

Geometry of classical particles on curved surfaces

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In this paper we consider a particle moving on a curved surface. From a variational principle, we write the equation of motion and the constraining force, both in terms of the Darboux frame adapted to the trajectory, that involves geometric information of the surface. By deformation of the trajectory on the surface, the constraining force and equation of motion of the perturbation are obtained. We show that the transversal deformation follows a generalized Raychaudhuri equation that contains extrinsic information besides the geodesic curvature. Results in the case of surface with axial symmetry can be parametrized in terms of the angular momenta.

Keywords: Curves; curved surfaces; particle on surfaces.

En este artículo consideramos el movimiento de una partícula sobre una superficie curvada. De un principio variacional, encontramos las ecuaciones de movimiento y la fuerza de restricción, en el marco de Darboux adaptado a la trayectoria. Deformando estas ecuaciones, encontramos que la perturbación transversal satisface una ecuación de Raychaudhuri generalizada que contiene información extrínseca además de la curvatura geodésica. En el caso de superficies con simetría axial, los resultados se pueden parametrizar en términos del momento angular.

Descriptores: Curvas; superficies curvas; partículas en superficies.

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1. Introduction

Recently the physics of particles and fields in curved surfaces in Euclidean space has become subject of interest, because many phenomena are reduced to one of them. For example, it is known that a liquid crystal on curved surfaces, look for smectic phases as parallel curved surface [1]. We also know, that some electronic properties of certain two-dimensional materials, are explained modeling particles, living in the surface, satisfying the Schrödinger or the Dirac equation [2]. Topological defects on surfaces [3], relativistic particle dynamics [4], and diffusion on surfaces [5] are also some examples of how the interaction of particles and fields with surfaces through geometry, can be used to model several natural phenomena. Although from a mathematical point of view, variational problems related to curves and surfaces, has been addressed [6], they have recently attracted attention, because has been possible to interpret the Euler-Lagrange equations in physical terms, as a balance of internal forces and moments, see *e.g.* [7] and [8].

Inspired by a recent formalism developed to investigate properties of elastic curves constrained on surfaces [9], in this work we obtain the basic, but generic, geometric elements of particles constrained on surfaces. Although it is well known the classical physics of particles, usually specific cases of particles moving on surfaces are solved. Here we present its description in terms of the geometric information of the surface, either intrinsic through the gaussian curvature or extrinsic information, encapsulated in the second fundamental form.

We obtain, aside to the equations of motion, the constraining force on the particle. The corresponding equations of small perturbations are also obtained, showing that they includes, even with no external forces, extrinsic information. Some general results if the surface has axial symmetry and the gravitational field taken into account are showed.

2. Lagrangian Classical Mechanics

Let us consider the action of a free particle, constrained on a surface through the vector Lagrange multiplier λ ,

$$L(\mathbf{x}, \lambda, \xi^a) = \frac{m}{2} \int \dot{\mathbf{x}}^2(t) dt + \int \lambda \cdot (\mathbf{x}(t) - \mathbf{X}(\xi^a)) dt. \quad (1)$$

Here $\mathbf{x}(t) = (x^1(t), x^2(t), x^3(t)) \in R^3$, the dot stands for the derivative with respect to the time t , and m is the particle's mass. The equation of the surface, parametrized by local coordinates ξ^a , $a = \{1, 2\}$, can be written as $\mathbf{x} = \mathbf{X}(\xi^a)$. By taking infinitesimal deformations $\mathbf{x} \rightarrow \mathbf{x} + \delta\mathbf{x}$ we have

$$\delta L = m \int \delta\dot{\mathbf{x}} \cdot \dot{\mathbf{x}} dt + \int \lambda \cdot \delta\mathbf{x} dt. \quad (2)$$

After integration by parts in the first term, the corresponding Euler-Lagrange equations are given by

$$m \ddot{\mathbf{x}} = \lambda. \quad (3)$$

Moreover, variation respect to the local variables, $\xi^a \rightarrow \xi^a + \delta\xi^a$, gives

$$\delta L = - \int \boldsymbol{\lambda} \cdot \mathbf{e}_a \delta\xi^a dt, \tag{4}$$

where $\mathbf{e}_a = \partial_a \mathbf{X}$, are tangent vectors to the surface. Therefore, the Lagrange multiplier $\boldsymbol{\lambda} = -\lambda \mathbf{n}$, being $\mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2 / \|\cdot\|$, the unit normal to the surface. Finally, under deformation of the Lagrange multiplier, $\delta\boldsymbol{\lambda}$, we have that

$$\delta L = \int \delta\boldsymbol{\lambda} \cdot (\mathbf{x}(t) - \mathbf{X}(\xi^a)) dt,$$

and the equation for the constraint is obtained.

However, the Eq. (3), is not useful in this form. Instead, we write it in the Darboux frame $\{\mathbf{T}, \mathbf{n}, \boldsymbol{\ell} = \mathbf{T} \times \mathbf{n}\}$, where \mathbf{T} is the unit tangent to the curve and $\boldsymbol{\ell}$ is orthogonal to the curve on the surface, see Fig. (1). This basis satisfies that $\mathbf{T}' = \kappa_n \mathbf{n} + \kappa_g \boldsymbol{\ell}$, $\mathbf{n}' = -\kappa_n \mathbf{T} + \tau_g \boldsymbol{\ell}$, $\boldsymbol{\ell}' = -\kappa_g \mathbf{T} - \tau_g \mathbf{n}$, where prime stands for the derivative respect to arc length s [10]. These equations defines de normal curvature $\kappa_n = \mathbf{T}' \cdot \mathbf{n}$, the geodesic curvature $\kappa_g = \mathbf{T}' \cdot \boldsymbol{\ell}$ in addition to the geodesic torsion $\tau_g = \mathbf{n}' \cdot \boldsymbol{\ell}$.

We also consider that the constraint does not depend on time, then the velocity of the particle is given by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{e}_a t^a v(t), \\ &= \mathbf{T} v(t), \end{aligned} \tag{5}$$

where $v(t) = ds/dt$, $t^a = d\xi^a/ds$ and s the arc length along the trajectory of the particle. The relation of time t , with the arc length of the curve is given by $ds^2 = g_{ab}(d\xi^a/dt)(d\xi^b/dt)dt^2$, that involves the induced metric on the surface, $g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b$. Then we can write $v^2(t) = g_{ab}(d\xi^a/dt)(d\xi^b/dt)$. In addition, using Eqs. (5) and (3), we see that $\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} = 0$, and therefore the conservation of the energy, $mv^2(t)/2 = E$, follows.

The second derivative can be obtained from

$$\ddot{\mathbf{x}} = \dot{\mathbf{T}}v(t) + \mathbf{T}\dot{v}(t). \tag{6}$$

In the first term we can get

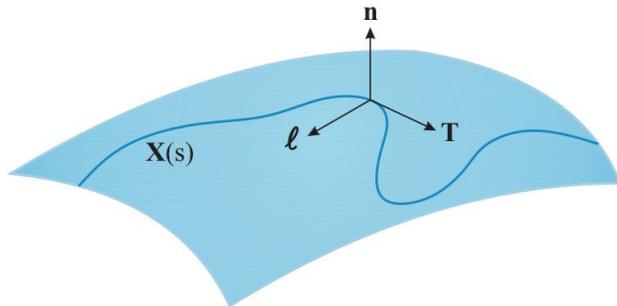


FIGURE 1. The Darboux frame.

$$\begin{aligned} \dot{\mathbf{T}} &= v(t) \frac{d\mathbf{T}}{ds}, \\ &= v(t) \left[-K_{ab} t^a t^b \mathbf{n} + \left(\frac{d^2 \xi^c}{ds^2} + \Gamma_{ab}^c t^a t^b \right) \mathbf{e}_c \right], \\ &= v(t) (\kappa_n \mathbf{n} + \kappa_g \boldsymbol{\ell}), \end{aligned} \tag{7}$$

in the second line, the second fundamental form of the surface $K_{ab} = -\partial_a \mathbf{e}_b \cdot \mathbf{n}$, has been introduced, and the Christoffel symbols Γ_{bc}^a compatible with g_{ab} also appear; there is no projection along \mathbf{T} , since it is a unit vector. Notice that we can identify $\kappa_n = -K_{ab} t^a t^b$ and $\kappa_g \boldsymbol{\ell} = \kappa_g^c \mathbf{e}_c$. Thus, we can write Eq. (3) in the form

$$mv^2(t) (\kappa_n \mathbf{n} + \kappa_g \boldsymbol{\ell}) + m a(t) \mathbf{T} = -\lambda \mathbf{n} \tag{8}$$

where $a(t) = dv/dt$. Therefore, we can read that the particle moves along a geodesic path, $\kappa_g = 0$, with $a(t) = 0$, i.e. $v(t) = v_0$, and feels a force, along the normal to the surface given by

$$\begin{aligned} \lambda &= -m v_0^2 \kappa_n, \\ &= m v_0^2 K_{ab} t^a t^b. \end{aligned} \tag{9}$$

We can write this force in terms of the principal curvatures κ_i , of the surface

$$\lambda = m v_0^2 (\kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta), \tag{10}$$

where, as usual, θ is the angle between \mathbf{T} and the principal vector \mathbf{v}_1 .

If there is an external force \mathbf{F} , acting on the particle, we can decompose the equation of motion in the local basis and we can write

$$\begin{aligned} m \ddot{\mathbf{x}} &= \mathbf{F} + \boldsymbol{\lambda}, \\ &= (-\lambda + F_n) \mathbf{n} + F_T \mathbf{T} + F_\ell \boldsymbol{\ell}, \end{aligned} \tag{11}$$

such that, the corresponding equations to solve are then given by

$$\begin{aligned} m v^2(t) \kappa_g &= F_\ell, \\ m a(t) &= F_T, \\ \lambda &= F_n - m v^2(t) \kappa_n \end{aligned} \tag{12}$$

and the energy is conserved, $mv^2(t)/2 + U = E$, with U the corresponding potential relative to the external force field. Notice that if we use the first and the third equations, then

$$\lambda = F_n - F_\ell \frac{\kappa_n}{\kappa_g}, \tag{13}$$

i.e. the force λ is written in terms of geometric information of the surface and of course, in general, it depends on time t . Therefore, a condition such that no constraining force exists is given by $F_n = m v^2 \kappa_n$, or

$$\kappa_n F_\ell = \kappa_g F_n. \tag{14}$$

This is a remarkable expression, which tell us that, in order that the constraining force vanishes, the ratio of non-tangential components of the external field must be equal to the ratio of the curvatures of the surface.

2.1. Perturbative expansion

To first order the Newton law (3) is given by (if the external force is a constant, *e.g* the gravitational field)

$$m \delta \ddot{\mathbf{x}} = \delta \boldsymbol{\lambda}, \tag{15}$$

where the deformations are along the surface,

$$\delta \mathbf{x} = \Phi \mathbf{T} + \Psi \boldsymbol{\ell}. \tag{16}$$

Then, since the energy is conserved, we see that $m \dot{\mathbf{x}} \cdot \delta \dot{\mathbf{x}} + \nabla U \cdot \delta \mathbf{x} = 0$. Derivative respect to the time is given by

$$\delta \dot{\mathbf{x}} = (\dot{\Phi} - v \Psi \kappa_g) \mathbf{T} + v(\Phi \kappa_n - \Psi \tau_g) \mathbf{n} + (\dot{\Psi} + \Phi v \kappa_g) \boldsymbol{\ell}. \tag{17}$$

Thus we have that $\dot{\mathbf{x}} \cdot \delta \dot{\mathbf{x}} = v(\dot{\Phi} - v \Psi \kappa_g)$, and therefore

$$\begin{aligned} m v (\dot{\Phi} - v \kappa_g \Psi) &= -\Phi \mathbf{T} \cdot \nabla U - \Psi \boldsymbol{\ell} \cdot \nabla U, \\ &= (\Phi F_T + \Psi F_\ell). \end{aligned} \tag{18}$$

If there is no external potential then $\dot{\Phi} - v \kappa_g \Psi = 0$, follows. Along geodesic trajectories we have that

$$m v \dot{\Phi} = 0. \tag{19}$$

Therefore, with no external forces the tangential deformation of geodesics is a rigid translation, $\Phi = C$, or is only a reparametrization of the path if $F_T \neq 0$. Thus, the only physical deformations is along the transversal direction $\boldsymbol{\ell}$, in Eq. (16), as we already know from invariance of the action under reparameterizations.

In order to obtain $\delta \ddot{\mathbf{x}}$, we notice that it is not just $(d/dt)\delta \dot{\mathbf{x}}$, but we have to take into account the Riemann curvature of the surface \mathcal{R}_{abcd} , so we get

$$\begin{aligned} \frac{d(\delta \dot{\mathbf{x}})}{dt} &= \delta \ddot{\mathbf{x}} + \mathcal{R}(v \mathbf{T}, \Phi \mathbf{T} + \Psi \boldsymbol{\ell}) \dot{\mathbf{x}}, \\ &= \delta \ddot{\mathbf{x}} + v^2 \mathcal{R}(\mathbf{T}, \Phi \mathbf{T} + \Psi \boldsymbol{\ell}) \mathbf{T}, \\ &= \delta \ddot{\mathbf{x}} + \Phi v^2 \mathcal{R}(\mathbf{T}, \mathbf{T}) \mathbf{T} + \Psi v^2 \mathcal{R}(\mathbf{T}, \boldsymbol{\ell}) \mathbf{T}. \end{aligned} \tag{20}$$

When projected into the tangent direction, the second term of the last equation is identically zero, as a consequence of the antisymmetry of the Riemann tensor. From Eq. (17) and (20), we can find

$$\begin{aligned} \delta \ddot{\mathbf{x}} &= [\ddot{\Phi} - 2v \kappa_g \dot{\Psi} - v^2(\kappa_g^2 + \kappa_n^2) \Phi \\ &\quad + (v^2 \tau_g \kappa_n - a(t) \kappa_g - v \dot{\kappa}_g) \Psi] \mathbf{T} \\ &\quad + [\ddot{\Psi} + 2v \kappa_g \dot{\Phi} - v^2(\kappa_g^2 + \tau_g^2) \Psi \\ &\quad + (v^2 \tau_g \kappa_n + a(t) \kappa_g + v \dot{\kappa}_g) \Phi - v^2 \mathcal{R}_{ab} \ell^a \ell^b \Psi] \boldsymbol{\ell} \\ &\quad + [2v \kappa_n \dot{\Phi} - 2v \tau_g \dot{\Psi} - (v^2 \tau_g \kappa_g - a(t) \kappa_n - v \dot{\kappa}_n) \Phi \\ &\quad - (v^2 \kappa_n \kappa_g + a(t) \tau_g + v \dot{\tau}_g) \Psi] \mathbf{n}. \end{aligned} \tag{21}$$

Since on two dimensional surfaces $\mathcal{R}_{ab} = g_{ab} \mathcal{R}/2$, we can rewrite $\mathcal{R}_{ab} \ell^a \ell^b = \mathcal{R}/2$, in the second line of Eq. (21). On

the other hand, considering that the constraint $\boldsymbol{\lambda} \cdot \mathbf{e}_a = 0$, remains under deformations, i.e. $\delta(\boldsymbol{\lambda} \cdot \mathbf{e}_a) = 0$, we have that

$$\delta \boldsymbol{\lambda} = -\delta \lambda \mathbf{n} + \delta_{\parallel} \boldsymbol{\lambda}, \tag{22}$$

where

$$\delta_{\parallel} \boldsymbol{\lambda} = -\lambda(-\kappa_n \Phi + \tau_g \Psi) \mathbf{T} - \lambda(\tau_g \Phi + K_{ab} \ell^a \ell^b \Psi) \boldsymbol{\ell}. \tag{23}$$

Therefore to first order, the Newton law (3) implies that the constraining force along the perturbed path is given by

$$\begin{aligned} \delta \lambda &= 2m v \tau_g \dot{\Psi} + (\tau_g F_\ell - m v \dot{\kappa}_n - 3 \kappa_n F_T) \Phi \\ &\quad + (\tau_g F_T + m v \dot{\tau}_g - 3 \kappa_n F_\ell) \Psi. \end{aligned} \tag{24}$$

Whereas the transversal projection implies that

$$\begin{aligned} m \ddot{\Psi} &+ (4 \kappa_g F_\ell - m v^2(\kappa_g^2 + \tau_g^2 + \mathcal{R}/2) + \lambda K_{ab} \ell^a \ell^b) \Psi \\ &= - (3 \kappa_g F_T + m v^2 \tau_g \kappa_n + m v \dot{\kappa}_g + \lambda \tau_g) \Phi. \end{aligned} \tag{25}$$

We see that it involves not only the transversal deformation Ψ , but also the longitudinal one Φ , that satisfies (18). In these equations, we can write the first and the second derivatives, of the field Ψ , in terms of the local coordinates as

$$\dot{\Psi} = v t^a \Psi_a, \tag{26}$$

and

$$\ddot{\Psi} = v^2(\kappa_g \Psi_a \ell^a + t^a t^b \nabla_b \Psi_a) + a(t) t^a \Psi_a, \tag{27}$$

where we have defined $\Psi_a = \partial_a \Psi$. In a later work, we will discuss specific solutions to these equations.

2.2. Geodesic trajectories

On perturbed geodesic curves, the deformation on the constraining force, is given by

$$\delta \lambda = 2m v \tau_g \dot{\Psi} + m v \dot{\tau}_g \Psi. \tag{28}$$

Without external forces it simplifies such that $\delta \lambda = m v(2 \tau_g \dot{\Psi} + \dot{\tau}_g \Psi)$. The transversal perturbation satisfies the Raychadhury-like equation [12]

$$\begin{aligned} m \ddot{\Psi} &- (m v^2(\tau_g^2 + \mathcal{R}/2) \\ &\quad + (m v^2 \kappa_n - F_n) K_{ab} \ell^a \ell^b) \Psi = 0. \end{aligned} \tag{29}$$

Besides intrinsic information, projections of the second fundamental form appear in this equation. Even without external forces, extrinsic information are required,

$$\ddot{\Psi} - (v^2(\tau_g^2 + \mathcal{R}/2) + v^2 \kappa_n K_{ab} \ell^a \ell^b) \Psi = 0. \tag{30}$$

A simple example is given by a particle falling on the unit sphere, starting from rest at the north pole, under the gravitational field. Since $F_\ell = 0$ then, according to the first equation in Eq. (12), we have $\kappa_g = 0$, and therefore the

trajectory is a great circle. For a sphere of radius r it is known that $K_{ab} = g_{ab}/r$, such that for a unit sphere we have $K_{ab} = g_{ab}$ and hence $\kappa_n = -1$, $\tau_g = 0$. Moreover, since $\mathcal{R} = K^2 - K^{ab}K_{ab}$ in this case $\mathcal{R} = 2$, and from the energy conservation we have $v^2 = 2g(1 - \cos \theta)$, where θ is the angle between the normal vector \mathbf{n} and the axis z . Due to the geodesic torsion vanishes we get $\delta\lambda = 0$. On the normal direction, $F_n = -mg \cos \theta$, and then the transversal deformation follows the equation $\ddot{\Psi} - g \cos \theta \Psi = 0$. In terms of the angle θ ,

$$v^2 \theta'^2 \nabla_\theta \Psi_\theta + g \theta' \sin \theta \Psi_\theta - g \cos \theta \Psi = 0. \quad (31)$$

Nevertheless, we note that a first integral is, $\dot{\Psi}^2/2 - g \cos \theta \Psi^2 = C$, and therefore

$$2g(1 - \cos \theta) \theta'^2 \Psi_\theta^2 - 2g \cos \theta \Psi^2 = C. \quad (32)$$

2.2.1. Axisymmetric surfaces

If the surface has axial symmetry, we can parametrize it in the form

$$\mathbf{X}(\varphi, l) = (\rho(l) \cos \varphi, \rho(l) \sin \varphi, z(l)). \quad (33)$$

The infinitesimal element of distance is given by $ds^2 = \rho^2 d\varphi^2 + (\rho'^2 + z'^2) dl^2$, here prime indicates derivative respect to l . The tangent vectors \mathbf{e}_a are

$$\begin{aligned} \mathbf{e}_\varphi &= (-\rho \sin \varphi, \rho \cos \varphi, 0), \\ \mathbf{e}_l &= (\rho' \cos \varphi, \rho' \sin \varphi, z'). \end{aligned} \quad (34)$$

The unit normal is then

$$\mathbf{n} = \frac{1}{\sqrt{\rho'^2 + z'^2}} (z' \cos \varphi, z' \sin \varphi, -\rho'). \quad (35)$$

The second fundamental form K_{ab} has components,

$$K_{\varphi\varphi} = \frac{\rho z'}{\sqrt{\rho'^2 + z'^2}}$$

and

$$K_{ll} = \frac{\rho' z'' - \rho'' z'}{\sqrt{\rho'^2 + z'^2}}.$$

The principal curvatures are found to be

$$K_\varphi^\varphi = \frac{z'}{\rho \sqrt{\rho'^2 + z'^2}}$$

and

$$K_l^l = \frac{K_{ll}}{\rho'^2 + z'^2}.$$

With these basic elements we can describe a particle restricted to lie along this surface. Because the symmetry under rotations about the z axis, $M_z = \mathbf{M} \cdot \mathbf{k}$ is conserved where $\mathbf{M} = m \mathbf{x} \times \dot{\mathbf{x}}$, is the angular momenta and $\dot{\mathbf{x}} = \mathbf{e}_\varphi \dot{\varphi} + \mathbf{e}_l \dot{l}$. We find

$$M_z = m \rho^2 \dot{\varphi}. \quad (36)$$

The normal curvature along the trajectory is given by

$$-\kappa_n = K_{\varphi\varphi} \left(\frac{d\varphi}{ds} \right)^2 + K_{ll} \left(\frac{dl}{ds} \right)^2. \quad (37)$$

This equation can be written in terms of M_z . In the first term we can use Eq. (36), in the second one we can use the induced metric on the surface, to obtain a first integral of $\kappa_g = 0$, in the form

$$\frac{1}{2} \left(\frac{dl}{ds} \right)^2 + U_{eff} = 0, \quad (38)$$

where

$$U_{eff}(M_z, l) = -\frac{1}{2(\rho'^2 + z'^2)} \left(1 - \frac{M_z^2}{m^2 v^2 \rho^2} \right). \quad (39)$$

Then we can write the normal curvature of geodesics

$$\begin{aligned} -\kappa_n &= \frac{M_z^2}{m^2 v^2} \frac{z'}{\sqrt{\rho'^2 + z'^2}} \frac{1}{\rho^3} \\ &+ \frac{\rho' z'' - \rho'' z'}{(\rho'^2 + z'^2)^{3/2}} \left(1 - \frac{M_z^2}{m^2 v^2 \rho^2} \right), \end{aligned} \quad (40)$$

and therefore, the force λ in Eq. (9), is completely determined. If the gravitational field $\mathbf{F} = -mg\mathbf{k}$ is taken into account, we can see that

$$\begin{aligned} F_n &= \frac{mg\rho'}{\sqrt{\rho'^2 + z'^2}}, \\ F_l &= \frac{mgz'\rho}{\sqrt{\rho'^2 + z'^2}} \frac{d\varphi}{ds}. \end{aligned} \quad (41)$$

The velocity $v(t)$, is determined by conservation of the energy

$$v^2(t) = \frac{2}{m} (E - mgz(l(t))). \quad (42)$$

An example is the catenoid, a minimal surface that we can parametrize through

$$\mathbf{X}(\varphi, l) = (\cosh l \cos \varphi, \cosh l \sin \varphi, l), \quad (43)$$

where $l \in (-\infty, \infty)$, $\varphi \in [0, 2\pi]$. The normal curvature of geodesic curves is parametrized by M_z

$$-\kappa_n = \text{sech}^2 l \left(\frac{2M_z^2 \text{sech}^2 l}{m^2 v^2} - 1 \right). \quad (44)$$

The curvature $\mathcal{R} = -2 \text{sech}^4 l$, and the second fundamental form has components $K_{\varphi\varphi} = 1$, and $K_{ll} = -1$. If a particle moves on this surface under the gravitational field, then the projections of the force are $F_n = mg \tanh l$, and $F_l = gM_z/v \cosh^2 l$. Because the symmetry, geodesics can be classified according the angular momentum M_z , see Fig. (2): If $0 < M_z < 1$, geodesic crosses parallels along the catenoid; for $M_z > 1$, the path is on one side of the catenoid; the equator corresponds to $M_z = 1$ and $M_z = 0$ is a meridian [11].

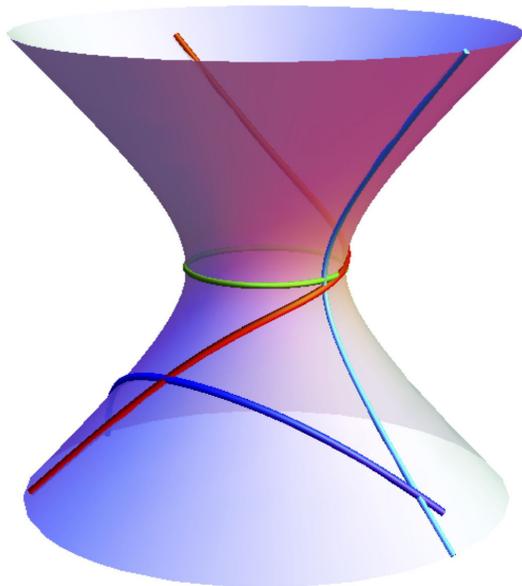


FIGURE 2. The several geodesic curves on the catenoid, classified according to the angular momenta.

An easy example is a particle falling along a meridian starting at l_0 , and $v^2 = 2g(l_0 - l)$. The constraining force λ is found to be

$$\lambda(l) = mg (\tanh l - 2(l_0 - l)\text{sech}^2 l). \tag{45}$$

If the particle leaves the catenoid at l_s , then $\lambda(l_s) = 0$, implies that $\tanh l_s = 2(l_0 - l_s)\text{sech}^2 l_s$. If for instance $l_0 = 1$, then $l_s \sim 0.61$.

Being $\varphi = \text{const}$, for this path $t^\varphi = 0$ and $t^l = \text{sech } l$. From the definition of ℓ , we note that $\ell_a = \sqrt{\bar{g}}\epsilon_{ab}t^b$. We have $\ell^l = 0$ and $\ell^\varphi = \text{sech } l$. Therefore the equation of the transversal deformation is given by

$$\ddot{\Psi} + g \text{sech}^2 l \tanh l \Psi = 0. \tag{46}$$

2.2.2. Particle on a surface in a curved space

If the particle moves onto a surface embedded into a curved space, with coordinates x^μ and metric $g_{\mu\nu}(x^\alpha)$, we have the constrained functional as

$$L(x^\mu, \xi^a, \lambda) = \frac{m}{2} \int v^2 dt + \int \lambda_\mu (x^\mu(t) - X^\mu(\xi^a)) dt. \tag{47}$$

The first term involves the background metric, $v^2 = g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu$, the second one the Lagrange multipliers λ_μ . It generalizes equation (1) to curved background. Under deformations δx we obtain as before, equation (3), but now in the form

$$m \left(\frac{d^2 x^\mu}{dt^2} + \Gamma_{\nu\alpha}^\mu \frac{dx^\nu}{dt} \frac{dx^\alpha}{dt} \right) = -\lambda n^\mu, \tag{48}$$

where the Christoffel symbols compatible with $g_{\mu\nu}$ appear. The projection on the Darboux frame follows the same lines as before. We then have

$$mv^2(t) (\kappa_n n^\mu + \kappa_g \ell^\mu + \Gamma_{\nu\alpha}^\mu T^\nu T^\alpha) + ma(t)T^\mu = -\lambda n^\mu, \tag{49}$$

where $\mathbf{T} = T^\mu e_\mu$, is the tangent vector field, and $e_\mu = \partial_\mu X$, the tangent vectors to the surface. If the particle moves freely we have, as before, that it follows a geodesic curve along the surface, with $a(t) = 0$. The force λ is given by

$$\lambda = -mv^2 \kappa_n. \tag{50}$$

In this case the second fundamental form K_{ab} is defined through the covariant derivative D_μ , compatible with $g_{\mu\nu}$: $D_a = e_a^\mu D_\mu$, i.e. $K_{ab} = -g(D_a e_b, \mathbf{n})$. In addition we can see that the projection of the Christoffel symbols along the tangent vector is null

$$\Gamma_{\alpha\beta}^\nu T^\alpha T^\beta = 0. \tag{51}$$

It is certainly interesting to know the effect of nontrivial background, that we will present in future work.

3. Summary and conclusions

In this paper we have presented the geometric elements in the description of particles on surfaces. Projection of the Newton second law along the normal to surface, gives the constraining force, that involves the normal curvature of the surface. Geodesic curvature (times v^2) gives the acceleration in the transversal direction to the movement. Classical perturbations of the path is equivalent to a field theory on curved surfaces. We show that from conservation of energy, a first order equation of the tangential deformation, is obtained. We also show that the equation of the transversal perturbation follows a Raychaudhuri-like equation that includes extrinsic information even if there is not external forces. If the particle moves onto a surface with axial symmetry, the results are parametrized in terms of the conserved projection of the angular momenta. The particular results in the case of geodesic curves are obtained. Using the formalism here presented, a problem to be addressed, is related to the motion of extended objects on surfaces, for elastic curves, the model must include the bending energy, quadratic in the curvature which has been extensively examined [13].

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