Entropy production: evolution criteria, robustness and fractal dimension

J.A. Betancourt-Mar\textsuperscript{a}, M. Rodríguez-Ricard\textsuperscript{b}, R. Mansilla\textsuperscript{c}, G. Cocho\textsuperscript{d,∗}, and J.M. Nieto-Villar\textsuperscript{e, a}

\textsuperscript{a}Mexican Institute of Complex Systems, Tamaulipas, México, \\
\textsuperscript{b}Departamento de Ecuaciones Diferenciales, \\
Facultad de Matemática y Computación, Universidad de La Habana, La Habana 10400 Cuba. \\
\textsuperscript{c}Centro de Investigaciones Interdisciplinarias en Ciencias y Humanidades, \\
Universidad Nacional Autónoma de México. \\
\textsuperscript{d}Departamento de Sistemas Complejos del Instituto de Física, \\
Universidad Nacional Autónoma de México. \\
\textsuperscript{∗}e-mail: cocho@fisica.unam.mx \\
\textsuperscript{e}Department of Chemical-Physics, M.V. Lomonosov Chemistry Division, Faculty of Chemistry, \\
& H. Poincaré Group of Complex Systems, Physics Faculty, University of Havana, Havana 10400 Cuba.

Received 13 October 2015; accepted 4 January 2016

It was proved through Rössler model, where the funnel case is more robust than spiral chaos, the entropy production per unit time is a Lyapunov’s function on the space of the control system parameters. It was established the conjecture of entropy production fractal dimension. The current theoretical framework will hopefully provide a better understanding of the relationship between thermodynamics and nonlinear dynamics and contribute to unify theses through complex systems theory.

Keywords: Irreversible thermodynamics; complex systems; fractal dimension.

PACS: 05.45.Pq; 05.45.Df; 05.70.Ln

1. Introduction

The unification of nonlinear dynamics and complexity through thermodynamics is already a challenge despite the many efforts trying to reach this goal [1,2]. Presently, it is needed to make an effort to develop formalism for thermodynamics of complex processes.

The aim of this work is to extend the thermodynamics formalism previously developed [3,4] and offers an approximation to the unification of nonlinear dynamics and complexity through thermodynamics. The manuscript is organized as follow: Section 2 we propose the relation of entropy production with the Lyapunov exponent spectra is a Lyapunov function. Moreover, it is a type of measure of dynamical system robustness. Section 3, a conjecture analogous to Lyapunov dimension is proposed to define a fractal dimension of entropy production as a method to measure the complexity of a dynamical system. Finally, some concluding remarks are presented.

2. Entropy production, Lyapunov exponent spectra and Lyapunov function

The seminal work of Nose-Hoover [5] and of more recent work [6] have showed that entropy production per time unit \( \dot{S}_{i} \) is related with the Lyapunov exponent spectra \( \lambda_{j} \) through a relation

\[
\frac{dS_{i}}{dt} = \dot{S}_{i} \approx - \sum_{j} \lambda_{j} > 0 \quad (1)
\]

The Eq. (1) is per se a natural link between the thermodynamics of irreversible processes formalism [7] and nonlinear dynamics [8], without the need of to know if the dynamical system is far or near the equilibrium.

In previous works [3,4], we showed that entropy production per time unit is a Lyapunov function by its dependence on control parameters. This dependence can be exemplified by numerical experiments with Rössler model (Eq. 2) [9] for some distinct values of control parameters (Table I).

\[
\dot{x} = -y - z \\
\dot{y} = x + ay \\
\dot{z} = b + (x - c)z \quad (2)
\]

As can be seen (Table I) there is a drastic dependence of the entropy production rate on the control parameters. This show our thesis [3,4] that the entropy production per time unit is a Lyapunov function that depends on control parameters. These parameters are constants along all the orbit of the ordinary differential equations system. We calculate Lyapunov spectrum and \( \dot{S}_{i} \) for each orbit with constant parameters.

About the specific case of Rössler model, it is known that its dynamics shows two levels of complexity in its robustness: the spiral chaos and funnel chaos [12]. These chaos types depend on the control parameters values. In another work, we showed that funnel chaos is more robust that spiral chaos [13].

So, it can be showed how the entropy production per time unit, as an extremal criterion, fulfills the necessary and sufficient conditions of a Lyapunov function [14], such that:

\[
\dot{S}_{i} = f(\Omega) > 0 \quad (3)
\]
where $\Omega$ is the control parameters vector $(a, b, c)$. The Eulerian derivative of (3) has to fulfill:

$$\frac{d\dot{S}_i}{dt} = \frac{d\dot{S}_i}{d\Omega} \frac{d\Omega}{dt} \leq 0;$$  \hspace{1cm} (4)

$\dot{S}_i = f(\Omega)$ is the Lyapunov function of the fixed point $\Omega_0$ of a system of ordinary differential equations $\ddot{\Omega} = g(\Omega)$, such as $\dot{\Omega} \in P$ and $P \subset \mathbb{R}^n$, where $P$ is the parameters space of the system of ordinary differential equations $\dot{x} = h(x), x \in \mathbb{R}^m$ (as the Rössler system). And we know that $\dot{S}_j = -\sum_{j=1}^{n} \lambda_j$, where $\lambda_j$ is the $j$th Lyapunov exponent of $\dot{x} = h(x)$.

If we fix $b = 0.1$ and $c = 18$ and let $a$ to increase monotonically in time, we have:

$$\frac{d\dot{S}_i}{dt} = \frac{d\dot{S}_i}{da} \frac{da}{dt} \leq 0;$$  \hspace{1cm} (5)

The control parameter $a$ is linked with the evolution of the spiral chaotic behavior to a funnel one [12]; as the value of is growing so the robustness of the system is growing too [13].

Because $\frac{da}{dt} > a$ during the evolution of the spiral chaotic behavior to a funnel one, this implies $d\dot{S}_i/da < 0$ as shown in Fig. 1.

This way, it can be seen that the entropy production per time unit not only satisfies Lyapunov function conditions; moreover, it is a magnitude to quantify the dynamical system robustness [15].

3. Kaplan-York dimension and entropy production

Fractal dimension is one the most important properties of an attractor [16], and it is a measure of the dynamical system complexity. A simple way to compute fractal dimension is

$$D_L = j + \frac{\sum_{j=1}^{n} \lambda_i}{|\sum_{i=j+1}^{n} \lambda_i|},$$  \hspace{1cm} (6)

Where $j$ is the largest integer number for which

$$\lambda_1 + \lambda_2 + \cdots + \lambda_j \geq 0$$

By analogy with the Eq. (6), we can establish the following conjecture: The fractal dimension of entropy production is defined as:

$$D_{\dot{S}_i} = j + \frac{\dot{S}_i}{(\sum_{i=j+1}^{n} \lambda_i)}$$  \hspace{1cm} (7)

where the entropy production per time unit $\dot{S}_i$, is evaluated from Eq. (1), $n$ is the number of all Lyapunov exponents, $j$ is the same as in Eq. (6) ($i$ in $\dot{S}_i$ is not an index, the symbol $\dot{S}_i$ stands for entropy production per time unit).

As an example, we used the Baier-Sahle model [18], a $N$-dimensional model of ordinary differential equations (see Eq. 8). This model is a generalization of the Rössler model. The Baier-Sahle model shows varied levels of complex behavior (see Fig. 2), including chaos and hyperchaos.

$$\dot{x}_1 = -x_2 + ax_1 \quad \dot{x}_i = -x_{i-1} - x_{i-1} \quad i \geq 2$$  \hspace{1cm} (8)

$$\dot{x}_N = e + bx_N(x_{N-1} - d)$$

### Table I. Lyapunov exponents and entropy production per time unit of the Rössler model for some distinct values of control parameters and fixed $b = 0.20$

<table>
<thead>
<tr>
<th>control parameters</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
<th>$\dot{S}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 0.1$</td>
<td>0.072</td>
<td>0</td>
<td>-13.79</td>
<td>13.718</td>
</tr>
<tr>
<td>$c = 14$</td>
<td>0.123</td>
<td>0</td>
<td>-25.79</td>
<td>25.67</td>
</tr>
<tr>
<td>$a = 0.15$</td>
<td>0.130</td>
<td>0</td>
<td>-14.1</td>
<td>13.967</td>
</tr>
<tr>
<td>$c = 14$</td>
<td>0.019</td>
<td>0</td>
<td>-25.5</td>
<td>25.48</td>
</tr>
<tr>
<td>$a = 0.2$</td>
<td>0.064</td>
<td>0</td>
<td>-4.98</td>
<td>4.918</td>
</tr>
<tr>
<td>$c = 14$</td>
<td>0.167</td>
<td>0</td>
<td>-25.26</td>
<td>25.1</td>
</tr>
</tbody>
</table>

For the numeric integration of the ordinary differential equations was used the Gear algorithm for stiff equations in Fortran, double precision and tolerance of $10^{-8}$ [10]. The system was compiled with Open Watcom v1.4 (www.openwatcom.org). The Lyapunov exponents were computed with the Wolf algorithm in Fortran [11].

### Table II. Lyapunov dimension $D_L$ and entropy production dimension $D_{\dot{S}_i}$, for the $N$-dimensional de Baier-Sahle model [18] ($b = 4, d = 2, c = 0.1$).

<table>
<thead>
<tr>
<th>$N(a)$</th>
<th>#($\lambda_i &gt; 0$)*</th>
<th>$D_L$</th>
<th>$D_{\dot{S}_i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5($a = 0.10$)</td>
<td>1</td>
<td>2.704</td>
<td>2.9977</td>
</tr>
<tr>
<td>5($a = 0.15$)</td>
<td>2</td>
<td>4.006</td>
<td>4.9937</td>
</tr>
<tr>
<td>5($a = 0.20$)</td>
<td>3</td>
<td>4.012</td>
<td>4.9900</td>
</tr>
<tr>
<td>7($a = 0.32$)</td>
<td>4</td>
<td>6.026</td>
<td>6.9740</td>
</tr>
<tr>
<td>9($a = 0.30$)</td>
<td>6</td>
<td>8.004</td>
<td>8.9959</td>
</tr>
</tbody>
</table>

*number of positive Lyapunov exponents
As can be seen, both fractal dimensions grow in proportion with the growing of the number of positive Lyapunov exponents \( \#(\lambda_i > 0) \). This way, the entropy production fractal dimension is a measure of system complexity [19] and robustness [20]. Figure 2 shows projections of the five-dimensional Baier-Sahle system. It can be seen the apparent increase in complexity.

4. Conclusions and remarks

In summary, in this paper we found:

1. It is shown how the rate of entropy production evaluated through the spectrum of Lyapunov exponents represents a Lyapunov’s function depending on the control system parameters. In fact it represents a physical magnitude which measures the robustness [15] of the dynamical system.

2. In the same way of the Lyapunov fractal dimension, it was established a conjecture and it was defined an Lyapunov entropy production fractal dimension which is a measure of complexity and robustness [15,21] of the dynamical systems.

The current theoretical framework will hopefully provide a better understanding of the relationship between thermodynamics and nonlinear dynamics and contribute to unify theses through complex systems theory.

Acknowledgments

Prof. Dr. A. Alzola in memoriam. We would like to thank Prof. Dr. Jacques Rieumont for support and encouragement for this research. One of the authors (JMNV) thanked the Institute of Physics of the UNAM Mexico for the warm hospitality and the financial support by SECITI DF and CLAF. Finally, the authors thank the anonymous reviewers for their helpful comments and interesting suggestions.

Table II shows the values of Lyapunov fractal dimension and those of entropy production fractal dimension for the Baier-Sahle model [18].


