields.

Los campos producidos por distribuciones de Bessel-Fourier de cargas eléctricas en un plano infinito se construyen integrando las ecuaciones diferenciales para el potencial electrostático y el campo de intensidad eléctrica, sujetos a las condiciones de frontera en el plano de la fuente. Similarmente, se construyen los campos producidos por corrientes eléctricas estacionarias en un plano infinito, por medio de la integración de las ecuaciones diferenciales para el potencial magnetostático y el campo de inducción magnética, sujetos a las condiciones de frontera correspondientes.

Descriptores: Distribuciones de Bessel-Fourier de fuentes; cargas estáticas y campo de intensidad eléctricas; corrientes eléctricas estacionarias y campos de inducción magnética.

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1. Introduction

Some electrostatic and magnetostatic configurations, based on simple geometries and source distributions, are studied in introductory courses, illustrating the concepts and laws of electrostatics and magnetostatics, respectively, and some of their applications [1,2]. The simplest and starting situation in electrostatics involves a point charge, with its Coulomb radial and inverse-distance electrostatic potential field. Through the superposition principle, other geometries with uniform charge distributions and their respective fields are identified. For a uniformly charged infinite straight line: radial and inverse-distance electric intensity field, and logarithm-of-the-distance electrostatic potential. For a uniformly charged infinite plane: normal and uniform electric intensity field and linear-in-the-distance electrostatic potential. The respective equipotential surfaces are concentric spheres, coaxial cylinders, and parallel planes. For a uniformly charged spherical surface, the electric intensity field is Coulombic outside and vanishes inside. Similarly, for a uniformly charged cylindrical surface, the electric intensity field vanishes inside, and is radial and inverse-distance outside. The superposition of the fields of two concentric spheres, two coaxial cylinders, and two parallel planes with equal charges of opposite signs leads to the vanishing of the fields outside, and their confinement inside the respective capacitors, for which the ratio of charge to potential difference, or *capacitance*, can be calculated.

In magnetostatics, the magnetic point charge situation does not exist. The simplest situation corresponds to an *infinite* straight wire carrying a stationary electric current, which was used by Oersted in his discovery that the electric current reorients a magnetic compass needle. Ampère formulated his right-hand rule to define the relative directions of the current and the needle: if the extended thumb represents the current direction, the fingers curling around the thumb indicate the direction in which the needle is oriented at each position. Ampère also formulated his circulation law: the circulation of the magnetic induction field around a curve C is proportional to the electric current intensity crossing the surface S of which C is the perimeter. The magnetic induction field for the straight wire geometry is inversely proportional to the distance, and its lines are coaxial circles. The reader's attention is called to recognizing the difference in directions of the field lines, and the same distance dependence of the field magnitudes, for the same straight line geometry in the electrostatic and magnetostatic cases. Again, by the application of the superposition principle, other geometries with uniform electric current distributions and their respective magnetic induction fields can be established. For an infinite straight circular cylinder with a stationary uniformly distributed current moving along its straight generatrices, the magnetic induction field vanishes inside, and coincides outside with that of the same total current moving along its axis. For the same *infinite* cylinder with a solenoidal winding, a stationary, uniformly-distributed current produces a magnetic induction field, which is uniform and along the axial direction inside, and vanishes outside. For an infinite plane with a uniformly distributed current, the magnetic induction field is uniform, parallel to the plane, and perpendicular to the direction of the current. The electrostatic and magnetostatic fields for the planes with uniform sources share the uniformity, and differ in directions.

The reader may ask if the simple and familiar situations just described can be modified in a simple way in order to generate other easy-to-understand and interesting situations of electrostatics and magnetostatics. The answer is yes, and is illustrated in the textbooks at the higher levels [3-7], specifically via the multipole expansions of the respective potential fields. However, most of the presentations in the books are limited to the fields outside a region where the sources are located, ignoring the fact that the fields inside are already present in the same expansion, and that there are geometrical and physical connections between the inner and outer fields. This was the motivation for writing the article *Multipole Expansions Inside and Outside* [8]. Other harmonic expansions in two dimensions in circular, elliptic, and parabolic cylindrical coordinates [9-12], and in three dimensions in circular cylindrical, parabolic, and spheroidal coordinates [13,14], in toroidal coordinates [15-18], and in bispherical coordinates [19,20] have been studied by our group in connection with the analysis of different electric, magnetic, and electronic devices.

In a recent work, harmonic static charge and stationary current distributions in planes were assumed to be of the circular and hyperbolic cosine types in one of the Cartesian coordinates, and the corresponding electrostatic and magnetostatic fields were constructed [21]. In this contribution, the harmonic source distributions on the planes are chosen to be of the Bessel-Fourier type, in circular cylindrical coordinates. The changes from [21] to this work in the type of distribution and in the coordinates translate also into changes in the dimensionality of the associated fields from two to three. There is also a change at the didactic and mathematical levels, requiring the use of Bessel radial functions instead of the (co)sine functions in the Cartesian coordinate. Section 2 presents the construction of the electrostatic potential and electric intensity fields from the assumed Bessel-Fourier static charge distributions in a plane. Section 3 illustrates the corresponding construction of the magnetostatic potential and magnetic induction fields for the Bessel-Fourier distributions of the stationary electric current in a plane. Section 4 contains a description of the results of the previous sections, including a comparison of the respective fields, and also their connections with other systems.

2. Construction of the electrostatic potential and electric intensity fields for Bessel-Fourier distributed static charges on an infinite plane

The electric intensity field $\vec{E}(\vec{r})$ is connected with its electrostatic charge sources via Gauss's law in differential equation and boundary condition forms:

$$\nabla \cdot \vec{E} = 4\pi\rho \tag{1}$$

$$(\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = 4\pi\sigma, \tag{2}$$

where ρ is the volume charge density, σ is the surface charge density, and $\hat{n}$ is a unit vector normal to the boundary surface. The conservative nature of the electrostatic field is expressed by the corresponding equations:

$$\nabla \times \vec{E} = 0 \tag{3}$$

$$(\vec{E}_2 - \vec{E}_1) \times \hat{n} = 0. \tag{4}$$

Integration of Eq. (3) is accomplished by introducing the electrostatic potential$\phi(\vec{r})$:

$$\vec{E} = -\nabla\phi. \tag{5}$$

Substitution of Eq. (5) into Eq. (1) leads to the Poisson equation for the potential,

$$\nabla^2\phi = -4\pi\rho. \tag{6}$$

For the situation of charge distributed in an infinite plane, the volume density ρ vanishes and Eqs. (1) and (6) become

$$\nabla \cdot \vec{E} = 0 \tag{7}$$

$$\nabla^2\phi = 0. \tag{8}$$

Consequently, the electric intensity field must be solenoidal, Eq. (7), and irrotational, Eq. (3); while the potential must be harmonic, Eq. (8).

The separability and integrability of Eq. (8) in circular cylindrical coordinates lead to the harmonic solutions

$$\phi(\rho,\varphi,z) = [A_m J_m(\kappa\rho) + B_m N_m(\kappa\rho)] \times [C_m \cos m\varphi + D_m \sin m\varphi][E_m e^{-\kappa z} + F_m e^{\kappa z}] \tag{9}$$

in terms of radial Bessel functions, angular (co)sine functions, and longitudinal exponential functions, with separation constants κ and $m = 0, 1, 2, 3, \ldots$ [22].

The Bessel-Fourier surface charge distributions in the $z = 0$ plane

$$\sigma_C = \sigma_0 J_m(\kappa\rho) \cos m\varphi \tag{10}$$

$$\sigma_S = \sigma_0 J_m(\kappa\rho) \sin m\varphi \tag{11}$$

determine the values of the parameters and coefficients in Eq. (9).

Specifically, for the charge distribution of Eq. (10), the potential becomes

$$\phi_C(\rho,\varphi,z \geq 0) = \phi_0 J_m(\kappa\rho) \cos m\varphi e^{-\kappa z} \tag{12}$$

$$\phi_C(\rho,\varphi,z \leq 0) = \phi_0 J_m(\kappa\rho) \cos m\varphi e^{\kappa z} \tag{13}$$

exhibiting its harmonic nature inherited from the source, its continuity at $z = 0$, and its vanishing for $|z| \to \infty$.

Then the electric intensity field follows directly from the substitution of Eqs. (12) and (13) into Eq. (5):

$$\vec{E}_C(\rho,\varphi,z\geq 0) = -\phi_0\left[\hat{\rho}\kappa J'_m(\kappa\rho)\cos m\varphi - \hat{\varphi}\frac{m}{\rho}J_m(\kappa\rho)\sin m\varphi - \hat{k}\kappa J_m(\kappa\rho)\cos m\varphi\right]e^{-\kappa z} \quad (14)$$

$$\vec{E}_C(\rho,\varphi,z\leq 0) = -\phi_0\left[\hat{\rho}\kappa J'_m(\kappa\rho)\cos m\varphi - \hat{\varphi}\frac{m}{\rho}J_m(\kappa\rho)\sin m\varphi + \hat{k}\kappa J_m(\kappa\rho)\cos m\varphi\right]e^{\kappa z} \quad (15)$$

Notice that the tangential components of the field at the source plane, $z = 0$, are continuous, so that Eq. (4) is satisfied, while the normal components in the $\hat{k}$ direction exhibit a discontinuity which, via Eqs. (2) and (10), determines the value of ϕ_0:

$$2\kappa\phi_0 = 4\pi\sigma_0. \quad (16)$$

For the charge distribution of Eq. (13), the final results are directly written, for the potential:

$$\phi_S(\rho,\varphi,z\geq 0) = \frac{2\pi\sigma_0}{\kappa}J_m(\kappa\rho)\sin m\varphi\, e^{-\kappa z} \quad (17)$$

$$\phi_S(\rho,\varphi,z\leq 0) = \frac{2\pi\sigma_0}{\kappa}J_m(\kappa\rho)\sin m\varphi\, e^{\kappa z} \quad (18)$$

and for the electric intensity field:

$$\vec{E}_S(\rho,\varphi,z\geq 0) = 2\pi\sigma_0\left[-\hat{\rho}J'_m(\kappa\rho)\sin m\varphi - \hat{\varphi}m\frac{J_m(\kappa\rho)}{\kappa\rho}\cos m\varphi + \hat{k}J_m(\kappa\rho)\sin m\varphi\right]e^{-\kappa z} \quad (19)$$

$$\vec{E}_S(\rho,\varphi,z\leq 0) = 2\pi\sigma_0\left[-\hat{\rho}J'_m(\kappa\rho)\sin m\varphi - \hat{\varphi}m\frac{J_m(\kappa\rho)}{\kappa\rho}\cos m\varphi - \hat{k}J_m(\kappa\rho)\sin m\varphi\right]e^{\kappa z} \quad (20)$$

The familiar case of the uniformly charged plane follows from Eq. (10) for the limit situation when $\kappa \to 0$ and $m = 0$, taking into account the fact that $J_0(\kappa\rho \to 0) \to 1$, $J'_0(\kappa\rho \to 0) \to 0$ and $e^{\mp\kappa z} \to 1 \mp \kappa z$. Then, the potential in Eqs. (12) and (13) becomes linear in the distance from the source plane, and the field in Eqs. (14) and (15) becomes uniform and normal to the source plane. with the discontinuity $4\pi\sigma_0$ associated with Gauss's law, Eq. (2).

The source distributions in Eqs. (19) and (11) for a chosen value of $m = 1, 2, 3, \ldots$, share the same form and differ only in their relative orientations. The same properties are naturally inherited by the potentials and the electric intensity fields, as follows from the comparisons of Eqs. (12)-(13) vs (17)-(18) and (14)-(15) vs (19)-(20).

3. Construction of the magnetic potential and magnetic induction fields for Bessel-Fourier distributed stationary currents on an infinite plane

The magnetic induction field $\vec{B}(\vec{r})$ is connected with its stationary electric current sources by Ampère's law in its differential equation and boundary condition forms:

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} \quad (21)$$

$$(\vec{B}_2 - \vec{B}_1)\times\hat{n} = \frac{4\pi}{c}\vec{K} \quad (22)$$

where $\vec{J}$ is the current per unit area and $\vec{K}$ is the current per unit length. The non-existence of magnetic monopoles is expressed by the magnetic Gauss law in its differential equation and boundary condition forms:

$$\nabla\cdot\vec{B} = 0 \quad (23)$$

$$(\vec{B}_2 - \vec{B}_1)\cdot\hat{n} = 0. \quad (24)$$

The solenoidal character of the magnetic induction field, Eq. (23), allows its expression as the curl of the magnetostatic vector potential$\vec{A}(\vec{r})$:

$$\vec{B} = \nabla\times\vec{A}(\vec{r}). \quad (25)$$

Substitution of this expression into Eq. (21) leads to the equation satisfied by the potential

$$\nabla\times(\nabla\times\vec{A}) = \frac{4\pi}{c}\vec{J}$$

$$\nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{J}. \quad (26)$$

By using the transverse gauge,

$$\nabla\cdot\vec{A} = 0,$$

equation (26) reduces to Poisson's equation

$$\nabla^2\vec{A} = -\frac{4\pi}{c}\vec{J}. \quad (27)$$

For the current sources distributed on the plane $z = 0$, the current density outside the plane vanishes, $\vec{J} = 0$. Then, the magnetic induction field becomes irrotational, and the potential satisfies Laplace's equation, since Eqs. (21) and (27) reduce to

$$\nabla\times\vec{B} = 0 \quad (28)$$

$$\nabla^2\vec{A} = 0. \quad (29)$$

The stationary electric currents on the source plane are chosen as the linear density distributions of the Bessel-Fourier types:

$$\vec{K}^C_m = I_0\nabla\times[\hat{k}J_m(\kappa\rho)\cos m\varphi] = I_0\left[-\hat{\rho}\frac{m}{\rho}J_m(\kappa\rho)\sin m\varphi - \hat{\varphi}\kappa J'_m(\kappa\rho)\cos m\varphi\right] \quad (30)$$

$$\vec{K}_m^C = I_0 \nabla \times [\hat{k} J_m(\kappa\rho) \sin m\varphi] = I_0 \left[\hat{\rho}\frac{m}{\rho} J_m(\kappa\rho)\cos m\varphi - \hat{\varphi}\kappa J_m'(\kappa\rho)\sin m\varphi\right]. \quad (31)$$

The harmonic magnetostatic potential, solutions of Eq. (29), share the Bessel-Fourier character of the sources. In the case of Eq. (30) we consider the potential above and below the source plane:

$$\vec{A}_m^C(\rho,\varphi,z\geq 0) = A_{m0}^C \times \left[-\hat{\rho}\frac{m}{\rho}J_m(\kappa\rho)\sin m\varphi - \hat{\varphi}\kappa J_m'(\kappa\rho)\cos m\varphi\right] e^{-\kappa z} \quad (32)$$

$$\vec{A}_m^C(\rho,\varphi,z\leq 0) = A_{m0}^C \times \left[-\hat{\rho}\frac{m}{\rho}J_m(\kappa\rho)\sin m\varphi - \hat{\varphi}\kappa J_m'(\kappa\rho)\cos m\varphi\right] e^{\kappa z}. \quad (33)$$

which is solenoidal and continuous at $z = 0$.

The magnetic induction field is obtained when the potential from these equations are substituted into Eq. (25):

$$\vec{B}_m^C(\rho,\varphi,z\geq 0) = A_{m0}^C\kappa \left[-\hat{\rho}\kappa J_m'(\kappa\rho)\cos m\varphi + \hat{\varphi}\frac{m}{\rho}J_m(\kappa\rho)\sin m\varphi + \hat{k}\kappa J_m(\kappa\rho)\cos m\varphi\right] e^{-\kappa z} \quad (34)$$

$$\vec{B}_m^C(\rho,\varphi,z\leq 0) = A_{m0}^C\kappa \left[\hat{\rho}\kappa J_m'(\kappa\rho)\cos m\varphi - \hat{\varphi}\frac{m}{\rho}J_m(\kappa\rho)\sin m\varphi + \hat{k}\kappa J_m(\kappa\rho)\cos m\varphi\right] e^{\kappa z} \quad (35)$$

Notice now that the normal components of the magnetic induction fields at $z = 0$ are continuous, Eq. (24) for $\hat{n} = \hat{k}$; while its tangential components are discontinuous and proportional to the linear current density, Eq. (30), consistent with Eq. (22) and determining the value of A_{0m}^C:

$$-2\kappa A_{0m}^C = \frac{4\pi}{c} I_0. \quad (36)$$

For the current distribution of Eq. (31), the corresponding magnetic potential and magnetic induction fields take the final forms:

$$\vec{A}_m^S(\rho,\varphi,z \gtrless 0) = -\frac{2\pi I_0}{c\kappa}\left[\hat{\rho}\frac{m}{\rho}J_m(\kappa\rho)\cos m\varphi - \hat{\varphi}\kappa J_m'(\kappa\rho)\cos m\varphi\right] e^{\mp\kappa z} \quad (37)$$

$$\vec{B}_m^S(\rho,\varphi,z \gtrless 0) = -\frac{2\pi I_0}{c\kappa}\left[\mp\hat{\rho}\kappa J_m'(\kappa\rho)\sin m\varphi \mp\hat{\varphi}\frac{m}{\rho}J_m(\kappa\rho)\cos m\varphi + \hat{k}\kappa J_m(\kappa\rho)\sin m\varphi\right] e^{\mp\kappa z} \quad (38)$$

It is instructive to analyze the purely Bessel $m = 0$ distribution in Eq. (30), which becomes tangential to concentric circles:

$$\vec{K}_0^C = -I_0\hat{\varphi}\kappa J_0'(\kappa\rho) = I_0\hat{\varphi}\kappa J_1(\kappa\rho), \quad (39)$$

and, in addition, the derivative of the zero-order Bessel function is replaced by its value in terms of the first-order Bessel function [22]. Then, the magnetostatic potential also becomes circular, Eqs. (32) and (36),

$$\vec{A}_0^C(\rho,\varphi,z \gtrless 0) = \frac{2\pi I_0}{c\kappa}\hat{\varphi}\kappa J_0'(\kappa\rho)e^{\mp\kappa z} = -\frac{2\pi I_0}{c\kappa}\hat{\varphi}J_1(\kappa\rho)e^{\mp\kappa z}. \quad (40)$$

And the magnetic induction field does not have any circular components, Eqs. (34) and (35):

$$\vec{B}_0^C(\rho,\varphi,z \gtrless 0) = \frac{2\pi I_0}{c} \times \left[\mp\hat{\rho}\kappa J_1(\kappa\rho) - \hat{k}\kappa J_0(\kappa\rho)\sin m\varphi\right] e^{\mp\kappa z}. \quad (41)$$

Of course, the case $m = 0$ in Eq. (31) gives a vanishing charge distribution. For the other values of $m = 1, 2, 3, \ldots$, the corresponding charge distributions, Eqs. (30)-(31), magnetostatic potentials, Eqs. (32), (33), (36) and (37), and magnetic induction fields, Eqs. (34), (35), (36) and (38), have the same shapes and differ in orientations.

4. Discussion

Both elements, the geometrical region and the harmonicity of the sources of electrostatic and magnetostatic fields, are important for the construction or identification of the respective potential and force fields, which inherit the corresponding harmonicity, as illustrated by the references in the Introduction. Specifically, the well-known textbook examples of spheres, cylinders and planes with uniformly surface distributed sources, corresponding to the lowest harmonicity, have equipotential surfaces with the respective geometries and straight field lines. The higher harmonicity sources, potential and force fields are the natural extensions of those simplest examples. For other geometries, the surface source distribution with lowest harmonicity is no longer uniform [9-20].

The harmonic distribution of sources in a plane was chosen to be of cosine types in one Cartesian coordinate in Ref. 21, and of Bessel-Fourier types in this article. Naturally, the differences between the Cartesian and circular harmonicities of the sources are inherited by the respective potential and force fields, as a comparison of Secs. 2 and 3 of both works show. The only common case corresponds to the uniformly charged plane, associated with the lowest harmonicities, $k \to 0$ in [21], and $m = 0$, $\kappa \to 0$, as discussed at the end of the respective Secs. 2. Another difference between [21] and this work is in the method of solution: in the Cartesian two-dimensional case, the force fields were constructed directly as solenoidal and irrotational with their components subject to the respective boundary conditions; in the present circular cylindrical three-dimensional case, the potentials were constructed first borrowing the harmonicity of the sources, then the solenoidal and irrotational character of the force fields is guaranteed, and the boundary conditions are applied to determining the magnitude parameter for the potential, Eq. (36).

In Ref. 21 the electromagnetic fields associated with time-harmonically-varying currents distributed harmonically on an infinite plane were constructed. The corresponding problem, for currents with Bessel-Fourier, Mathieu, and Weber distributions on an infinite plane, has also been investigated using circular, elliptic, and parabolic cylindrical coordinates, respectively. It is recognized that the corresponding electromagnetic radiation fields correspond to the respective Propagation Invariant Optical Fields [23]. Their static limits correspond to the Bessel-Fourier cases studied here, and to the respective harmonic sources, potentials, and force fields in elliptic and parabolic geometries.

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