

On the anisotropic advection-diffusion equation with time dependent coefficients

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Received 10 March 2016; accepted 14 October 2016

The advection-diffusion equation with time dependent velocity and anisotropic time dependent diffusion tensor is examined in regard to its non-classical transport features and to the use of a non-orthogonal coordinate system. Although this equation appears in diverse physical problems, particularly in particle transport in stochastic velocity fields and in underground porous media, a detailed analysis of its solutions is lacking. In order to study the effects of the time-dependent coefficients and the anisotropic diffusion on transport, we solve analytically the equation for an initial Dirac delta pulse. We discuss the solutions to three cases: one based on power-law correlation functions where the pulse diffuses faster than the classical rate $\sim t$, a second case specifically designed to display slower rate of diffusion than the classical one, and a third case to describe hydrodynamic dispersion in porous media.

Keywords: Time-dependent diffusion; anisotropic media; tracer and pollutant transport.

En este trabajo se analiza la ecuación de advección-difusión en la cual se tiene la velocidad y el tensor de difusión anisotrópico dependientes del tiempo, y se examinan los efectos no-clásicos del transporte y el uso de una base vectorial no ortogonal. Esta ecuación aparece en diversas áreas de la física, particularmente en transporte de partículas en campos estocásticos de velocidad y en medios porosos subterráneos, sin embargo, hace falta un análisis más profundo de sus soluciones. A fin de examinar el efecto de los coeficientes dependientes del tiempo y de la anisotropía en la difusión hemos obtenido analíticamente la solución general del modelo para el caso de un pulso inicial tipo delta de Dirac. Aplicamos la ecuación a tres casos: uno basado en funciones de correlación que siguen leyes de potencias que da lugar a super-difusión, el cual ha sido resuelto numéricamente con anterioridad, otro que hemos construido específicamente para exhibir sub-difusión, y un tercero desarrollado para describir dispersión hidrodinámica en medios porosos.

Descriptores: Difusión dependiente del tiempo; medios anisotrópicos; transporte de trazador y contaminantes.

PACS: 05.60.Cd; 05.45.Df; 47.53.+n; 47.56.+r

1. Introduction

The transport equation for a concentration field, C , analyzed in this work is obtained from the conservation law

$$\frac{\partial C}{\partial t} = -\nabla \cdot \mathbf{q} \quad (1)$$

and a Fourier-Fick's advective-diffusive closure relation of the form

$$\mathbf{q} = -\mathbb{D}(t) \cdot \nabla C + \mathbf{v}(t)C, \quad (2)$$

where $\mathbb{D}(t)$ is a symmetric diffusion tensor, and $\mathbf{v}(t)$ is a velocity vector field, both functions of time (not space). The resulting transport equation is

$$\frac{\partial C}{\partial t} = \nabla \cdot (\mathbb{D}(t) \cdot \nabla C - \mathbf{v}(t)C). \quad (3)$$

Our main interest regarding the structure of the last equation concerns the time-dependence on advection and diffusion, and the anisotropy of the diffusion tensor. This equation and diverse variants of it can be found in many applications, including particle transport in anisotropic time-dependent stochastic velocity fields, where the diffusion tensor is related to velocity correlation functions. Also, tracer or pollutant transport in underground porous formations can be

captured within this last approach by associating the stochastic velocity variations to permeability changes of the porous media [1-6] and \mathbb{D} is the hydrodynamic dispersion tensor that displays larger diffusion along the velocity direction than in the transversal direction. Equation (3) can be found in many other problems too, such as : (i) transport in porous materials with mixed-pore-size [7], (ii) transport in periodic porous media [8], (iii) transport of contaminants carried by the wind in the atmosphere [9-11], (iv) low- and high-temperature magnetically confined plasmas, where charged particles can displace rapidly along the magnetic lines but diffuse slowly in the perpendicular (with respect to the magnetic field) direction [12-14], (v) gas transport in cylindrical industrial ducts [15-16], (vi) transport of molecular spins in bio-tissues for nuclear magnetic resonance applications [17-19], and (vii) Brownian particles with memory, where the diffusion tensor's time-dependence can be related to long correlations of the randomly fluctuating forces acting on the particle [20,21].

In this work we obtain an analytical solution of Eq. (3) and explore the implications of the anisotropy on the transport process by examining the natural coordinate systems involved. It is to be mentioned that some practical approaches propose the description of the so called *anomalous transport* via a time-dependent diffusion coefficient (as in Eq. (3)),

since the resulting mean square standard deviation can increase above or below the linear time dependence (as we will show later), it is interpreted as *sub-* or *super-diffusion*. As it is well-known, non-diffusive transport also naturally arises in the context of Continuous Time random Walk models with non-Gaussian and/or non-Markovian statistics see for example [22,23] and references therein. However, the present work focuses on a different source of non-diffusive behavior. In particular, we are interested in systems that despite being described by Fourier-Fick's closure relations exhibit non-diffusive transport because of the anomalous time dependence and anisotropy in the diffusion tensor.

To formulate the anisotropic transport problem on a general setting let $\{\mathbf{f}_i\}$, $i = 1, \dots, N$, denote a normalized basis of vectors in \mathbb{R}^N (not necessarily orthogonal), and write the flux as

$$\mathbf{q} = \sum_{i=1}^N q_i \mathbf{f}_i. \quad (4)$$

Notice that the i -th component q_i is not necessarily equal to the projection of vector \mathbf{q} onto the basis vector \mathbf{f}_i , *i.e.*, $q_i = \sum_{j=1}^N (\mathbf{q} \cdot \mathbf{f}_j) g_{ji}^{-1}$, where $g_{ij} = (\mathbf{f}_i \cdot \mathbf{f}_j)$ is the Euclidean metric written in the $\{\mathbf{f}_i\}$ basis. This difference corresponds to the fact that q_i are contravariant components while $\mathbf{q} \cdot \mathbf{f}_j$ are covariant ones. If (u_1, \dots, u_N) denotes the coordinate system corresponding to the basis $\{\mathbf{f}_i\}$, then $\mathbf{f}_i \cdot \nabla = \partial/\partial u_i$.

In this work we restrict attention to homogeneous anisotropic media, such that the diffusion tensor, \mathbb{D} , the velocity, $\mathbf{v} = \sum_i v_i \mathbf{f}_i$, and the basis $\{\mathbf{f}_i\}$ are space-independent quantities. Following the stochastic transport approach [2,4,24], however, we keep the time dependence in both, the diffusion tensor, $\mathbb{D}(t)$, and the velocity, $\mathbf{v} = \mathbf{v}(t)$. Furthermore, we focus on a particular class of systems characterized by the condition that their diffusion tensor can be brought to a diagonal form (with time-dependent elements), when written in the coordinate system corresponding to the basis $\{\mathbf{f}_i\}$ through a time-independent linear transformation (see Eq. (9)). When, moreover, such a transformation is an element of the $SO(N)$ group, and thus $\{\mathbf{f}_i\}$ is itself an orthonormal basis, then the previous condition implies that $\mathbb{D}(t) \cdot \mathbb{D}(t') = \mathbb{D}(t') \cdot \mathbb{D}(t)$ for $t \neq t'$ (this can be verified in the particular case considered in Sec. 4, see Eq. (27)). Of course this is not the most general situation by far and the physical meaning of the particular cases considered might not be straightforward at all, but for the purposes of this work, it is general enough. In fact, as will be discussed in the following sections, the “diagonalizability” condition allows us to find exact solutions to our model. The rest of the paper is organized as follows: in Sec. 2 we make some physical assumptions on the flux to define the model and we find its general solution analytically, Sec. 3 is devoted to clarify the physical interpretation of the model and in Sec. 4 we show its main features through particular examples that exhibit either normal, super or sub-diffusion. Finally, conclusions and directions of possible future work are presented in Sec. 5.

2. Model

To construct the model we assume that \mathbb{D} is diagonal in the u -coordinate system, *i.e.*, $D_{ij} = \chi_i(t)\delta_{ij}$, where δ_{ij} is the Kronecker delta, and thus

$$q_i = -\chi_i (\mathbf{f}_i \cdot \nabla) C + v_i C. \quad (5)$$

Substituting Eq. (5) into Eq. (1), and using $\nabla \cdot \mathbf{f}_i = 0$, we get the transport equation

$$\frac{\partial C}{\partial t} = \sum_i \left(\chi_i (\mathbf{f}_i \cdot \nabla)^2 C - v_i (\mathbf{f}_i \cdot \nabla) C \right), \quad (6)$$

or equivalently,

$$\frac{\partial C}{\partial t} = \sum_i \left(\chi_i \frac{\partial^2 C}{\partial u_i^2} - v_i \frac{\partial C}{\partial u_i} \right). \quad (7)$$

Note that in the $\{\mathbf{f}_i\}$ basis the transport equation in (7) is diagonal, *i.e.*, it has no cross terms in the directional derivatives $\mathbf{f}_i \cdot \nabla$. However, as we will discuss in Sec. 3 this does not necessarily imply that $\{\mathbf{f}_i\}$ are the principal directions (eigen-vectors) of \mathbb{D} .

Clearly, if a different coordinate system is adopted, cross terms (involving the directional derivatives along the corresponding coordinate directions) will in general appear. As a particular important example, consider a Cartesian coordinate system (x_1, \dots, x_N) with orthonormal basis vectors \mathbf{e}_i , $i = 1, \dots, N$. In this case,

$$\mathbf{f}_i \cdot \nabla = \sum_j a_{ij} \frac{\partial}{\partial x_j}, \quad (8)$$

where the constants

$$a_{ij} = \mathbf{f}_i \cdot \mathbf{e}_j, \quad (9)$$

define the change of bases with the corresponding change of coordinates given by the linear transformation

$$x_i = \sum_j a_{ji} u_j. \quad (10)$$

With inverse

$$u_i = \sum_j (a^T)_{ij}^{-1} x_j = \sum_j A_{ij} x_j. \quad (11)$$

Notice that the metric tensor can be written as $g_{ij} = \sum_k a_{ik} a_{jk} = (a \cdot a^T)_{ij}$.

In the coordinates (x_1, \dots, x_N) and under the assumptions made, Eq. (3) becomes

$$\frac{\partial C}{\partial t} = \sum_{j,k} \mathcal{D}_{jk} \frac{\partial^2 C}{\partial x_j \partial x_k} - \sum_j \mathbf{v}_j \frac{\partial C}{\partial x_j}, \quad (12)$$

where $v_j = \sum_i a_{ij} v_i$ and the diffusion tensor, \mathcal{D}_{jk} , is defined as

$$\mathcal{D}_{jk} = \sum_i \chi_i a_{ij} a_{ik}. \quad (13)$$

Note that, by construction, \mathcal{D}_{jk} is symmetric, $\mathcal{D}_{jk} = \mathcal{D}_{kj}$.

When written in the (u_1, \dots, u_N) coordinates, the solution of the transport problem given by Eq. (7) can be obtained analytically even when χ_i and v_i are time dependent functions. Notice that an advective-diffusive equation with time-dependent mean drift velocity, like Eq. (7), can always be written as a purely diffusive equation in terms of the rigid-translated coordinates

$$u'_i = u_i - \int_0^t v_i(s) ds, \quad t' = t. \quad (14)$$

If the ansatz $C(u', t') = \prod_{i=1}^N h_i(u'_i, t')$ is then assumed, Eq. (7) is equivalent to the set of N equations:

$$\frac{\partial h_i}{\partial t'} = \chi_i(t') \frac{\partial^2 h_i}{\partial u'^2}, \quad (15)$$

with $i = 1, \dots, N$, in the sense that any solution of (15) is also a solution of (7). Along this work there will be sum over repeated indices only when explicitly stated by the summation symbol. Furthermore, by introducing the temporal variables

$$\tau_i(t') = \int_0^{t'} \chi_i(s) ds, \quad (16)$$

(15) reduces to

$$\frac{\partial h_i}{\partial \tau_i} = \frac{\partial^2 h_i}{\partial u'^2}, \quad (17)$$

for each $i = 1, \dots, N$. In order for τ_i to be a genuine temporal variable, it has to be a monotonic function of t , which is guaranteed since $\chi_i(t)$ is a positive definite function. For the initial condition $h_i(u'_i, \tau_i = 0) = \delta(u'_i)$, the solution of (17) is given by

$$h_i(u'_i, \tau_i) = \frac{1}{\sqrt{4\pi\tau_i}} \exp\left(-\frac{u'^2_i}{4\tau_i}\right), \quad (18)$$

and the solution of Eq. (12) is

$$C(x, t) = \prod_i \left(\frac{\exp\left(-\frac{(\sum_j A_{ij} x_j - \int_0^t v_i(s) ds)^2}{4\tau_i(t)}\right)}{\sqrt{4\pi g \tau_i(t)}} \right), \quad (19)$$

where $g = \det g_{ij}$ and Eq. (11) was used. The components of the drift velocity vector can analogously be written in the Cartesian frame according to $v_i = \sum_j A_{ij} v_j$.

To calculate the mean standard deviation of a pulse with respect to the normalized distribution $C(u_i, t)$ we define the expectation value $\langle F \rangle$ of a scalar function F as

$$\langle F \rangle = \int C(u'_i, t') F(u') \sqrt{g} d^N u', \quad (20)$$

where the translated coordinate system in Eq. (14) has been used. By noticing that $\langle u'_i \rangle = 0$, it follows that

$$\langle u_i \rangle = \int_0^t v_i(s) ds, \quad (21)$$

which means, $d\langle u_i \rangle / dt = v_i$, i.e., the mean pulse position in the f_i direction drifts at the speed v_i . The mean square deviation in the f_i direction, σ_i^2 , is given by

$$\sigma_i^2 = \langle [u_i - \langle u_i \rangle]^2 \rangle = \langle [u'_i]^2 \rangle = 2 \int_0^t \chi_i(s) ds. \quad (22)$$

Provided the diffusion tensor components, $\chi_i(t)$, Eq. (22) gives the corresponding anisotropic diffusion which, depending on χ_i , can be normal or anomalous.

3. Interpretation

When the f_i basis is orthonormal (*i.e.*, Cartesian), q_i and $f_i \cdot q$ are equal, and Eq. (5) implies that the flux along the f_i direction depends only on the gradient of C along that same direction. However, when $\{f_i\}$ is not orthogonal, it can be the case that even if q_i depends only on the gradient of C along direction f_i , the projection $f_i \cdot q$ does depend on the gradient of C along the j -th direction, for $j \neq i$.

By choosing q_i to obey the relation (5) and therefore the \mathbb{D} 's components (in the $\{f_i\}$ basis) to satisfy $D_{ij} = \chi_i \delta_{ij}$, one might be tempted to conclude that the vectors of the basis $\{f_i\}$ are eigenvectors of the diffusion tensor. As it will be shown in what follows, it is not the case in general. In vector notation, a general eigenvalue problem can be written as

$$\mathbb{D} \cdot \mathbf{d}_i = \lambda_i \mathbf{d}_i, \quad (23)$$

where \mathbf{d}_i and λ_i are the i -th eigenvector and the corresponding eigenvalue of the second order tensor \mathbb{D} , respectively. In particular, the dispersion tensor can be written in terms of dyadic products of the non-orthogonal basis vectors $\{f_i\}$, as $\mathbb{D} = \sum_{i,j} D_{ij} f_i f_j$, where

$$D_{ij} = \sum_{k,l} g_{ik}^{-1} g_{lj}^{-1} (f_k \cdot \mathbb{D} \cdot f_l), \quad (24)$$

and we remind the reader that $g_{jk}^{-1} = (f_j \cdot f_k)^{-1}$ is the inverse of $g_{jk} = f_j \cdot f_k$, which in general is different to the identity. The existence of $(f_j \cdot f_k)^{-1}$ is guaranteed by the fact that $\{f_i\}$, $i = 1, 2, \dots, N$, is a basis.

From expression (24) it is clear that if $\{f_i\}$ were eigenvectors of \mathbb{D} , then the dispersion tensor components would be given by $D_{ij} = \lambda_i g_{ij}^{-1}$. Therefore we conclude that the flux model in (5) does not imply that $\{f_i\}$ are the eigenvectors of

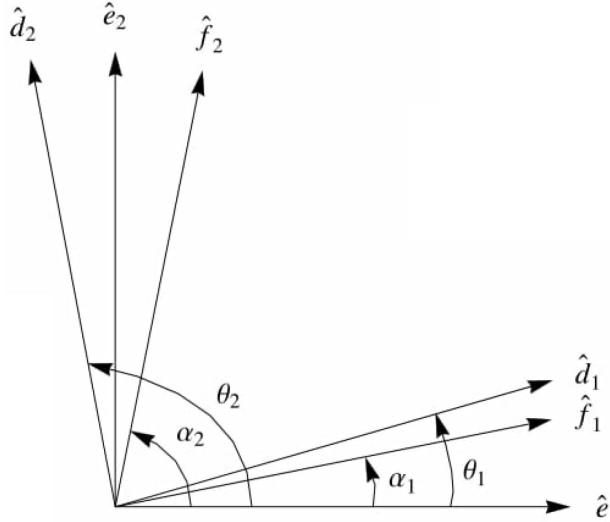


FIGURE 1. Illustration of the three basis of vectors: $\{\mathbf{e}_i\}$, $\{\mathbf{f}_i\}$ and $\{\mathbf{d}_i\}$.

Moreover, since the tensor \mathbb{D} is symmetric, its eigenvectors have to be orthogonal, and therefore they cannot be in general equal to $\{\mathbf{f}_i\}$.

Summarizing, in general there are three different basis (see Fig. 1): (i) the non orthonormal constant basis in terms of which the problem takes its simplest form, $\{\mathbf{f}_i\}$; (ii) the Cartesian time-independent basis where the standard form of transport equation holds and our physical intuition works better, $\{\mathbf{e}_i\}$; and (iii) the time-dependent Cartesian basis defined by \mathbb{D} 's eigenvectors, $\{\mathbf{d}_i\}$ (assuming they are not degenerate), which diagonalizes the diffusion tensor. It is interesting to notice that in the proposed model the coordinate system

where the equation takes its simplest form is not $\{\mathbf{d}_i\}$, but an auxiliary basis that allows us to solve the problem. Notice that Standard diagonalization methods are not directly applicable in this case because the diffusion tensor depends on time, its principal directions (or eigenvectors) – and therefore the associated coordinates – depend on time as well. Consequently, when written in this non-inertial coordinate system, the advection-diffusion equation contains a space and time-dependent advective term which renders it more complicated than the original one. Although these three bases are different in general, $\{\mathbf{f}_i\}$ and $\{\mathbf{d}_i\}$ are related. This will become clear in the next examples, where we restrict attention to the case $N = 2$ for the sake of simplicity.

4. Applications

Let us first consider a general two-dimensional anisotropic situation. We consider a non-orthogonal basis spanned by the vectors $\mathbf{f}_i = \sum_j a_{ij} \mathbf{e}_j$, with $i = 1, 2$, as shown in Fig. 1, with

$$a_{ij} = \begin{pmatrix} \cos \alpha_1 & \sin \alpha_1 \\ \cos \alpha_2 & \sin \alpha_2 \end{pmatrix}. \quad (25)$$

In such a basis, we propose a diffusion tensor whose components are written as $D_{ij} = \chi_i(t) \delta_{ij}$. In this case, the metric tensor in the u coordinates is:

$$g_{ij} = \begin{pmatrix} 1 & \cos(\alpha_1 - \alpha_2) \\ \cos(\alpha_1 - \alpha_2) & 1 \end{pmatrix}. \quad (26)$$

According to (13) the diffusion tensor written in Cartesian coordinates takes the form:

$$\mathcal{D}_{ij} = \begin{pmatrix} \chi_1 \cos^2 \alpha_1 + \chi_2 \cos^2 \alpha_2 & \chi_1 \cos \alpha_1 \sin \alpha_1 + \chi_2 \cos \alpha_2 \sin \alpha_2 \\ \chi_1 \cos \alpha_1 \sin \alpha_1 + \chi_2 \cos \alpha_2 \sin \alpha_2 & \chi_1 \sin^2 \alpha_1 + \chi_2 \sin^2 \alpha_2 \end{pmatrix} \quad (27)$$

and its eigenvalues and eigenvectors are:

$$\lambda_i = \frac{1}{2} \left(\chi_1 + \chi_2 + (-1)^{i+1} \sqrt{\chi_1^2 + \chi_2^2 + 2\chi_1\chi_2 \cos[2(\alpha_1 - \alpha_2)]} \right), \quad (28)$$

$$\mathbf{d}_i = \frac{1}{n_i} \begin{pmatrix} \chi_1 \cos 2\alpha_1 + \chi_2 \cos 2\alpha_2 + (-1)^{i+1} \sqrt{\chi_1^2 + \chi_2^2 + 2\chi_1\chi_2 \cos[2(\alpha_1 - \alpha_2)]} \\ \chi_1 \sin 2\alpha_1 + \chi_2 \sin 2\alpha_2 \end{pmatrix}, \quad (29)$$

respectively, where n_i is a normalization factor. Consistent with the general discussion in Sec. 3., the eigenvectors \mathbf{d}_i are orthonormal and different from \mathbf{f}_i . The vector \mathbf{d}_1 corresponds to the direction of maximum diffusion (since $\lambda_1 \geq \lambda_2$). For general functions $\chi_i(t)$ there is not much more that can be said about \mathbf{d}_1 . However, for the situation described below in cases 4.1 and 4.2, it can be shown that for large values of time $t \gg T$ (for a given characteristic decay time T) $\lim_{t \rightarrow \infty} \mathbf{d}_1 = \mathbf{f}_2$ for $\kappa = \chi_1/\chi_2 < 1$, $\lim_{t \rightarrow \infty} \mathbf{d}_1 = \mathbf{f}_1$ for $\kappa > 1$, and $\mathbf{d}_1 \propto \mathbf{f}_1 + \mathbf{f}_2$ for $\kappa = 1$, shown in Fig. (2).

4.1. Anisotropic Sub-Diffusion

As a first example let us consider a case in which sub-diffusion takes place. To this end we provide an adequate form of $\chi_i(t)$ in the diffusion tensor. We have designed this function with the help of Eq. (22), such that σ_i^2 exhibits the desired sub-diffusive behavior at short and long times. The resulting, at-first-glance complicated expression, is

$$\chi_i(t_D) = D_{0i} e^{-t_D} t_D^{\gamma_i - \frac{1}{2}} I_{\frac{1}{2} - \gamma_i}(t_D), \quad (30)$$

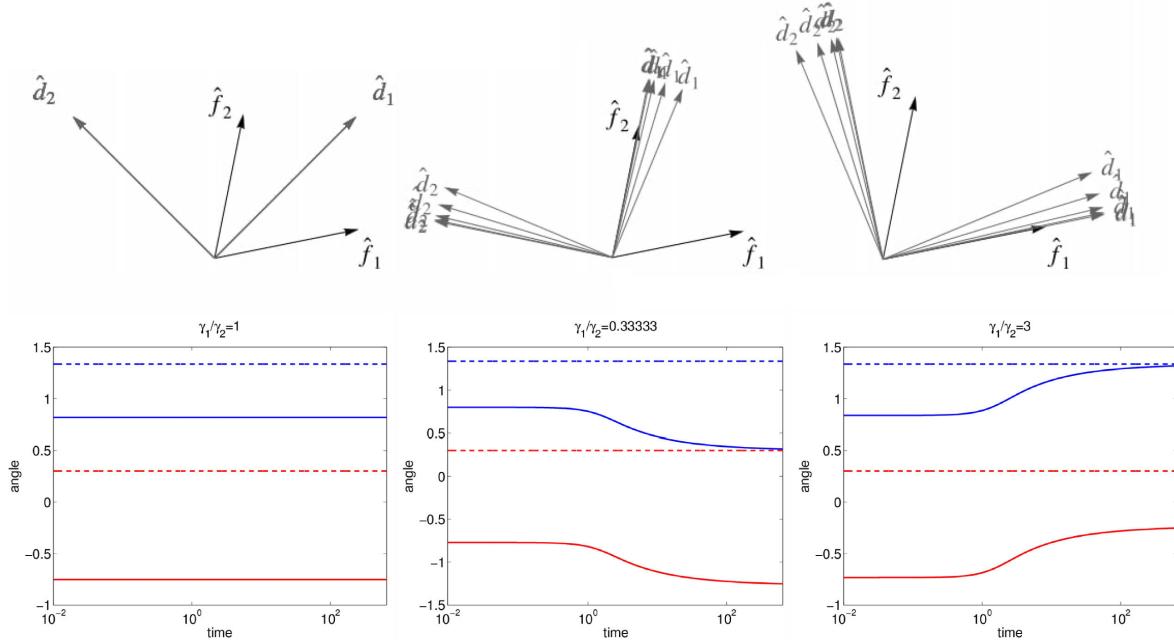


FIGURA 2. (Color online) Results corresponding to model in Eq. (30). (Top) Eigenvectors of the diffusion tensor \mathcal{D}_{ij} in the limit $t \rightarrow \infty$ (in the figure such a limit is represented by opacity tending to 1). The eigenvector \mathbf{d}_1 corresponds to the maximum diffusion direction while \mathbf{d}_2 to the minimum one. The figure on the left corresponds to the case $\gamma_1/\gamma_2 = 1$, the one in the middle to $\gamma_1/\gamma_2 = 1/3$ and the one on the right to $\gamma_1/\gamma_2 = 3$. (Bottom) Time evolution of angle θ_i formed by the eigenvectors of the diffusion tensor with respect to the x -axis, for three different γ_1/γ_2 ratios. The solid upper (blue) lines denote angle of \mathbf{d}_1 and the solid lower (red) lines denote angle of \mathbf{d}_2 given in Eq. (29). The top (blue) dashed line denotes the angle $\alpha_1 = 0.85\pi/2$ and the bottom (red) dashed line denotes the angle $\alpha_1 = 0.3$. According to Fig. 1, these are the angles of \mathbf{f}_1 and \mathbf{f}_2 . The asymptotic behavior of the angles is consistent with the text.

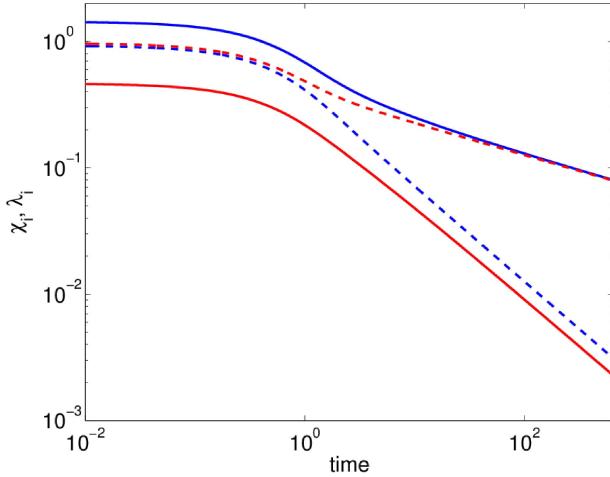


FIGURE 3. (Color online) Time evolution of the tensor components χ_i and its eigenvalues λ_i for different values of γ_i for the model (30). The lower dashed (blue) corresponds to χ_1 with $\gamma_1 = 1/4$ and the upper dashed (red) to χ_2 with $\gamma_2 = 3/4$. The solid upper (blue) line shows the eigenvalue λ_1 and the solid lower (red) shows the eigenvalue λ_2 .

where $t_D = t/T$, $\gamma_i \in (0, 1)$, $I_\alpha(x)$ is the modified Bessel function of order α and T is the system's characteristic decay correlation time. The behavior in time of χ_i is relatively simple, as displayed in Fig. 3 for some values of γ . Asymptotically, it is observed that $\chi_i(t) \sim t_D^{\gamma_i-1}$ for $t_D \gg 1$.

The corresponding mean square displacement is,

$$\frac{\sigma_i^2}{D_{0i}T} = \frac{1}{\gamma_i} \left[e^{-t_D} t_D^{\gamma_i-1/2} \times \left((t_D + 1 - 2\gamma_i) I_{\frac{1}{2}-\gamma_i}(t_D) + t_D I_{\frac{3}{2}-\gamma_i}(t_D) \right) - \frac{2^{\gamma_i+1/2}}{\Gamma(1/2-\gamma_i)} \right], \quad (31)$$

for $\gamma_i \neq 1/2$. For the special case of $\gamma_i = 1/2$, the mean square displacement takes the simpler form

$$\frac{\sigma_i^2}{D_{0i}T} = 2e^{-t_D} t_D \left(I_0(t_D) + I_1(t_D) \right). \quad (32)$$

For short times, $t_D \ll 1$, the mean square deviation increases linearly with time for all values of γ_i , which corresponds to normal diffusion. However, for $t_D \gg 1$ it asymptotically behaves as

$$\frac{\sigma_i^2}{D_{0i}T} \sim \frac{1}{\gamma_i} \sqrt{\frac{2}{\pi}} t_D^{\gamma_i}, \quad (33)$$

which corresponds to sub-diffusion since $0 < \gamma_i < 1$, as can be observed in Fig. 4 for $t_D \gg 1$ where we show the mean square displacement as a function of time for different values of γ_i .

From expressions (27), (28) and (29) we know that \mathbf{d}_1 corresponds to the direction of maximum diffusion, and also

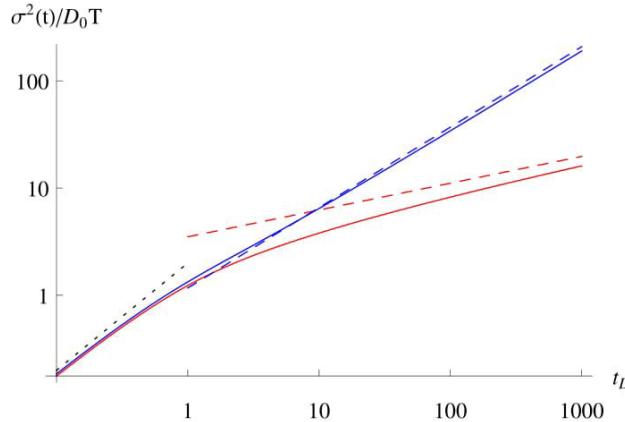


FIGURE 4. (Color online) Mean square displacement as a function of time for different values of γ for model (30). From bottom to top, the solid lines correspond to the exact value of σ^2 for $\gamma = 1/4$ (red) and $\gamma = 3/4$ (blue), while the dashed lines show the corresponding asymptotic values according to expression (33). The dotted (black) line corresponds to normal diffusion $\sigma^2 \sim t$.

that, as shown in Fig. 2, it satisfies the asymptotic κ -dependent behaviour discussed previously, namely $\lim_{t \rightarrow \infty} \mathbf{d}_1 = \mathbf{f}_2$ for $\kappa < 1$, $\lim_{t \rightarrow \infty} \mathbf{d}_1 = \mathbf{f}_1$ for $\kappa > 1$, and $\mathbf{d}_1 \propto \mathbf{f}_1 + \mathbf{f}_2$ for $\kappa = 1$, where $\kappa = \chi_1/\chi_2$. In all cases, of course, the concentration pulse sub-diffuses according to relation (33).

4.2. Anisotropic Super-Diffusion

Let us next assume that $\chi_i(t)$ is given by an expression typically encountered in the theory of particle transport in random stochastic velocity fields [2,4], where the diffusion coefficient is the integral of the velocity correlation function over time. If a power law correlation function is considered [2,3]: $\xi_i(t) = \xi_{0i}(1+t_D)^{\beta_i}$, it then follows

$$\chi_i = \begin{cases} \frac{D_{0i}}{\beta_i+1} [(1+t_D)^{\beta_i+1} - 1] & \beta_i \neq -1 \\ D_{0i} \ln(1+t_D) & \beta_i = -1. \end{cases} \quad (34)$$

In this case, the mean square displacement takes the form:

$$\frac{\sigma_i^2}{4D_{0i}T} = \begin{cases} \frac{1}{2+\beta_i} ((1+t_D)^{\beta_i+2} - 1) - t_D & \beta_i \neq -1, -2 \\ t_D - \ln(1+t_D) & \beta_i = -2 \\ (t_D + 1) \ln(1+t_D) & \beta_i = -1 \end{cases} \quad (35)$$

For short times, $t_D \ll 1$, the mean square deviation grows as t_D for $\beta_i = -1$, which is diffusive; for other values of β_i it increases as t_D^2 , which corresponds to ballistic behaviour. However, for large times, $t_D \gg 1$,

$$\frac{\sigma_i^2}{4D_{0i}T} \sim \begin{cases} \frac{t_D}{|\beta_i+1|} & \beta_i < -2 \\ t_D & \beta_i = -2 \\ \frac{t_D}{|\beta_i+1||\beta_i+2|} & -2 < \beta_i < -1 \\ t_D \ln(t_D) & \beta_i = -1 \\ \frac{t_D^{\beta_i+2}}{(\beta_i+1)(\beta_i+2)} & -1 < \beta_i < 0 \end{cases} \quad (36)$$

For large values of t_D and $\beta_i = 0$ the diffusion is ballistic, i.e., $\sigma^2 \sim t_D^2$. As shown in Fig. 5, this model describes three different asymptotic transport regimes: (i) normal diffusion for $\beta_i < -1$, (ii) logarithmic-type diffusion for $\beta_i = -1$ and (iii) super-diffusion for $-1 < \beta_i < 0$.

This example has been previously studied by Numbere and Erkal [3] by solving the corresponding transport equation, Eq. (12), numerically. Our analytical results confirm to certain extent their findings. In particular, for $\beta < -3$ they report “Fickian” (diffusive) behaviour in all cases. However, according to Eqs. (35) and (36), this is only true in the asymptotic regime. They describe the region $-3 \leq \beta \leq -1$ as scale dependent or weakly diffusive, since it converges asymptotically to diffusive; this results coincides with what we analytically found, however, they logically missed (since their results are numeric) the particular case $\beta = -1$. Finally, they report the region $-1 \leq \beta \leq 0$ as “non-Fickian” (non-diffusive), which agrees with our findings. From our results,

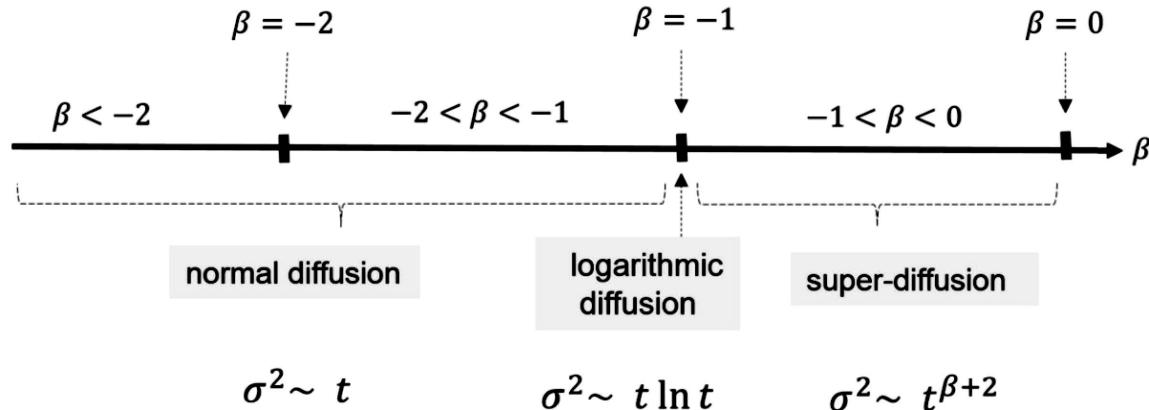


FIGURE 5. Three different asymptotic transport regimes for different $\beta < 0$ values, in model (34).

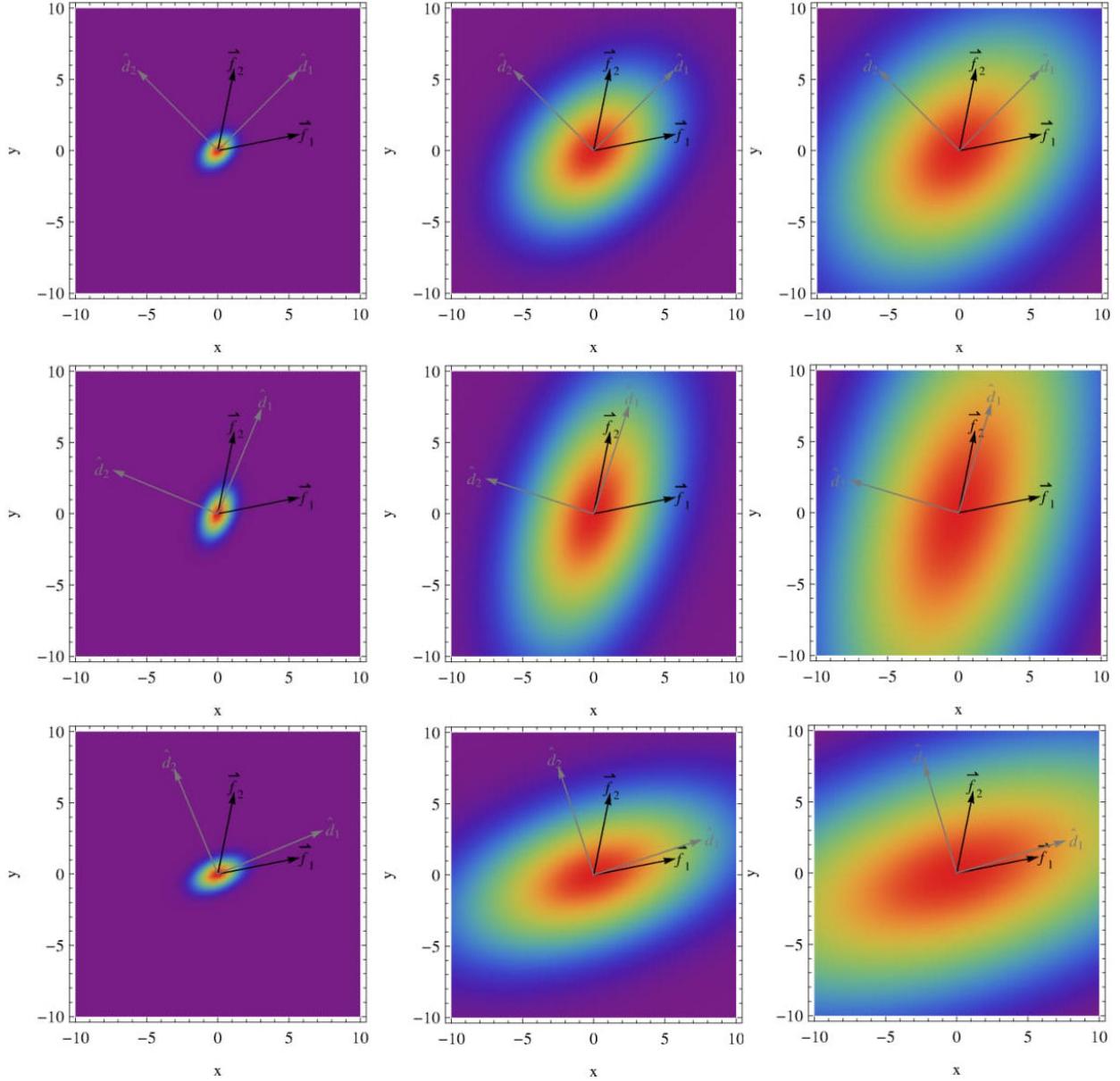


FIGURE 6. (Color online) Density of concentration C for the model (34) for different values of time: $t = (1T, 6T, 11T)$ (from left to right), for $v_i = 0$ and $\chi_i(t)$ given by Eq. (34) – with $i = 1, 2$, which correspond to non-orthogonal coordinates (u_i) associated with the matrix in Eq. (25). Three cases are considered: $(\beta_1, \beta_2) = (-0.5, -0.5)$, $(\beta_1, \beta_2) = (-0.5, 0)$ and $(\beta_1, \beta_2) = (0, -0.5)$, (from top to bottom). The pattern asymptotically gets oriented along a fixed direction depending on the ratio $\kappa = \chi_1(t)/\chi_2(t)$: for $\kappa < 1$ it orients along \mathbf{f}_2 , while for $\kappa > 1$ along the direction \mathbf{f}_1 , when $\kappa = 1$, the pattern gets oriented along $\mathbf{f}_1 + \mathbf{f}_2$. The eigenvectors of the diffusion tensor \mathbb{D} are also shown. It can be observed that the direction of largest diffusion corresponds in all cases to \mathbf{d}_1 and for $\kappa < 1$, $\lim_{t \rightarrow \infty} \mathbf{d}_1 = \mathbf{f}_2$, while for $\kappa > 1$, $\lim_{t \rightarrow \infty} \mathbf{d}_1 = \mathbf{f}_1$ and for $\kappa = 1$, $\mathbf{d}_1 \propto \mathbf{f}_1 + \mathbf{f}_2$.

the dynamics of a pulse, as illustrated in some of their figures, can be exactly evaluated using Eq. (19).

The explicit form of the concentration pulse $C(x, y, t)$ can be found directly through expression (19). In Fig. (6) we show the time evolution of the concentration profile for three sets of parameter values (β_1, β_2) . It can be noted that the concentration mainly diffuses along the direction \mathbf{d}_1 which depends on the ratio κ , as we claimed: for $\kappa < 1$ it tends to the direction \mathbf{f}_2 , while for $\kappa > 1$ it tends to in the direction \mathbf{f}_1 , when $\kappa = 1$, the concentration orients along $\mathbf{f}_1 + \mathbf{f}_2$.

4.3. Hydrodynamic Dispersion

Let us finally analyse the case of hydrodynamic dispersion in an homogeneous isotropic porous media, where anisotropic dispersion appears intrinsically. The anisotropy exhibits a maximum in the velocity direction and the minimum in the perpendicular directions. In this case, the dispersion tensor is written as [25,26]

$$\mathbb{D} = \alpha_T |\mathbf{v}| \mathbb{I} + (\alpha_L - \alpha_T) \frac{\mathbf{v}\mathbf{v}}{|\mathbf{v}|} \quad (37)$$

where α_L and α_T are the longitudinal and transverse dispersivity respectively. Since anisotropy axes are orthogonal, we choose an orthonormal vector basis with \mathbf{f}_1 in the direction of the velocity (direction of maximum dispersion). Thus $\mathbf{v} = v\mathbf{f}_1$, and therefore in a 2-dimensional system we write

$$D_{ij} = v \begin{pmatrix} \alpha_L & 0 \\ 0 & \alpha_T \end{pmatrix}, \quad (38)$$

and Eq. (5) gives

$$q_1 = -\alpha_L v \frac{\partial C}{\partial u_1} + vC \quad (39)$$

$$q_2 = -\alpha_T v \frac{\partial C}{\partial u_2} \quad (40)$$

Thus the model of the previous subsections can be applied with $\chi_1 = \alpha_L v$ and $\chi_2 = \alpha_T v$. Here the transformation matrix a_{ij} is a rigid rotation of angle Ω from the Cartesian reference system $\{\mathbf{e}_i\}$ to the $\{\mathbf{f}_i\}$ system. Therefore

$$a_{ij} = \begin{pmatrix} \cos \Omega & \sin \Omega \\ -\sin \Omega & \cos \Omega \end{pmatrix}. \quad (41)$$

and $A_{ij} = (a_{ij}^T)^{-1} = a_{ij}$.

Non-diffusive transport can be introduced in this model by assuming that $\alpha_L = \alpha_L(t)$ and $\alpha_T = \alpha_T(t)$ are functions of t conveniently chosen as it was done in the previous examples. The solution for the tracer concentration $C(x_1, x_2, t)$ will be that in Eq. (19) where the proper corresponding variables are to be introduced ($A_{ij} = a_{ij}$, $g = 1$, $v_2 = 0$). In this example it is easy to see that $\mathbf{d}_i = \mathbf{f}_i$.

5. Summary and Conclusions

In this paper we have analytically solved an N -dimensional advection-diffusion equation with time-dependent coeffi-

cients and diffusion anisotropy. The problem has been formulated in terms of a non-orthogonal basis in order to accommodate anisotropy. There appear naturally two basis vector: the non-orthonormal constant basis in terms of which the problem takes its simplest form, $\{\mathbf{f}_i\}$, and the time-dependent Cartesian one defined by \mathbb{D} 's eigenvectors, $\{\mathbf{d}_i\}$, which diagonalizes the diffusion tensor. It has been shown that the coordinate system where the equation takes its simplest form is not $\{\mathbf{d}_i\}$ but $\{\mathbf{f}_i\}$. Although these basis are different in general, the direction of maximum dispersion, $\{\mathbf{d}_i\}$, at large times, asymptotically approaches to \mathbf{f}_1 , \mathbf{f}_2 or $\mathbf{f}_1 + \mathbf{f}_2$, depending on the value of the ratio $\kappa = \chi_1/\chi_2$

We have particularly discussed the advection-diffusion equation in the context of its application to particle or solute transport in stochastic velocity fields. Here the diffusion tensor is given in terms of a time integral of the correlation function. The resulting transport behavior can display features of the so called sub- or super-diffusion, where the standard deviation increase in time with a power lower or larger than the unity, respectively. A known case that consider power law anisotropic dispersion coefficients [2,3,24,25] was examined, and we analytically reproduced the numerical results they obtained [3].

Finally, an interesting direction of future work might be to extend this model to the domain of fractional diffusion, where non-locality (in space and time) naturally arises.

Acknowledgments

This work has been partially supported by Conacyt-Sener-Hidrocarburos Fund through the project No. 143935. DdCN acknowledges support from the U.S. Department of Energy at Oak Ridge National Laboratory, managed by UT-Battelle, LLC, for the U.S. Department of Energy under contract DE-AC05-00OR22725.

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