Soliton solutions for Fokas-Lenells equation by (G'/G)-expansion method

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In this paper we investigate the Fokas-Lenells equation via the (G'/G)-expansion method. To convert this nonlinear model into ODEs, we utilize an intelligible wave transformation. The solutions show that considered method fit well for Fokas-Lenells equation with complex structure. With the view of the results, new improvements can happen for applications of the model.

Keywords: Fokas-Lenells (FL) equation; (G'/G)-expansion method; soliton.

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1. Introduction

One of the main nonlinear problem is the Fokas-Lenells (FL) equation

$$\begin{aligned} &iq_t + a_1 q_{xx} + a_2 q_{xt} + |q|^2(bq + i\sigma q_x) \\ &\quad = i(\alpha q_x + \gamma[|q|^{2n}q)_x + \eta[|q|^{2n}]_x q). \end{aligned} \tag{1}$$

Where a_1 and a_2 are coefficients of group velocity dispersion, this model is the famous one which constructed as applications for current-fed string interacting with an external magnetic field. The perturbation terms $\eta, \gamma, \alpha,$ and on the right-hand side of Eq. (1) represent nonlinear dispersion, self-steepening effect, and, self-steepening effect, and inter-modal dispersion, respectively. There have been numerous approaches put forward by several authors to handle the approximate and exact solutions these nonlinear problems which are G'/G-expansion method, direct algebraic method, Sine-cosine method, tanh method, F-expansion method, and so on [1-15]. To construct the analytical solutions of these equations for realized the dynamics structure is the mainly suitable way [16-18]. The exact solution in explicit form of these models assist the confirmation of numerical researchers and also help in investigate the stability. The purpose of this work is to apply the G'/G-expansion method to solve the Fokas-Lenells (FL) equation . Such a study has not been considered before to our knowledge. This paper is outlined as follows: Section 2 gives a brief review of (G'/G)-expansion method. Section 3 contains the discussions. Finally, conclusion is given in Sec. 4. This paper is organized as follows: In Sec. 2, we describe the (G'/G)-expansion method while the application of the method has been presented in Sec. 3. The conclusions are drawn in Sec. 4.

2. Brief of the methodology

In this section, we briefly explain the application of (G'/G)-expansion to determine an exact solution for the partial differential equation. For given NLEEs of the form

$$\mathbf{P}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_t, \mathbf{u}_{xx}, \mathbf{u}_{xt}, \mathbf{u}_{tt}, \ldots) = 0. \tag{2}$$

Consider the wave transformation, $u(x,t) = U(\xi)$, $\xi = x - vt$, Eq. (2) can be reduced to the following ODE:

$$P(U, U', -vU', U'', -vU'', v^2U'', \ldots) = 0, \tag{3}$$

Where $U = U(\xi)$, and its total derivatives. the exact solution for the nonlinear ordinary differential equation can be written in the following:

$$u(\xi) = \sum_{n=1}^{m} \alpha_n \left(\frac{\mathrm{G'}(\xi)}{\mathrm{G}(\xi)}\right)^n \alpha_0, \quad \alpha_m \neq 0, \tag{4}$$

where G(ξ) satisfies a second order linear ordinary differential equation:

$$+\frac{d^2\mathrm{G}(\xi)}{d\xi^2} + \lambda\frac{d\mathrm{G}(\xi)}{d\xi} + \mu G(\xi) = 0. \tag{5}$$

And $\alpha_n, n = 0, 1, 2, \ldots, m,$ are constants to be determined later.

Using the solutions of Eq. (5), we obtain

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right]+C_2\cosh\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right]}{C_1\cosh\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right]+C_2\sinh\left[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right]}\right)-\frac{\lambda}{2}, \quad \lambda^2-4\mu>0\,, \\ \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sinh\left[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right]+C_2\cosh\left[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right]}{C_1\cosh\left[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right]+C_2\sinh\left[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right]}\right)-\frac{\lambda}{2}, \quad \lambda^2-4\mu<0\,, \end{cases} \tag{6}$$

and from (4) and (5), we have

$$U' = -\sum_{n=1}^{m} n\alpha_n\left(\left[\frac{G'}{G}\right]^{n+1}+\lambda\left[\frac{G'}{G}\right]^{n}+\mu\left[\frac{G'}{G}\right]^{n-1}\right),$$

$$U'' = \sum_{n=1}^{m} n\alpha_n\Bigg([n+1]\left[\frac{G'}{G}\right]^{n+2}+[2n+1]\lambda\left[\frac{G'}{G}\right]^{n+1}$$
$$+n[\lambda^2+2\mu]\left[\frac{G'}{G}\right]^{n}+[2n-1]\lambda\mu\left[\frac{G'}{G}\right]^{n-1}+[n-1]\mu^2\mu^2\left[\frac{G'}{G}\right]^{n-2}\Bigg),$$

Therefore, here the prime denotes the derivative with respective to ξ.

3. Discussions

Let us use the transformation

$$q(x,t) = u(\xi)e^{i\phi(\xi)-\Omega t}, \tag{7}$$

where $u(\xi)$ and $\phi(\xi)$ are real functions of the traveling coordinate $\xi = x - vt$. Here, v is the group velocity while Ω is the frequency of the wave oscillation.

Substituting the transformation (7) into Eq. (1), Thus we can easily get,

$$-(\alpha+v+a_2\Omega)u'+2(a_1-a_2v)u'\phi'+(a_1-a_2v)u\phi''+\sigma u^2u'-([2n+1]\gamma+2n\eta)u^{2n}u'=0, \tag{8}$$

$$(\alpha+v+a_2\Omega)u\phi'+\Omega u+u''-(a_1-a_2v)u\phi^{2'}+bu^3-\sigma u^3\phi'+\gamma u^{2n+1}\phi'=0. \tag{9}$$

Equation (9) can be integrated after multiplying on u to induce

$$\phi' = \frac{\alpha+v+a_2\Omega}{2(a_1-a_2v)}-\frac{\sigma u^2}{4(a_1-a_2v)}+\frac{([2n+1]\gamma+2n\eta)}{(a_1-a_2v)(2n+2)}, \tag{10}$$

where the integration constant is taken to be zero. We set $n = 1$. Therefore, Eq. (9) reduces to

$$\phi' = \frac{\alpha+v+a_2\Omega}{2(a_1-a_2v)}+\frac{(3\gamma+2\eta-\sigma)u^2}{4(a_1-a_2v)}. \tag{11}$$

Thus we can easily obtain,

$$\delta\omega(x,t) = -\frac{\alpha+v+a_2\Omega}{2(a_1-a_2v)}-\frac{(3\gamma+2\eta-\sigma)u^2}{4(a_1-a_2v)}. \tag{12}$$

Now, substituting Eq. (12) into Eq. (8) leads to

$$u''+\frac{s_2}{4}u+\frac{s_4}{2}u^3+\frac{s_6}{16}u^5=0 \tag{13}$$

where

$$s_2 = \frac{(\alpha + v + a_2\Omega)^2 + 4\Omega(a_1 - a_2 v)}{(a_1 - a_2 v)^2}, \tag{14}$$

$$s_4 = \frac{(\gamma - \sigma)(\alpha + v + a_2\Omega) + 2b(a_1 - a_2 v)}{(a_1 - a_2 v)^2}, \tag{15}$$

$$s_6 = \frac{(3\gamma + 2\mu - \sigma)(\gamma - 2\eta - 3\sigma)}{(a_1 - a_2 v)^2}. \tag{16}$$

According to Step 1, we put $5m = m + 2$, hence $m = 1$. Therefore, we assume that the solution of Eq. (13) can be expressed by a polynomial in (G$'$/G) as follows:

$$u = \alpha_1 \left(\frac{\mathrm{G}'}{\mathrm{G}}\right) + \alpha_0, \qquad \alpha_1 \neq 0. \tag{17}$$

Substituting Eq. (14) into Eq. (13) and collecting all terms with the same order of (G$'$/G) together , the left-hand side of Eq. (13) is conserved into a polynomial in (G$'$/G). Equating each coefficient of this polynomial to zero yields a set of simultaneous algebraic equations for λ, μ, ω, α_0, α_1, and α_2. Solving the system of algebraic equations with the aid of Maple 16, we obtain the following three general results.

Case 1.

The first set of obtained results is

$$v = \mp\frac{2}{3}\sqrt{s_2} - \frac{1}{2}\lambda^2, \qquad \mu = \pm\frac{2}{3}\sqrt{s_2}, \qquad \alpha_0 = 0, \qquad \alpha_1 = 3\sqrt{s_2}. \tag{18}$$

Where λ is an arbitrary constant. Therefore, substituting the above case in Eq. (17), we get solution of Eq. (1) for $\lambda^2 - 4\mu > 0$,

$$q_1(x,t) = 3\sqrt{s_2}\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \times \left[\frac{C_1 \sinh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\left(x - \left[\mp\frac{2}{3}\sqrt{s_2} - \frac{1}{2}\lambda^2\right]t\right)\right\} - C_2\cosh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\left(x - \left[\mp\frac{2}{3}\sqrt{s_2} - \frac{1}{2}\lambda^2\right]t\right)\right\}}{C_1 \cosh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\left(x - \left[\mp\frac{2}{3}\sqrt{s_2} - \frac{1}{2}\lambda^2\right]t\right)\right\} + C_2\sinh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\left(x - \left[\mp\frac{2}{3}\sqrt{s_2} - \frac{1}{2}\lambda^2\right]t\right)\right\}}\right] - \frac{\lambda}{2}\right) \times e^{i(\phi[x-\{\mp(2/3)\sqrt{s_2}-(1/2)\lambda^2\}t]-\Omega t)}.$$

And for $\lambda^2 - 4\mu < 0$

$$q_2(x,t) = 3\sqrt{s_2}\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \times \left[\frac{-C_1 \sin\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\left(x - \left[\mp\frac{2}{3}\sqrt{s_2} - \frac{1}{2}\lambda^2\right]t\right)\right\} + C_2\cos\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\left(x - \left[\mp\frac{2}{3}\sqrt{s_2} - \frac{1}{2}\lambda^2\right]t\right)\right\}}{C_1 \cos\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\left(x - \left[\mp\frac{2}{3}\sqrt{s_2} - \frac{1}{2}\lambda^2\right]t\right)\right\} + C_2\sin\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\left(x - \left[\mp\frac{2}{3}\sqrt{s_2} - \frac{1}{2}\lambda^2\right]t\right)\right\}}\right] - \frac{\lambda}{2}\right) \times e^{i(\phi[x-\{\mp(2/3)\sqrt{s_2}-(1/2)\lambda^2\}t]-\Omega t)}.$$

The second set of obtained results is

$$v = \mp\frac{2}{3}\sqrt{s_4} - \frac{1}{2}\lambda^2, \qquad \mu = \pm\sqrt{s_4}, \qquad \alpha_0 = \frac{2}{3}\sqrt{s_6}, \qquad \alpha_1 = 3\sqrt{s_4},$$

we get solution of Eq. (1) for $\lambda^2 - 4\mu > 0$

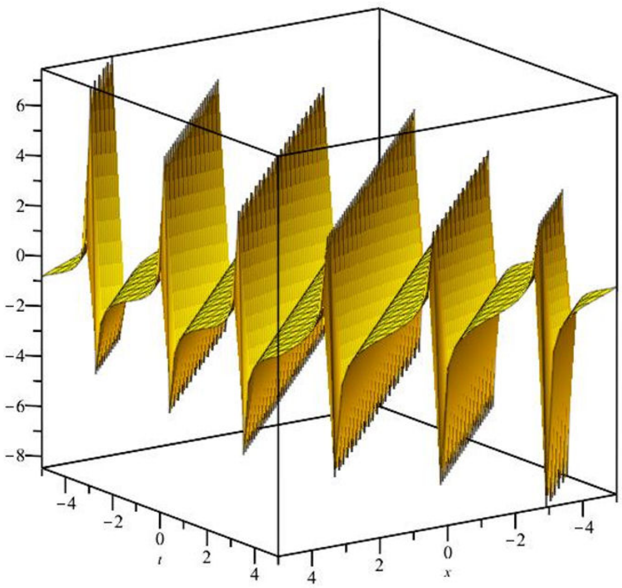

FIGURE 1. Graphical behavior for q_1.

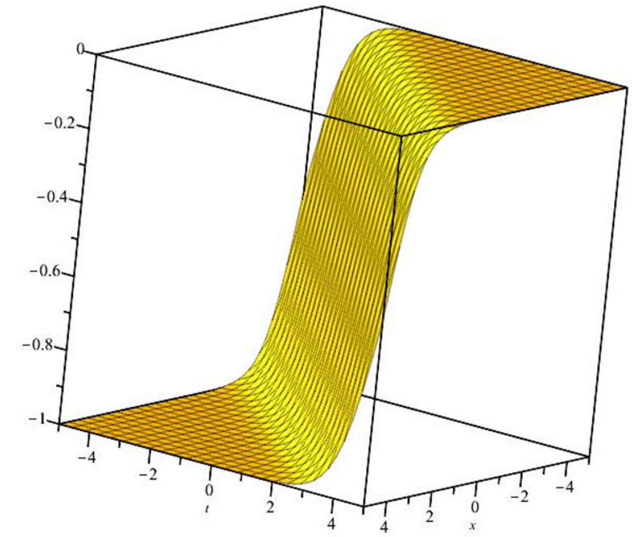

FIGURE 2. Graphical behavior for q_2.

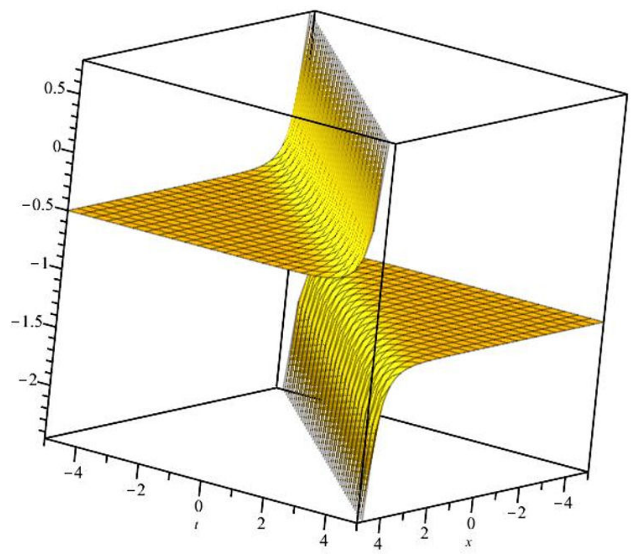

FIGURE 3. Graphical behavior for q_3.

$$
\begin{aligned}
q_1(x,t) = 3\sqrt{s_4}\Bigg(&\frac{\sqrt{\lambda^2-4\mu}}{2} \\
&\times\left[\frac{C_1\sinh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\left(x-\left[\mp\frac{2}{3}\sqrt{s_4}-\frac{1}{2}\lambda^2\right]t\right)\right\}+C_2\cosh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\left(x-\left[\mp\frac{2}{3}\sqrt{s_4}-\frac{1}{2}\lambda^2\right]t\right)\right\}}{C_1\cosh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\left(x-\left[\mp\frac{2}{3}\sqrt{s_4}-\frac{1}{2}\lambda^2\right]t\right)\right\}+C_2\sinh\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}\left(x-\left[\mp\frac{2}{3}\sqrt{s_4}-\frac{1}{2}\lambda^2\right]t\right)\right\}}\right]-\frac{\lambda}{2}\Bigg) \\
&\times e^{i(\phi[x-\{\mp(2/3)\sqrt{s_4}-(1/2)\lambda^2\}t]-\Omega t)}+\frac{2}{3}\sqrt{s_6}\,,
\end{aligned}
$$

And for $4\mu-\lambda^2<0$

$$q_2(x,t) = 3\sqrt{s_2}\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right.$$
$$\times\left[\frac{-C_1\sin\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\left(x-\left[\mp\frac{2}{3}\sqrt{s_4}-\frac{1}{2}\lambda^2\right]t\right)\right\}+C_2\cos\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\left(x-\left[\mp\frac{2}{3}\sqrt{s_4}-\frac{1}{2}\lambda^2\right]t\right)\right\}}{C_1\cos\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\left(x-\left[\mp\frac{2}{3}\sqrt{s_4}-\frac{1}{2}\lambda^2\right]t\right)\right\}+C_2\sin\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}\left(x-\left[\mp\frac{2}{3}\sqrt{s_4}-\frac{1}{2}\lambda^2\right]t\right)\right\}}\right]\left.-\frac{\lambda}{2}\right)$$
$$\times e^{i(\phi[x-\{\mp(2/3)\sqrt{s_4}-(1/2)\lambda^2\}t]-\Omega t)}+\frac{2}{3}\sqrt{s_6}\,,$$

4. Conclusions

In this paper, we investigated the Fokas-Lenells equation via the G'/G- Expansion method. Using some wave transformations, the PDE system is turned into an ODE system. The solutions of the ODE system are assumed in the forms of the G'/G- Expansion method. In the same vein and parallel the FL has been applied to achieve other new visions to the soliton solutions of this model Figs. 1-3. The solutions will be considered for the new application areas of the generalized form of modified NSE. Consequently a positive forward future studies have been introduced for the given model. We also conclude that the effectively, powerful of the suggested method.

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