# Hamiltonian dynamics: four dimensional BF-like theories with a compact dimension

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Received 29 July 2015; accepted 23 October 2015

A detailed Dirac's canonical analysis for a topological four dimensional $BF$-like theory with a compact dimension is developed. By performing the compactification process we find out the relevant symmetries of the theory, namely, the full structure of the constraints and the extended action. We show that the extended Hamiltonian is a linear combination of first class constraints, which means that the general covariance of the theory is not affected by the compactification process. Furthermore, in order to carry out the correct counting of physical degrees of freedom, we show that must be taken into account reducibility conditions among the first class constraints associated with the excited KK modes. Moreover, we perform the Hamiltonian analysis of Maxwell theory written as a $BF$-like theory with a compact dimension, we analyze the constraints of the theory and we calculate the fundamental Dirac's brackets, finally the results obtained are compared with those found in the literature.

*Keywords:* Topological theories; extra dimensions; Hamiltonian dynamics.

PACS: 98.80.-k;98.80.Cq

## 1. Introduction

Models that involve extra dimensions have introduced completely new ways of looking up on old problems in theoretical physics; the possible existence of a dimension extra beyond the fourth dimension was considered around 1920's, when Kaluza and Klein (KK) tried to unify electromagnetism with Einstein's gravity by proposing a theory in 5D, where the fifth dimension is a circle $S^1$ of radius $R$, and the gauge field is contained in the extra component of the metric tensor (see Ref. 1 and references therein). Nowadays, the study of models involving extra dimensions have an important activity in order to explain and solve some fundamental problems found in theoretical physics, such as, the problem of mass hierarchy, the explanation of dark energy, dark matter and inflation [2]. Moreover, extra dimensions become also important in theories of grand unification trying of incorporating gravity and gauge interactions consistently. In this respect, it is well known that extra dimensions have a fundamental role in the developing of string theory, since all versions of the theory are natural and consistently formulated only in a spacetime of more than four dimensions [3,4]. For some time, however, it was conventional to assume that in string theory such extra dimensions were compactified to complex manifolds of small sizes about the order of the Planck length, $\ell_P \sim\ 10^{-33}$ cm [4,5], or they could be even of lower size independently of the Plank Length [6-8]; in this respect, the compactification process is a crucial step in the construction of models with extra dimensions [9].

On the other hand, there are phenomenological and theoretical motivations to quantize a gauge theory in extra dimensions, for instance, if there exist extra dimensions, then their effects could be tested in the actual LHC collider, and in the International Linear Collider [10].

We can find several works involving extra dimensions, for instance, in Refs. 4, 5, and 9 is developed the canonical analysis of Maxwell theory in five dimensions with a compact dimension, after performing the compactification and fixing the gauge parameters, the final theory describes to Maxwell theory plus a tower of KK excitations corresponding to massive Proca fields. Furthermore, in the context of Yang-Mills (YM) theories, in Ref. 11 it has been carry out the canonical analysis of a 5D YM theory with a compact dimension; in that work were obtained different scenarios for the 4D effective action obtained after the compactification; if the gauge parameters propagate in the bulk, then the excited KK modes are gauge fields, and they are matter vector fields provided that those parameters are confined in the 3-brane.

On the other side, the study of alternative models describing Maxwell and YM theories expressed as the coupling of topological theories have attracted attention recently because of its close relation with gravity. In fact, the study of topological actions has been motived in several contexts of theoretical physics given their interesting relation with physical theories. One example of this is the well-known MacDowell-Maunsouri formulation of gravity (see Ref. 12 and references therein). In this formulation, breaking the $SO(5)$ symmetry of a $BF$-theory for $SO(5)$ group down to $SO(4)$ we can obtain the Palatini action plus the sum of second Chern and Euler topological invariants. Due to these topological classes have trivial local variations that do not contribute classically to the dynamics, we thus obtain essentially general relativity [13,14]. Furthermore, in Refs. 15 and 16 , an analysis of specific limits in the gauge coupling of topological theories yielding a pure YM dynamics in four and three dimensions has been reported. In this respect, in the four-dimensional case, nonperturbative topological configurations of the gauge fields are defined as having an important role in realistic theories, *e.g.* quantum chromodynamics. Moreover, the 3D case is analyzed at the Lagrangian level, and the action becomes the coupling of $BF$-like terms in order to generalize the quantum dynamics of YM [15].

Because of the ideas expressed above, in this paper we analyze a four dimensional $BF$-like theory and the Maxwell theory written as a $BF$-like theory (it is also called first order Maxwell action [17]) with a compact dimension. First, we perform the analysis for the $BF$ term; in this case we are interested in knowing the symmetries of a topological theory defined in four dimensions with a compact dimension. We shall show that in order to obtain the correct counting of physical degrees of freedom, we must take into account reducibility conditions among the first class constraints of the KK excitations; hence, in this paper we present the study of a model with reducibility conditions in the KK modes. Finally, we perform the Hamiltonian analysis of first order Maxwell action with a compact dimension, and we compare our results with those found in the literature. In addition, we have added as appendix the fundamental Dirac's brackets of the theories under study, thus we develop the first steps for studying the quantization aspects.

## 2. Hamiltonian dynamics of a BF-like topological theory with a compact dimension

In the following lines, we shall study the Hamiltonian dynamics for a four dimensional $BF$-like topological theory with a compact dimension; then we develop the canonical analysis of a four dimensional Maxwell theory written as a $BF$-like theory with a compact dimension.

Let us start with the following action reported in Sundermeyer's book [18] (also Yang-Mills theory is written as a $BF$-like theory in that book) defined in four dimensions

$$S_1\left[A,\mathbf{B}\right]=\int d^4x\left\{\frac{1}{4}B^{MN}B_{MN}-\frac{1}{2}B^{MN}\left(\partial_M A_N-\partial_N A_M\right)\right\}, \tag{1}$$

where $B^{MN}=-B^{NM}$. The equations of motion obtained from (1) are given by

$$\partial^M B_{MN}=0, \tag{2}$$

$$B_{MN}=\partial_M A_N-\partial_N A_M. \tag{3}$$

By taking into account (3) in (2), we obtain the Maxwell's free field equations. The action (1) is written in the first order form, it have been analysed without a compacta dimension in Ref. 17. In that paper was worked the context of S-duality transformation by taking into account the general covariance of the Dirac algorithm and also the Dirac brackets of the theory were constructed by using the field redefinition method. Moreover, a pure Dirac's analysis of the action (1) has been reported in Ref. 19, where it was showed that the action can be split in two terms lacking physical degrees of freedom, the complete action, however, does have physical degrees of freedom, the Maxwellian degrees of freedom. The studio of action (1) with a compact dimension becomes important because could expose some information among the topological sector given in the second term on the right hand side of (1) (the $BF$ term), and the dynamical sector given in the full action. Furthermore, our study could be useful for extending the work reported in Ref. 17 in order to find the dual theory of Maxwell with extra dimensions.

Hence, we first analyze the following action

$$S_2\left[A,\mathbf{B}\right]=\int d^4x\left\{B^{MN}\left(\partial_M A_N-\partial_N A_M\right)\right\}. \tag{4}$$

The action $S_2$ is a topological theory, and its study in the context of extra dimensions become relevant. We need to remember that topological field theories are characterized by being devoid of local degrees of freedom. That is, the theories are susceptible only to global degrees of freedom associated with non-trivial topologies of the manifold in which they are defined and topologies of the gauge bundle, thus the next question arises; it is affected the topological nature of $S_2$ because of the compactification process?. Moreover, in order to carry out the counting of physical degrees of freedom of (4) without a compact dimension, we must take into account reducibility conditions among the constraints [19,20], hence, it is interesting to investigate if reducible constraints are still present after performing the compactification process. In fact, the Hamiltonian analysis of theories with reducibility conditions among the constraints in the context of extra dimensions has not been performed, and we shall answer these questions along this paper.

For simplicity, we shall work with a four dimensional action. Then we will perform the compactification process in order to obtain a three dimensional effective Lagrangian. It is straightforward perform the extension of our results to dimensions higher than four. The notation that we will use along the paper is the following: the capital latin indices $M,N$ run over $0,1,2,3$ here 3 label the compact dimension and these indices can be raised and lowered by the four-dimensional Minkowski metric $\eta_{MN}=(-1,1,1,1)$; $z$ will represent the coordinate in the compact dimension and $\mu,\nu=0,1,2$ are spacetime indices, $x^\mu$ the coordinates that label the points for the three-dimensional manifold $M_3$; furthermore we will suppose that the compact dimension is a $S^1/\mathbf{Z_2}$ orbifold whose radius is $R$; then any dynamical variable defined on $M_3\times S^1/\mathbf{Z_2}$ can be expanded in terms of the complete set of harmonics [4,5,11,21]

$$B^{3\mu}(x,z)=\frac{1}{\sqrt{\pi R}}\sum_{n=1}^{\infty}B^{3\mu}_{(n)}(x)\sin\left(\frac{nz}{R}\right),$$

$$
\begin{aligned}
B^{\mu\nu}(x,z) &= \frac{1}{\sqrt{2\pi R}} B^{\mu\nu}_{(0)}(x) \\
&\quad + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} B^{\mu\nu}_{(n)}(x) \cos\left(\frac{nz}{R}\right), \\
A_3(x,z) &= \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A_3^{(n)}(x) \sin\left(\frac{nz}{R}\right), \\
A_\mu(x,z) &= \frac{1}{\sqrt{2\pi R}} A_\mu^{(0)}(x) \\
&\quad + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A_\mu^{(n)}(x) \cos\left(\frac{nz}{R}\right).
\end{aligned} \tag{5}
$$

The dynamical variables of the theory are given by $A_i^{(0)}$, $A_0^{(0)}$, $B^{0i}_{(0)}$, $B^{ij}_{(0)}$, $A_3^{(n)}$, $A_i^{(n)}$, $A_0^{(n)}$, $B^{03}_{(n)}$, $B^{i3}_{(n)}$, $B^{0i}_{(n)}$, $B^{ij}_{(n)}$, with $i,j=1,2$.

Let us perform the Hamiltonian analysis of the topological term given by $S_2$

$$
S_2\left[A,\mathbf{B}\right] = \int d^3x \int_0^{2\pi R} dz \left\{ B^{MN}\left(\partial_M A_N - \partial_N A_M\right)\right\}. \tag{6}
$$

First, we start the analysis by performing the 3+1 decomposition and we use explicitly the expansions given in (5); then we perform the compactification process on a $S^1/\mathbf{Z_2}$ orbifold, we obtain the following effective Lagrangian

$$
\begin{aligned}
\mathcal{L}_2 &= 2B^{0i}_{(0)}\dot{A}_i^{(0)} + 2A_0^{(0)}\partial_i B^{0i}_{(0)} + B^{ij}_{(0)}F^{(0)}_{ij} \\
&\quad + \sum_{n=1}^{\infty}\Bigg[ 2A_0^{(n)}\partial_i B^{0i}_{(n)} + 2B^{0i}_{(n)}\dot{A}_i^{(n)} \\
&\quad + 2B^{i3}_{(n)}\left(\partial_i A_3^{(n)} + \frac{n}{R}A_i^{(n)}\right) \\
&\quad + 2B^{03}_{(n)}\left(\partial_0 A_3^{(n)} + \frac{n}{R}A_0^{(n)}\right) + B^{ij}_{(n)}F^{(n)}_{ij}\Bigg],
\end{aligned} \tag{7}
$$

where $F^{(m)}_{ij} = \partial_i A^{(m)}_j - \partial_j A^{(m)}_i$. The first three terms on the left hand side are called the zero modes and the theory describes a topological theory [19,20,22], the following terms correspond to a KK tower; in fact, both $B^{\alpha\beta}_{(n)}$ and $A^{(n)}_\alpha$ are called Kaluza-Klein (KK) modes. In the following, we shall suppose that the number of KK modes is given by $k$, taking the limit $k\to\infty$ at the end of the calculations.

The theory under study is a singular system, it is easy to observe that the Hessian is a $10k-4\times 10k-4$ matrix, it has a determinant equal to zero. Hence, the Hamiltonian formalism calls for the definition of the momenta $\left(\Pi^{(n)}_{MN}, \Pi^M_{(n)}\right)$ canonically conjugate to $\left(A^{(n)}_M, B^{MN}_{(n)}\right)$,

$$
\Pi^M_{(n)} = \frac{\delta L_2}{\delta\left(\partial_0 A^{(n)}_M\right)}, \qquad \Pi^{(n)}_{MN} = \frac{\delta L_2}{\delta\left(\partial_0 B^{MN}_{(n)}\right)}, \tag{8}
$$

here, $n=1,2,3,..,k-1$. We also can observe that the rank of the Hessian is zero, so we expect $10k-4$ primary constraints; from the definition of the momenta (8). We identify the following primary constraints:

$$
\begin{array}{ll}
\text{zero-modes} & k\text{-modes} \\
\phi^{(0)}_{0j} \equiv \Pi^{(0)}_{0j} \approx 0, & \phi^{(n)}_{03} \equiv \Pi^{(n)}_{03} \approx 0, \\
\phi^{(0)}_{ij} \equiv \Pi^{(0)}_{ij} \approx 0, & \phi^{(n)}_{i3} \equiv \Pi^{(n)}_{i3} \approx 0, \\
\phi^{i}_{(0)} \equiv \Pi^{i}_{(0)} - 2B^{0i}_{(0)} \approx 0, & \phi^{(n)}_{0i} \equiv \Pi^{(n)}_{0i} \approx 0, \\
\phi^{0}_{(0)} \equiv \Pi^{0}_{(0)} \approx 0, & \phi^{(n)}_{0i} \equiv \Pi^{(n)}_{0i} \approx 0, \\
 & \phi^{3}_{(n)} \equiv \Pi^{3}_{(n)} - 2B^{03}_{(n)} \approx 0, \\
 & \phi^{i}_{(n)} \equiv \Pi^{i}_{(n)} - 2B^{0i}_{(n)} \approx 0, \\
 & \phi^{0}_{(n)} \equiv \Pi^{0}_{(n)} \approx 0.
\end{array} \tag{9}
$$

Furthermore, the canonical Hamiltonian is given by

$$
\begin{aligned}
H_c = \int d^2x \Bigg( &- A_0^{(0)}\partial_i \Pi^i_{(0)} - B^{ij}_{(0)}F^{(0)}_{ij} \\
&+ \sum_{n=1}^{\infty}\Big[ -2B^{i3}_{(n)}\left(\partial_i A_3^{(n)} + \frac{n}{R}A_i^{(n)}\right) \\
&- A_0^{(n)}\left(\partial_i \Pi^i_{(n)} + \frac{n}{R}\Pi^3_{(n)}\right) B^{ij}_{(n)}F^{(n)}_{ij}\Big]\Bigg).
\end{aligned}
$$

Thus by using the primary constraints (9), we define the primary Hamiltonian given by

$$
\begin{aligned}
H_P = H_c + \int dx^2 \Bigg[ &\lambda^{0j}_{(0)}\phi^{(0)}_{0j} + \lambda^{ij}_{(0)}\phi^{(0)}_{ij} + \lambda^{(0)}_i \phi^i_{(0)} \\
&+ \lambda^{(0)}_0 \phi^0_{(0)} + \sum_{n=1}^{\infty}\Big( \lambda^{03}_{(n)}\phi^{(n)}_{03} + \lambda^{j3}_{(n)}\phi^{(n)}_{j3} + \lambda^{(n)}_3 \phi^3_{(n)} \\
&+ \lambda^{0j}_{(n)}\phi^{(n)}_{0j} + \lambda^{ij}_{(n)}\phi^{(n)}_{ij} + \lambda^{(n)}_i \phi^i_{(n)} + \lambda^{(n)}_0 \phi^0_{(n)}\Big)\Bigg],
\end{aligned} \tag{10}
$$

where $\lambda^{03}_{(n)}$, $\lambda^{j3}_{(n)}$, $\lambda^{(n)}_3$, $\lambda^{0j}_{(n)}$, $\lambda^{ij}_{(n)}$, $\lambda^{(n)}_i$, $\lambda^{(n)}_0$ and $\lambda^{0j}_{(0)}$, $\lambda^{ij}_{(0)}$, $\lambda^{(0)}_i$, $\lambda^{(0)}_0$ are Lagrange multipliers enforcing the constraints. The non-vanishing fundamental Poisson brackets for the theory under study are given by

$$
\begin{aligned}
\{A^{(m)}_M(x^0,x), \Pi^N_{(n)}(x^0,y)\} &= \delta^M{}_N \delta^m{}_n \delta^2(x-y), \\
\{B^{MN}_{(m)}(x^0,x), \Pi^{(n)}_{IJ}(x^0,y)\} &= \frac{1}{2}\delta^m{}_n\big(\delta^M{}_I\delta^N{}_J \\
&\quad - \delta^N{}_I\delta^M{}_J\big)\delta^2(x-y).
\end{aligned} \tag{11}
$$

Let us now analyze if secondary constraints arise from the consistency conditions over the primary constraints. For this aim, we construct the $(10k-4)\times(10k-4)$ matrix formed by the Poisson brackets among the primary constraints; the non-vanishing Poisson brackets between primary constraints are given by

$$\begin{aligned}
\{\phi^{(0)}_{0i}(x),\phi^{j}_{(0)}(y)\} &= \delta^{j}{}_{i}\delta^{2}(x-y),\\
\{\phi^{(m)}_{03}(x),\phi^{3}_{(n)}(y)\} &= \delta^{m}{}_{n}\delta^{2}(x-y),\\
\{\phi^{(m)}_{0i}(x),\phi^{j}_{(n)}(y)\} &= \delta^{m}{}_{n}\delta^{j}{}_{i}\delta^{2}(x-y).
\end{aligned} \tag{12}$$

If we write to (12) in matrix form, it is straightforward to observe that has rank=$6k-2$ and $4k-2$ null vectors. From consistency and by using the null vectors, we find the following $4k-2$ secondary constraints

$$\begin{aligned}
\dot{\phi}^{0}_{(0)}(x) = \{\phi^{0}_{(0)}(x),H_P\}\approx 0 \qquad &\Rightarrow \qquad \psi_{(0)} = \partial_k\Pi^{k}_{(0)},\approx 0.\\
\dot{\phi}^{(0)}_{ij}(x) = \{\phi^{(0)}_{ij}(x),H_P\}\approx 0 \qquad &\Rightarrow \qquad \psi^{(0)}_{ij} = F^{(0)}_{ij}\approx 0,\\
\dot{\phi}^{(m)}_{k3}(x) = \{\phi^{(m)}_{k3}(x),H_P\}\approx 0 \qquad &\Rightarrow \qquad \psi^{(m)}_{k3} = \partial_k A^{(m)}_{3} + \frac{m}{R}A^{(m)}_{k}\approx 0,\\
\dot{\phi}^{0}_{(m)}(x) = \{\phi^{0}_{(m)}(x),H_P\}\approx 0 \qquad &\Rightarrow \qquad \psi^{3}_{(m)} = \partial_k\Pi^{k}_{(m)} + \frac{m}{R}\Pi_{(m)}\approx 0,\\
\dot{\phi}^{(m)}_{ij}(x) = \{\phi^{(m)}_{ij}(x),H_P\}\approx 0 \qquad &\Rightarrow \qquad \psi^{(m)}_{ij} = F^{(m)}_{ij}\approx 0;
\end{aligned} \tag{13}$$

and the rank allows us to fix the following $6k-2$ Lagrange multipliers

$$\begin{array}{ll}
\text{zero-modes} & k\text{-modes}\\
\lambda^{0j}_{(0)} = -2\partial_i B^{ij}_{(0)}, & \lambda^{03}_{(n)} = -2\partial_i B^{i3}_{(n)},\\
\lambda^{(0)}_{i} = 0, & \lambda^{(n)}_{3} = 0,\\
 & \lambda^{0k}_{(m)} = -2\partial_i B^{ik}_{(m)} + \frac{2m}{R}B^{k3}_{(m)},\\
 & \lambda^{(m)}_{i} = 0.
\end{array} \tag{14}$$

For this theory there are not third constraints. Hence, this completes Dirac's consistency procedure for finding the complete set of constraints. Explicitly the set of constraints primary and secondary obtained are given by

$$\begin{array}{ll}
\text{zero-modes} & k\text{-modes}\\
\phi^{(0)}_{0j} \equiv \Pi^{(0)}_{0j}\approx 0, & \phi^{(n)}_{03}\equiv \Pi^{(n)}_{03}\approx 0,\\
\phi^{(0)}_{ij} \equiv \Pi^{(0)}_{ij}\approx 0, & \phi^{(n)}_{i3}\equiv \Pi^{(n)}_{i3}\approx 0,\\
\phi^{i}_{(0)} \equiv \Pi^{i}_{(0)} - 2B^{0i}_{(0)}\approx 0, & \phi^{(n)}_{0i}\equiv \Pi^{(n)}_{0i}\approx 0,\\
\phi^{0}_{(0)} \equiv \Pi^{0}_{(0)}\approx 0, & \phi^{(n)}_{ij}\equiv \Pi^{(n)}_{ij}\approx 0,\\
\psi^{0}_{(0)} \equiv \partial_k\Pi^{k}_{(0)}\approx 0, & \phi^{3}_{(n)}\equiv \Pi^{3}_{(n)} - 2B^{03}_{(n)}\approx 0,\\
\psi^{(0)}_{ij} \equiv F^{(0)}_{ij}\approx 0, & \phi^{i}_{(n)}\equiv \Pi^{i}_{(n)} - 2B^{0i}_{(n)}\approx 0,\\
 & \phi^{0}_{(n)}\equiv \Pi^{0}_{(n)}\approx 0,\\
 & \psi^{(n)}_{k3}\equiv \partial_k A^{(n)}_{3} + \frac{n}{R}A^{(n)}_{k}\approx 0,\\
 & \psi^{3}_{(n)}\equiv \partial_k\Pi^{k}_{(n)} + \frac{n}{R}\Pi^{3}_{(n)}\approx 0,\\
 & \psi^{(n)}_{ij}\equiv F^{(n)}_{ij}\approx 0.
\end{array} \tag{15}$$

Once identified all the constraints as primary, secondary etc., we may verify which ones correspond to first and second class. For this purpose we will construct the matrix formed by the Poisson brackets among the primary and secondary constraints; in order to achieve this aim, we find that the non-zero Poisson brackets among primary and secondary constraints are given by

$$
\begin{aligned}
\{\phi_{0i}^{(0)}(x), \phi^{(0)j}(y)\} &= \delta^j{}_i \delta^2(x-y), \\
\{\phi^j_{(0)}(x), \psi^{(0)}_{ls}(y)\} &= -\left(\delta^j{}_s \partial^y_l - \delta^j{}_l \partial^y_s\right)\delta^2(x-y), \\
\{\phi_{03}^{(m)}(x), \phi^3_{(n)}(y)\} &= \delta^m{}_n \delta^2(x-y), \\
\{\phi_{0i}^{(m)}(x), \phi^j_{(n)}(y)\} &= \delta^m{}_n \delta^i{}_j \delta^2(x-y), \\
\{\phi^3_{(m)}(x), \psi^{(n)}_{k3}(y)\} &= -\delta^m{}_n \partial^y_k \delta^2(x-y), \\
\{\phi^j_{(m)}(x), \psi^{(n)}_{k3}(y)\} &= -\frac{n}{R}\delta^m{}_n \delta^k{}_j \delta^2(x-y), \\
\{\phi^j_{(m)}(x), \psi^{(n)}_{ls}(y)\} &= -\delta^m{}_n \left(\delta^j{}_s \partial^y_l - \delta^j{}_l \partial^y_s\right)\delta^2(x-y).
\end{aligned} \tag{16}
$$

Again, if we write to (16) in matrix form, we find that has a rank $= 6k-2$ and $8k-4$ null vectors. Thus, by using the rank and the null vectors, we find the following 4 first class constraints for the zero modes

$$
\begin{aligned}
\tilde{\gamma}^{(0)}_{ij} &= F^{(0)}_{ij} - \partial_i \Pi^{(0)}_{0j} + \partial_j \Pi^{(0)}_{0i} \approx 0, \\
\gamma_{(0)} &= \partial_i \Pi^i_{(0)} \approx 0, \\
\gamma^{(0)}_{ij} &= \Pi^{(0)}_{ij} \approx 0, \\
\gamma^0_{(0)} &= \Pi^0_{(0)} \approx 0,
\end{aligned} \tag{17}
$$

and the following 4 second class constraints for the zero modes

$$
\begin{aligned}
\chi^{(0)}_{0i} &= \Pi^{(0)}_{0i} \approx 0, \\
\chi^i_{(0)} &= \Pi^i_{(0)} - 2B^{0i}_{(0)} \approx 0.
\end{aligned} \tag{18}
$$

Furthermore, we identifying the following $8k-8$ first class constraints for the KK-modes

$$
\begin{aligned}
\tilde{\gamma}^{(m)}_{i3} &= \partial_i A^{(m)}_3 + \frac{m}{R} A^{(m)}_i - \partial_i \Pi^{(m)}_{03} - \frac{m}{R}\Pi^{(m)}_{0i} \approx 0, \\
\tilde{\gamma}^{(m)}_{ij} &= F^{(m)}_{ij} - \partial_i \Pi^{(m)}_{0j} + \partial_j \Pi^{(m)}_{0i} \approx 0, \\
\gamma^{(m)}_{i3} &= \Pi^{(m)}_{i3} \approx 0, \\
\gamma^{(m)}_{ij} &= \Pi^{(m)}_{ij} \approx 0, \\
\gamma^0_{(m)} &= \Pi^0_{(m)} \approx 0, \\
\gamma_{(m)} &= \partial_i \Pi^i_{(m)} + \frac{m}{R}\Pi^3_{(m)} \approx 0,
\end{aligned} \tag{19}
$$

and $6k-6$ second class constraints

$$
\begin{aligned}
\chi^{(m)}_{03} &= \Pi^{(m)}_{03} \approx 0, \\
\chi^3_{(m)} &= \Pi^3_{(m)} - 2B^{03}_{(m)} \approx 0, \\
\chi^i_{(m)} &= \Pi^i_{(m)} - 2B^{0i}_{(m)} \approx 0, \\
\chi^{(m)}_{0i} &= \Pi^{(m)}_{0i} \approx 0.
\end{aligned} \tag{20}
$$

With all this information at hand, the counting of degrees of freedom is carry out as follows: there are $20k-8$ dynamical variables, $8k-4$ first class constraints and $6k-2$ second class constraints, therefore the number of degrees of freedom is given by

$$
G = \frac{1}{2}\left(20k - 8 - \left(2(8k-4) + 6k - 2\right)\right) = -\left(k-1\right). \tag{21}
$$

This is an interesting fact, the counting of degrees of freedom is negative and this can not be correct. It is important to comment, that in a four dimensional $BF$ theory without a compact dimension, in order to carry out the correct counting of physical degrees of freedom, we must take into account the reducibility conditions among first class constraints [20,22] Hence, if we observe the constraints found above, we can see that the reducibility among the constraints is also present; however, there exist reducibility conditions in the first class constraints of the KK excitations and there are not in the zero mode. In fact, it can be showed that the reducibility conditions are identified by the following $k-1$ relations

$$
\partial_i \tilde{\gamma}^{(m)}_{j3} - \partial_j \tilde{\gamma}^{(m)}_{i3} - \frac{m}{R}\tilde{\gamma}^{(m)}_{ij} = 0. \tag{22}
$$

In this manner, the number of independent first class constraints are $(8k-4-k+1 = 7k-3)$; then, this implies that the number of physical degrees of freedom is

$$
G = \frac{1}{2}\left(20k - 8 - \left(2(7k-3) + 6k - 2\right)\right) = 0. \tag{23}
$$

Therefore, the $BF$-like theory with a compact dimension is still topological one. It is important to comment that if we perform the counting of physical degrees of freedom for the zero mode, then we find that it is devoid of local degrees of freedom as expected; for the zero mode defined in three dimensions there are not reducibility conditions. All this information become relevant, because after performing the compactification process there are already reducibility conditions; we need to remember that the correct identification of the constraints is a relevant step because they allows us identify observables and constraints are the best guideline to perform the quantization; similarly the reducibility conditions in the KK modes must be taken into account in that process.

With all this information, we can identify the extended action; thus, we use the first class constraints (20), the second class constraints (18), the Lagrange multipliers (14) and we find that the extended action takes the form

$$S_E\Big(Q_K, P_K, \lambda_K\Big) = \int d^3x \Big[\dot{A}^{(0)}_\nu \Pi^\nu_{(0)} + \dot{B}^{\nu\mu}_{(0)}\Pi^{(0)}_{\nu\mu} - \mathcal{H}^{(0)} - \tilde{\alpha}^{ij}_{(0)}\tilde{\gamma}^{(0)}_{ij} - \alpha^{(0)}\gamma_{(0)} - \alpha^{ij}_{(0)}\gamma^{(0)}_{ij} - \alpha^{(0)}_0\gamma^0_{(0)} - \lambda^{0i}_{(0)}\chi^{(0)}_{0i}$$
$$- \lambda^{(0)}_i\chi^i_{(0)} + \sum_{n=1}^{k}\Big\{\dot{A}^{(n)}_N\Pi^N_{(n)} + \dot{B}^{MN}_{(n)}\Pi^{(n)}_{MN} - \mathcal{H}^{(n)} - \alpha^{i3}_{(n)}\tilde{\gamma}^{(n)}_{i3} - \alpha^{ij}_{(n)}\tilde{\gamma}^{(n)}_{ij} - \lambda^{i3}_{(n)}\gamma^{(n)}_{i3}$$
$$- \lambda^{ij}_{(n)}\gamma^{(n)}_{ij} - \lambda^{(n)}_0\gamma^0_{(n)} - \alpha^{(n)}\gamma_{(n)} - \lambda^{(n)}_i\chi^i_{(n)} - \lambda^{0i}_{(n)}\chi^{(n)}_{0i} - \lambda^{03}_{(n)}\chi^{(n)}_{03} - \lambda^{(n)}_3\chi^3_{(n)}\Big\}\Big], \tag{24}$$

where we abbreviate with $Q_K$ y $P_K$ all the dynamical variables and the generalized momenta; $\lambda_K$ stand for all Lagrange multipliers associated with the first and second class constraints. From the extended action, it is possible to identify the extended Hamiltonian which is given by

$$H_{\text{ext}} = \int d^2x\Big[ - A^{(0)}_0\gamma^{(0)} - B^{ij}_{(0)}\tilde{\gamma}^{(0)}_{ij} + \sum_{n=1}^{k}\Big[ - A^{(n)}_0\gamma^3_{(n)} - 2B^{i3}_{(n)}\tilde{\gamma}^{(n)}_{i3} - B^{ij}_{(n)}\tilde{\gamma}^{(n)}_{ij}\Big] + \alpha^{ij}_{(0)}\tilde{\gamma}^{(0)}_{ij} + \alpha^{(0)}\gamma_{(0)} + \lambda^{ij}_{(0)}\gamma^{(0)}_{ij}$$
$$+ \alpha^{(0)}_0\gamma^0_{(0)} + \sum_{n=1}^{k}\Big\{\alpha^{i3}\tilde{\gamma}^{(n)}_{i3} + \alpha^{ij}_{(n)}\tilde{\gamma}^{(n)}_{ij} + \lambda^{i3}_{(n)}\gamma^{(n)}_{i3} + \lambda^{ij}_{(n)}\gamma^{(n)}_{ij} + \lambda^{(n)}_0\gamma^0_{(n)} + \alpha^{(n)}\gamma_{(n)}\Big\}\Big]. \tag{25}$$

We can observe that this expression is a linear combination of constraints. In fact, they are first class constraints of the zero mode and first class constrains of the $KK$-modes. It is well-known, that for the action (6) without compact dimensions, its extended Hamiltonian is a linear combination of first class constraints [20,22]. Thus, we can notice that the general covariance of the theory is not affected by the compactification process. Hence, in order to perform a quantization of the theory, it is not possible to construct the Schrodinger equation because the action of the Hamiltonian on physical states is annihilation. In Dirac's quantization of systems with general covariance, the restriction on physical states is archived by demanding that the first class constraints in their quantum form must be satisfied; thus in this paper we have all tools for studying the quantization of the theory by means a canonical framework.

By following with our analysis, we need to know the gauge transformations on the phase space. For this important step, we shall define the following gauge generator in terms of the first class constraints (20)

$$G = \int_\Sigma d^2x\left[\varepsilon^{i3}_{(n)}\tilde{\gamma}^{(n)}_{i3} + \varepsilon^{ij}_{(n)}\tilde{\gamma}^{(n)}_{ij} + \varepsilon^{(n)}_0\gamma_{(n)} + \dot{\varepsilon}^{i3}_{(n)}\gamma^{(n)}_{i3} + \dot{\varepsilon}^{ij}_{(n)}\gamma^{(n)}_{ij} + \dot{\varepsilon}^{(n)}_0\gamma^0_{(n)} + \varepsilon^{ij}_{(0)}\tilde{\gamma}^{(0)}_{ij} + \dot{\varepsilon}^{ij}_{(0)}\gamma^{(0)}_{ij} + \varepsilon^{(0)}_0\gamma^0_{(0)} + \dot{\varepsilon}^{(0)}_0\gamma_{(0)}\right]. \tag{26}$$

Thus we obtain that the gauge transformations on the phase space are given by

$$\begin{array}{ll}
\text{zero-mode} & \text{k-mode} \\
\delta A^{(0)}_\mu = -\partial_\mu\varepsilon^{(0)}_0, & \delta A^{(n)}_\mu = -\partial_\mu\varepsilon^{(n)}_0, \\
\delta B^{0i}_{(0)} = \partial_k\varepsilon^{ki}_{(0)}, & \delta A^{(n)}_3 = \frac{n}{R}\varepsilon^{(n)}_0, \\
\delta B^{ij}_{(0)} = \partial_0\varepsilon^{ij}_{(0)}, & \delta B^{03}_{(n)} = \frac{1}{2}\partial_i\varepsilon^{i3}_{(n)}, \\
\delta\Pi^i_{(0)} = \partial_k\varepsilon^{ki}_{(0)}, & \delta B^{0i}_{(n)} = -\partial_k\varepsilon^{ik}_{(n)} - \frac{n}{2R}\varepsilon^{i3}_{(n)}, \\
\delta\Pi^0_{(0)} = 0, & \delta B^{i3}_{(n)} = \frac{1}{2}\partial_0\varepsilon^{i3}_{(n)}, \\
\delta\Pi^{MN}_{(0)} = 0, & \delta B^{ij}_{(n)} = \partial_0\varepsilon^{ij}_{(n)}, \\
 & \delta\Pi^3_{(n)} = -\partial_i\varepsilon^{i3}_{(n)}, \\
 & \delta\Pi^i_{(n)} = \partial_k\varepsilon^{ki}_{(n)} - \frac{n}{R}\varepsilon^{i3}_{(n)}, \\
 & \delta\Pi^0_{(n)} = 0, \\
 & \delta\Pi^{MN}_{(n)} = 0.
\end{array} \tag{27}$$

We notice that the fields $B^{MN}$ and $A_M$ are gauge fields; there are not degrees of freedom, thus, it is not relevant to fix the gauge parameters. In the following lines, we shall perform the Hamiltonian analysis of the action (1) and we will find that the field $B^{MN}$ is not a gauge field anymore. There are not reducibility conditions among the constraints, moreover, there exist physical degrees of freedom and the fixing of the gauge parameters will allow us to find massive Proca fields and pseudo-Goldston bosons as expected. Furthermore, we have added in the Appendix B the Dirac brackets of the theory which are important for studying the quantization.

## 3. Hamiltonian analysis of the four-dimensional Maxwell theory written as a BF-like theory with a compact dimension

By following the steps developed in previous section, we can perform the Hamiltonian analysis of (1). In this section we shall resume the complete analysis; thus by performing the $3+1$ decomposition. Using the expansion of the fields (5), and developing the compactification process on a $S^1/\mathbf{Z_2}$ orbifold, we obtain the following effective Lagrangian written as

$$\mathcal{L} = \frac{1}{4}B^{\mu\nu}_{(0)}B^{(0)}_{\mu\nu} - \frac{1}{2}B^{\mu\nu}_{(0)}F^{(0)}_{\mu\nu} + \sum_{n=1}^{\infty}\left[\frac{1}{2}B^{\nu 3}_{(n)}B^{(n)}_{\nu 3} - B^{\mu 3}_{(n)}\left(\partial_\mu A^{(n)}_3 + \frac{n}{R}A^{(n)}_\mu\right) + \frac{1}{4}B^{\mu\nu}_{(n)}B^{(n)}_{\mu\nu} - \frac{1}{2}B^{\mu\nu}_{(n)}F^{(n)}_{\mu\nu}\right]. \tag{28}$$

We identify the zero mode given by

$$\frac{1}{4}B^{\mu\nu}_{(0)}B^{(0)}_{\mu\nu} - \frac{1}{2}B^{\mu\nu}_{(0)}F^{(0)}_{\mu\nu}$$

and the following terms are identified as the KK excitations. We have commented above, that the action (28) describes Maxwell theory in three dimensions (zero mode) plus a tower of KK-modes. The theory is singular, there exists the same number of dynamical variables defined above. Hence, after developing a pure Dirac's analysis, we find a set of $2k$ first class constraints given by

$$\begin{aligned}
\gamma^0_{(0)} &= \Pi^0_{(0)} \approx 0,\\
\gamma_{(0)} &= \partial_i \Pi^i_{(0)} \approx 0,\\
\gamma^0_{(n)} &= \Pi^0_{(n)} \approx 0,\\
\gamma_{(n)} &= \partial_i \Pi^i_{(n)} + \frac{n}{R}\Pi^3_{(n)} \approx 0,
\end{aligned} \tag{29}$$

and the following $12k-6$ second class constraints

$$\begin{aligned}
\chi^{(0)}_{0i} &= \Pi^{(0)}_{0i} \approx 0,\\
\chi^{(0)}_{ij} &= \Pi^{(0)}_{ij} \approx 0,\\
\chi^{j}_{(0)} &= \Pi^{j}_{(0)} + B^{0j}_{(0)} \approx 0,\\
\tilde{\chi}^{(0)}_{ij} &= \frac{1}{2}\left(B^{(0)}_{ij} - F^{(0)}_{ij}\right) \approx 0,\\
\chi^{(n)}_{03} &= \Pi^{(n)}_{03} \approx 0,\\
\chi^{(n)}_{i3} &= \Pi^{(n)}_{i3} \approx 0,\\
\chi^{(n)}_{0j} &= \Pi^{(n)}_{0j} \approx 0,\\
\chi^{(n)}_{ij} &= \Pi^{(n)}_{ij} \approx 0,\\
\chi^{3}_{(n)} &= \Pi^{3}_{(n)} + B^{03}_{(n)} \approx 0,\\
\chi^{i}_{(n)} &= \Pi^{i}_{(n)} + B^{0i}_{(n)} \approx 0,\\
\tilde{\chi}^{(n)}_{i3} &= \frac{1}{2}\left(B^{(n)}_{i3} - \left(\partial_i A^{(n)}_3 + \frac{n}{R}A^{(n)}_i\right)\right) \approx 0,\\
\tilde{\chi}^{(n)}_{ij} &= \frac{1}{2}\left(B^{(n)}_{ij} - F^{(n)}_{ij}\right) \approx 0.
\end{aligned} \tag{30}$$

The identification of second class constraints, allows us to fix the following $12k-6$ Lagrange multipliers

$$\begin{array}{ll}
\text{zero-modes} & k\text{-modes}\\
\lambda^{0k}_{(0)} = -4\partial_i B^{ik}_{(0)} + 2\partial_i F^{ik}_{(0)}, & \lambda^{03}_{(n)} = -4\partial_i B^{i3}_{(n)} + 2\partial_i\left(\partial_i A^{(n)}_3 + \frac{n}{R}A^{(n)}_i\right),\\
\lambda^{ij}_{(0)} = \partial^i \Pi^j_{(0)} - \partial^j \Pi^i_{(0)}, & \lambda^{i3}_{(n)} = 2\left(\partial^i \Pi^3_{(n)} + \frac{n}{R}\Pi^i_{(n)}\right),\\
\lambda^{(0)}_i = 0, & \lambda^{0i}_{(n)} = -4\partial_k B^{ki}_{(n)} + 2\partial_k F^{ki}_{(n)} + \frac{2n}{R}\left(\partial_i A^{(n)}_3 + \frac{n}{R}A^{(n)}_i\right),\\
\beta^{ij}_{(0)} = B^{ij}_{(0)} - F^{ij}_{(0)}, & \lambda^{ij}_{(n)} = \partial^i \Pi^j_{(n)} - \partial^j \Pi^i_{(n)},\\
 & \lambda^{(n)}_3 = 0,\\
 & \lambda^{(n)}_i = 0,\\
 & \beta^{i3}_{(n)} = 2\left(B^{i3}_{(n)} - F^{i3}_{(n)}\right),\\
 & \beta^{ij}_{(n)} = B^{ij}_{(n)} - F^{ij}_{(n)}.
\end{array} \tag{31}$$

By using all this information it is possible to carry out the counting of degrees of freedom as follows; there are $10k-4$ dynamical variables, $2k$ first class constraints and $12k-6$ second class constraints, thus

$$G = \frac{1}{2}\left(20k - 8 - (2(2k) + 12k - 6)\right) = 2k - 1.$$

We observe that if $k=1$ we obtain one degree of freedom as expected for Maxwell theory in three dimensions.

By using the first class constraints (29), the second class constraints (30), and the Lagrange multipliers we find that the extended action takes the form

$$S_E\Big(Q_K,P_K,\lambda_K\Big)=\int\Big[\dot{A}^{(0)}_{\nu}\Pi^{\nu}_{(0)}+\dot{B}^{(0)}_{\nu\mu}\Pi^{\nu\mu}_{(0)}-\mathcal{H}^{(0)}-\beta^{(0)}\gamma_{(0)}-\lambda^{(0)}_0\gamma^0_{(0)}-\lambda^{(0)}_i\chi^i_{(0)}-\lambda^{ij}_{(0)}\chi^{(0)}_{ij}-\beta^{ij}_{(0)}\tilde{\chi}^{(0)}_{ij}$$
$$+\sum_{n=1}^{\mathcal{N}}\Big\{\dot{A}^{(n)}_N\Pi^N_{(n)}+\dot{B}^{(n)}_{MN}\Pi^{MN}_{(n)}-\mathcal{H}^{(n)}-\lambda^{(n)}_0\gamma^0_{(n)}-\beta^{(n)}\gamma_{(n)}-\lambda^{(n)}_i\chi^i_{(n)}-\lambda^{(n)}_3\chi^3_{(n)} \tag{32}$$
$$-\lambda^{03}_{(n)}\chi^{(n)}_{03}-\lambda^{0i}_{(n)}\chi^{(n)}_{0i}-\lambda^{i3}_{(n)}\chi^{(n)}_{i3}-\lambda^{ij}_{(n)}\chi^{(n)}_{ij}-\beta^{i3}_{(n)}\tilde{\chi}^{(n)}_{i3}-\beta^{ij}_{(n)}\tilde{\chi}^{(n)}_{ij}\Big\}\Big]dx^3,$$

where the corresponding extended Hamiltonian is given by

$$H_{\text{ext}}=H+\int\Big[\beta_{(0)}\gamma^{(0)}+\lambda^{(0)}_0\gamma^0_{(0)}+\sum_{n=1}^{\mathcal{N}}\Big\{\lambda^{(n)}_0\gamma^0_{(n)}+\beta^{(n)}\gamma_{(n)}\Big\}\Big]dx^3. \tag{33}$$

Here we define

$$H=\int d^2x\Bigg(\frac{1}{2}\Pi^i_{(0)}\Pi^{(0)}_i+\frac{1}{4}B^{ij}_{(0)}B^{(0)}_{ij}-A^{(0)}_0\gamma_{(0)}+\left(-4\partial_jB^{ji}_{(0)}+2\partial_jF^{ji}_{(0)}\right)\chi^{(0)}_{0i}+2\chi^{(0)}_{ij}\partial_j\Pi^{(0)}_i-\tilde{\chi}^{(0)}_{ij}F^{ij}_{(0)}$$
$$+\sum_{n=1}^{\mathcal{N}}\Bigg[\frac{1}{2}\Pi^i_{(n)}\Pi^{(n)}_i+\frac{1}{2}\Pi^3_{(n)}\Pi^{(n)}_3+\frac{1}{4}B^{ij}_{(n)}B^{(n)}_{ij}+\frac{1}{2}B^{i3}_{(n)}B^{(n)}_{i3}-A^{(n)}_0\gamma_{(n)}-F^{ij}_{(n)}\tilde{\chi}^{(n)}_{ij}+2\chi^{(n)}_{ij}\partial_i\Pi^{(n)}_j$$
$$-B^{i3}_{(n)}\left(\partial_iA^{(n)}_3+\frac{n}{R}A^{(n)}_i\right)+\left(\partial_iA^{(n)}_3+\frac{n}{R}A^{(n)}_i\right)\left(\partial_iA^{(n)}_3+\frac{n}{R}A^{(n)}_i\right)+2\left(\partial_i\Pi^{(n)}_3+\frac{n}{R}\Pi^{(n)}_i\right)\chi^{(n)}_{i3} \tag{34}$$
$$+\left(-4\partial_jB^{ji}_{(n)}+2\partial_jF^{ji}_{(n)}+\frac{2n}{R}\left(\partial_iA^{(n)}_3+\frac{n}{R}A^{(n)}_i\right)\right)\chi^{(n)}_{0i}+\left(-4\partial_iB^{i3}_{(n)}+2\partial_i\left(\partial_iA^{(n)}_3+\frac{n}{R}A^{(n)}_i\right)\right)\chi^{(n)}_{03}\Bigg]\Bigg)$$
$$=\int d^2x\left(\mathcal{H}^{(0)}+\sum_{n=1}^{\mathcal{N}}\mathcal{H}^{(n)}\right).$$

Note, that the extended Hamiltonian is not a linear combination of constraints anymore, the term $B^{MN}B_{MN}$ of the action (1) breaks down the general covariance of the theory and eliminates the reducibility relations present in the $BF$-like term.

Now, the first class constraints allows us to know the fundamental gauge transformations. For this important step, we use the Castellani's procedure [23,24] to construct the gauge generators

$$G=\int_{\Sigma}\left[\varepsilon^{(n)}_0\gamma^0_{(n)}+\varepsilon^{(n)}\gamma_{(n)}+\varepsilon^{(0)}_0\gamma^0_{(0)}+\varepsilon^{(0)}\gamma_{(0)}\right]dx^2. \tag{35}$$

Thus, we find that the gauge transformations on the phase space are given for

zero modes

$$\begin{aligned}\delta A^{(0)}_{\mu}&=-\partial_{\mu}\varepsilon_{(0)},\\ \delta B^{(0)}_{\mu\nu}&=0,\\ \delta\Pi^{\mu}_{(0)}&=0,\\ \delta\Pi^{\mu\nu}_{(0)}&=0,\end{aligned} \tag{36}$$

and the gauge transformation for the $KK$ modes

$$\delta A^{(n)}_{\mu}=-\partial_{\mu}\varepsilon_{(n)}, \tag{37}$$
$$\delta A^{(n)}_3=\frac{n}{R}\varepsilon_{(n)}, \tag{38}$$
$$\delta B^{(n)}_{\mu\nu}=0, \tag{39}$$
$$\delta\Pi^{\mu}_{(n)}=0, \tag{40}$$
$$\delta\Pi^{\mu\nu}_{(n)}=0, \tag{41}$$

we can observe that the gauge transformations for the zero mode are the same given for Maxwell theory written in the standard form [24], and we also observe that the $B$ filed is not a gauge field anymore. Finally, the transformations of the fields $A^n_{\mu}$, $A^n_3$ corresponding for the $k$-th mode are the same to those reported in the literature (see [4,24] and the cites therein). Hence, by fixing the gauge parameters by $\varepsilon_{(n)} = -(R/n)A_3^{(n)}$ and considering the second class constraints as strong identities, the effective action (28) is reduced to that reported in Ref. 4 and 25, namely

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}_{(0)}F^{(0)}_{\mu\nu} + \sum_{n=1}^{\infty}\left[-\frac{1}{4}F^{\nu\mu}_{(n)}F^{(n)}_{\nu\mu} + \frac{1}{2}\left(\frac{2n}{R}\right)^2 A^{(n)}_{\mu}A^{\mu(n)}\right], \quad (42)$$

where we able to observe that the KK-modes are massive Proca fields, $A_3^{(n)}$ has been absorbed and it is identified as a pseudo-Goldstone boson [4,25]. Furthermore, we have added in the appendix A the Dirac brackets, thus we have developed a full Hamiltonian analysis of the theory under study.

## 4. Conclusions

In this paper, the Hamiltonian analysis for a topological $BF$-like theory and for Maxwell theory expressed as a $BF$-like theory with a compact dimension has been performed. For the former, we performed the compactification process on a $S^1/\mathbf{Z_2}$ orbifold, then we analysed the effective theory and all the constraints, gauge transformations and the extended Hamiltonian have been obtained. We also found that the extended Hamiltonian is given by a linear combination of first class constraints of the zero mode and first class constraints of the KK modes, this indicates that the compactification process does not break the general covariance of the theory. Moreover, we observed that reducibility relations among the constraints are preserved before and after performing the compatification process, however, the reducibility is given among the first class constraints of the excited modes, there is not reducibility in the zero modes. This important fact allowed us to conclude that the theory is a topological one.

Finally, for Maxwell theory written in the first order formalism with a compact dimension, we found the constraints, the gauge transformations and the extended action. We observed that the theory do not present reducibility conditions among the constraints, and the theory is not topological anymore. In fact, the theory has the same symmetries and degrees of freedom than Maxwell theory with a compact dimension [4,9]. Finally by fixing the gauge parameters we noted that the theory is reduced to Maxwell theory in three dimensions described by the zero mode plus a tower of massive Proca fields excitations.

We would to comment that our results are generic and can be extended to a 5D theory and models with a close relation to YM and general relativity. In fact, we have commented above that there are topological generalizations of Maxwell and Yang-Mills theories in three and four dimensions, that could provide generalized QCD theories as it is claimed in Ref. 15. In this manner, our results can be used for studying those generalizations in the context of extra dimensions. Furthermore, we have commented that our results can be used for extending those reported in Ref. 17. In fact, in this paper we have at hand all the tools for study S-Duality of Maxwell theory with a compact dimension. Moreover, our results also can be used for studying models that are present in string theory such as those models described by Kalb-Ramond fields. Finally, all the results presented in this work will be useful in order to compare the Dirac quantization with other schemes, for instance, with the Faddeev-Jackiw quantization (see the Ref. 26 and cites therein). In fact, in Faddeev-Jackiw approach it is possible to obtain all the relevant Dirac's results; the generalized Faddeev-Jackiw brackets coincide with the Dirac ones, basically in this formulation we only choose the symplectic variables either the configuration space or the phase space and by fixing the appropriated gauge, we can invert the symplectic matrix in order to obtain a complete analysis. In this respect, in order to make progress in the quantization, we will work with the Faddeev-Jackiw formulation and we will confirm the results obtained along this paper. All these ideas are in progress and will be the subject of future works.

## Appendix

### A.

In this section we will compute the Dirac brackets for the $BF$ theory with a compact dimension given by the action (7). By using the constraints given in (17), (18) and the fixed gauge $\partial_i A_i^{(0)} \approx 0$, $A_0^{(0)} \approx 0$, $2\eta_{ij}\partial^i B^{0j}_{(0)} \approx 0$ and $2B^{ij}_{(0)} \approx 0$ we obtain the following set of second class constraints

$$\begin{aligned}
\hat{\chi}^{(0)} &= \partial_i A_i^{(0)} \approx 0, \\
\chi_0^{(0)} &= A_0^{(0)} \approx 0, \\
\chi_{(0)} &= \partial_i \Pi^i_{(0)} \approx 0, \\
\bar{\chi}^0_{(0)} &= \Pi^0_{(0)} \approx 0, \\
\tilde{\chi}^{(0)} &= \frac{1}{2}\eta^{ij}F^{(0)}_{ij} - \eta^{ij}\partial_i \Pi^{(0)}_{0j} \approx 0, \\
\hat{\chi}_{(0)} &= 2\eta^{ij}\partial_i B^{0j}_{(0)} \approx 0, \\
\chi^{ij}_{(0)} &= 2B^{ij}_{(0)} \approx 0, \\
\tilde{\chi}^{(0)}_{ij} &= \Pi^{(0)}_{ij} \approx 0, \\
\chi^i_{(0)} &= \Pi^i_{(0)} - 2B^{0i}_{(0)} \approx 0, \\
\chi^{(0)}_{0i} &= \Pi^{(0)}_{0i} \approx 0.
\end{aligned} \quad \text{(A.1)}$$

Thus, the matrix whose entries are given by the Poisson brackets among the constraints (A.1) takes the form

$$
G^{(0)}_{\alpha\nu} = \begin{array}{c} \\ \hat{\chi}^{(0)} \\ \chi^{(0)}_0 \\ \chi_{(0)} \\ \bar{\chi}^0_{(0)} \\ \tilde{\chi}^{(0)} \\ \hat{\chi}_{(0)} \\ \chi^{ij}_{(0)} \\ \tilde{\chi}^{(0)}_{ij} \\ \chi^i_{(0)} \\ \chi^{(0)}_{0i} \end{array}
\begin{array}{cccccccccc}
\hat{\chi}^{(0)} & \chi^{(0)}_0 & \chi_{(0)} & \bar{\chi}^0_{(0)} & \tilde{\chi}^{(0)} & \hat{\chi}_{(0)} & \chi^{kl}_{(0)} & \tilde{\chi}^{(0)}_{kl} & \chi^k_{(0)} & \chi^{(0)}_{0k} \\
\left( 0 \right. & 0 & -\nabla^2 & 0 & 0 & 0 & 0 & 0 & \partial_k & \left. 0 \right) \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\nabla^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\nabla^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \nabla^2 & 0 & 0 & 0 & 0 & \eta^{ik}\partial_i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \left(\delta^i{}_k\delta^j{}_l - \delta^i{}_l\delta^j{}_k\right) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\left(\delta^i{}_k\delta^j{}_l - \delta^i{}_l\delta^j{}_k\right) & 0 & 0 & 0 \\
-\partial_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta^k{}_i \\
0 & 0 & 0 & 0 & 0 & -\eta^{ji}\partial_j & 0 & 0 & \delta^k{}_i & 0.
\end{array} \delta^2(x-y)
\tag{A.2}
$$

Hence, the inverse is given

$$
G^{(0)-1}_{\alpha\nu} = \begin{array}{c} \\ \hat{\chi}^{(0)} \\ \chi^{(0)}_0 \\ \chi_{(0)} \\ \bar{\chi}^0_{(0)} \\ \tilde{\chi}^{(0)} \\ \hat{\chi}_{(0)} \\ \chi^{ij}_{(0)} \\ \tilde{\chi}^{(0)}_{ij} \\ \chi^i_{(0)} \\ \chi^{(0)}_{0i} \end{array}
\begin{array}{cccccccccc}
\hat{\chi}^{(0)} & \chi^{(0)}_0 & \chi_{(0)} & \bar{\chi}^0_{(0)} & \tilde{\chi}^{(0)} & \hat{\chi}_{(0)} & \chi^{kl}_{(0)} & \tilde{\chi}^{(0)}_{kl} & \chi^k_{(0)} & \chi^{(0)}_{0k} \\
0 & 0 & \frac{1}{\nabla^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{\nabla^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial_k}{\nabla^2} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\nabla^2} & 0 & 0 & \frac{\eta^{ik}\partial_i}{\nabla^2} & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{\nabla^2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\left(\delta^i{}_k\delta^j{}_l - \delta^i{}_l\delta^j{}_k\right) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \left(\delta^i{}_k\delta^j{}_l - \delta^i{}_l\delta^j{}_k\right) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{\eta^{ji}\partial_j}{\nabla^2} & 0 & 0 & 0 & 0 & \delta^k{}_i \\
0 & 0 & -\frac{\partial_i}{\nabla^2} & 0 & 0 & 0 & 0 & 0 & -\delta^k{}_i & 0
\end{array} \delta^2(x-y). \tag{A.3}
$$

In this manner, the nonzero Dirac's brackets are given by

$$
\begin{aligned}
\{A^{(0)}_i(x), \Pi^j_{(0)}(y)\}_D =& \delta^j{}_i\delta^2(x-y) - \frac{1}{\nabla^2}\left(\partial_i\partial_j - \eta^{ki}\eta^{lj}\partial_k\partial_l\right)\delta^2(x-y), \\
\{B^{0i}_{(0)}(x), A^{(0)}_j(y)\}_D =& -\frac{1}{2}\delta^j{}_i\delta^2(x-y) - \frac{1}{2\nabla^2}\left(\partial_i\partial_j - \eta^{ki}\eta^{lj}\partial_k\partial_l\right)\delta^2(x-y), \\
\{B^{0i}_{(0)}(x), \Pi^{(0)}_{0j}(y)\}_D =& 0, \\
\{B^{ij}_{(n)}(x), \Pi^{(n)}_{kl}(y)\}_D =& 0.
\end{aligned} \tag{A.4}
$$

We can obtain similar results for the excited modes. Therefore, we have in this work all the elements for studying the quantization of the theories under study. It is important to comment that all these results are not reported in the literature.

## B.

In this appendix, we calculate the Dirac brackets for Maxwell theory written as a BF-like theory. For our aims we will calculate the Dirac brackets for the zero mode, then we will calculate the brackets for the excited modes. Hence, by using the following fixed gauge $\partial^i A^{(0)}_i \approx 0$ and $A^{(0)}_0 \approx 0$, we obtain the following set of second class constraints

$$
\begin{aligned}
\bar{\chi}^{(0)} =& A_0^{(0)} \approx 0, \\
\hat{\chi}_{(0)} =& \Pi^0_{(0)} \approx 0, \\
\tilde{\chi}^{(0)} =& \partial_i A_i^{(0)} \approx 0, \\
\chi_{(0)} =& \partial_i \Pi^i_{(0)} \approx 0, \\
\chi^{(0)}_{0i} =& \Pi^{(0)}_{0i} \approx 0, \\
\chi^{(0)}_{ij} =& \Pi^{(0)}_{ij} \approx 0, \\
\chi^i_{(0)} =& \Pi^i_{(0)} + B^{0i}_{(0)} \approx 0, \\
\tilde{\chi}^{(0)}_{ij} =& \frac{1}{2}\left(B^{(0)}_{ij} - F^{(0)}_{ij}\right) \approx 0.
\end{aligned} \tag{B.1}
$$

Thus, we can calculate the following matrix whose entries are given by the Poisson brackets between these constraints, obtaining

$$
C^{(0)}_{\alpha\nu}(x,y) = \begin{array}{c} \\ \bar{\chi}^{(0)} \\ \hat{\chi}_{(0)} \\ \tilde{\chi}^{(0)} \\ \chi_{(0)} \\ \chi^{(0)}_{0i} \\ \chi^{(0)}_{ij} \\ \chi^i_{(0)} \\ \tilde{\chi}^{(0)}_{ij} \end{array}
\begin{array}{cccccccc}
\bar{\chi}^{(0)} & \hat{\chi}_{(0)} & \tilde{\chi}^{(0)} & \chi_{(0)} & \chi^{(0)}_{0j} & \chi^{(0)}_{kl} & \chi^k_{(0)} & \tilde{\chi}^{(0)}_{kl} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\nabla^2 & 0 & 0 & \partial_k & 0 \\
0 & 0 & \nabla^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\delta^i{}_k & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4}\left(\delta^i{}_k\delta^j{}_l - \delta^i{}_l\delta^j{}_k\right) \\
0 & 0 & -\partial_i & 0 & \frac{1}{2}\delta^i{}_j & 0 & 0 & \frac{1}{2}\left(\delta^i{}_l\partial_k - \delta^i{}_k\partial_l\right) \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4}\left(\delta^i{}_k\delta^j{}_l - \delta^i{}_l\delta^j{}_k\right) & -\frac{1}{2}\left(\delta^k{}_j\partial_i - \delta^k{}_i\partial_j\right) & 0
\end{array}
\delta^2(x-y). \tag{B.2}
$$

The inverse of this matrix is given by

$$
C^{(0)-1}_{\alpha\nu}(x,y) = \begin{array}{c} \\ \bar{\chi}^{(0)} \\ \hat{\chi}_{(0)} \\ \tilde{\chi}^{(0)} \\ \chi_{(0)} \\ \chi^{(0)}_{0i} \\ \chi^{(0)}_{ij} \\ \chi^i_{(0)} \\ \tilde{\chi}^{(0)}_{ij} \end{array}
\begin{array}{cccccccc}
\bar{\chi}^{(0)} & \hat{\chi}_{(0)} & \tilde{\chi}^{(0)} & \chi_{(0)} & \chi^{(0)}_{0k} & \chi^{(0)}_{kl} & \chi^k_{(0)} & \tilde{\chi}^{(0)}_{kl} \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\nabla^2} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\nabla^2} & 0 & -\frac{2\partial_k}{\nabla^2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2\partial_i}{\nabla^2} & 0 & 4\left(\delta^i{}_l\partial_k - \delta^i{}_k\partial_l\right) & 2\delta^i{}_k & 0 \\
0 & 0 & 0 & 0 & -4\left(\delta^k{}_j\partial_i - \delta^k{}_i\partial_j\right) & 0 & 0 & 4\left(\delta^i{}_k\delta^j{}_l - \delta^i{}_l\delta^j{}_k\right) \\
0 & 0 & 0 & 0 & -2\delta^i{}_k & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -4\left(\delta^i{}_k\delta^j{}_l - \delta^i{}_l\delta^j{}_k\right) & 0 & 0
\end{array}
\delta^2(x-y). \tag{B.3}
$$

Thus, we obtain the following Dirac's brackets for the zero mode

$$
\begin{aligned}
\{A_i^{(0)}(x), \Pi^j_{(0)}(y)\}_D &= \left(\delta^j{}_i - \frac{\partial_j \partial_i}{\nabla^2}\right)\delta^2(x-y), \\
\{B^{ij}_{(0)}(x), \Pi^{(0)}_{kl}(y)\}_D &= 0 \\
\{A_i^{(0)}(x), A_j^{(0)}(y))\}_D &= 0 \\
\{\Pi^i_{(0)}(x), \Pi^j_{(0)}(y)\}_D &= 0 \\
\{B^{ij}_{(0)}(x), \Pi^k_{(0)}(y)\}_D &= 2\left(\delta^k{}_j \partial_i - \delta^k{}_i \partial_j\right)\delta^2(x-y), \\
\{B^{ij}_{(0)}(x), B^{0k}_{(0)}(x)\}_D &= -2\left(\delta^k{}_j \partial_i - \delta^k{}_i \partial_j\right)\delta^2(x-y), \\
\{A_i^{(0)}(x), B^{0j}_{(0)}(x)\}_D &= -\left(\delta^j{}_i - \frac{\partial_j \partial_i}{\nabla^2}\right)\delta^2(x-y).
\end{aligned} \tag{B.4}
$$

Observe that the Dirac brackets among the fields $A_i^{(0)}, \Pi^j_{(0)}$ are those knew for Maxwell theory [24].
Now we calculate Dirac's brackets for the excited modes of the Maxwell $BF$-like theory. By working with the following fixed gauge $\partial_i A_i^{(n)} \approx 0$ and $\Pi^3_{(n)} + \frac{n}{R} A_0^{(n)} \approx 0$, we obtain the following set of second class constraints

$$
\begin{aligned}
\tilde{\chi}^{(n)} &= \partial_i A_i^{(n)} \approx 0, \\
\chi^0_{(n)} &= \Pi^0_{(n)} \approx 0, \\
\tilde{\chi}^3_{(n)} &= \Pi^3_{(n)} + \frac{n}{R} A_0^{(n)} \approx 0, \\
\chi_{(n)} &= \partial_i \Pi^i_{(n)} + \frac{n}{R}\Pi^3_{(n)} \approx 0, \\
\chi^{(n)}_{0j} &= \Pi^{(n)}_{0j} \approx 0, \\
\chi^{(n)}_{ij} &= \Pi^{(n)}_{ij} \approx 0, \\
\chi^i_{(n)} &= \Pi^i_{(n)} + B^{0i}_{(n)} \approx 0, \\
\tilde{\chi}^{(n)}_{ij} &= \frac{1}{2}\left(B^{(n)}_{ij} - F^{(n)}_{ij}\right) \approx 0, \\
\chi^{(n)}_{03} &= \Pi^{(n)}_{03} \approx 0, \\
\chi^{(n)}_{i3} &= \Pi^{(n)}_{i3} \approx 0, \\
\chi^3_{(n)} &= \Pi^3_{(n)} + B^{03}_{(n)} \approx 0, \\
\tilde{\chi}^{(n)}_{i3} &= \frac{1}{2}\left(B^{(n)}_{i3} - \left(\partial_i A_3^{(n)} + \frac{n}{R} A_i^{(n)}\right)\right) \approx 0.
\end{aligned} \tag{B.5}
$$

Thus, we obtain the following matrix whose entries are given by the Poisson brackets between these second class constraints, obtaining

$$
G^{(n)}_{\alpha\nu} = \begin{array}{c|cccccccccccc}
 & \tilde{\chi}^{(n)} & \chi^0_{(n)} & \tilde{\chi}^3_{(n)} & \chi_{(n)} & \chi^{(n)}_{0k} & \chi^{(n)}_{kl} & \chi^k_{(n)} & \tilde{\chi}^{(n)}_{kl} & \chi^{(n)}_{03} & \chi^{(n)}_{k3} & \chi^3_{(n)} & \tilde{\chi}^{(n)}_{k3} \\
\hline
\tilde{\chi}^{(n)} & 0 & 0 & 0 & -\nabla^2 & 0 & 0 & \partial_k & 0 & 0 & 0 & 0 & 0 \\
\chi^0_{(n)} & 0 & 0 & -\frac{n}{R} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{\chi}^3_{(n)} & 0 & \frac{n}{R} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\partial_k \\
\chi_{(n)} & \nabla^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\chi^{(n)}_{0i} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\delta^i{}_k & 0 & 0 & 0 & 0 & 0 \\
\chi^{(n)}_{ij} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4}\left(\delta^i{}_k \delta^j{}_l - \delta^i{}_l \delta^j{}_k\right) & 0 & 0 & 0 & 0 \\
\chi^i_{(n)} & -\partial_i & 0 & 0 & 0 & \frac{1}{2}\delta^i{}_k & 0 & 0 & \frac{1}{2}\left(\delta^i{}_l \partial_k - \delta^i{}_k \partial_l\right) & 0 & 0 & 0 & \frac{n}{2R}\delta^i{}_k \\
\tilde{\chi}^{(n)}_{ij} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4}\left(\delta^i{}_k \delta^j{}_l - \delta^i{}_l \delta^j{}_k\right) & -\frac{1}{2}\left(\delta^k{}_j \partial_i - \delta^k{}_i \partial_j\right) & 0 & 0 & 0 & 0 & 0 \\
\chi^{(n)}_{03} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\
\chi^{(n)}_{i3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4}\delta^i{}_k \\
\chi^3_{(n)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2}\partial_k \\
\tilde{\chi}^{(n)}_{i3} & 0 & 0 & -\frac{1}{2}\partial_i & 0 & 0 & 0 & -\frac{n}{2R}\delta^i{}_k & 0 & 0 & \frac{1}{4}\delta^i{}_k & -\frac{1}{2}\partial_i & 0
\end{array} \delta^2(x-y). \tag{B.6}
$$

Hence, the inverse matrix is given by

$$
G^{(n)-1}_{\alpha\nu} = \begin{array}{c} \\ \tilde{\chi}^{(n)} \\ \chi^{0}_{(n)} \\ \tilde{\chi}^{3}_{(n)} \\ \chi_{(n)} \\ \chi^{(n)}_{0i} \\ \chi^{(n)}_{ij} \\ \chi^{i}_{(n)} \\ \tilde{\chi}^{(n)}_{ij} \\ \chi^{(n)}_{03} \\ \chi^{(n)}_{i3} \\ \chi^{3}_{(n)} \\ \tilde{\chi}^{(n)}_{i3} \end{array}
\begin{array}{cccccccccccc}
\tilde{\chi}^{(n)} & \chi^{0}_{(n)} & \tilde{\chi}^{3}_{(n)} & \chi_{(n)} & \chi^{(n)}_{0k} & \chi^{(n)}_{kl} & \chi^{k}_{(n)} & \tilde{\chi}^{(n)}_{kl} & \chi^{(n)}_{03} & \chi^{(n)}_{k3} & \chi^{3}_{(n)} & \tilde{\chi}^{(n)}_{k3} \\
0 & 0 & 0 & \frac{1}{\nabla^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{R}{n} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2R}{n}\partial_k & 0 & 0 \\
0 & -\frac{R}{n} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{\nabla^2} & 0 & 0 & 0 & -2\frac{\partial_k}{\nabla^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2\frac{\partial_k}{\nabla^2} & 0 & 4\left(\delta^i{}_l\partial_k - \delta^i{}_k\partial_l\right) & 2\delta^i{}_k & 0 & 0 & \frac{4n\delta^i{}_k}{R} & 0 & 0 \\
0 & 0 & 0 & 0 & -4\left(\delta^k{}_j\partial_i - \delta^k{}_i\partial_j\right) & 0 & 0 & 4\left(\delta^i{}_k\delta^j{}_l - \delta^i{}_l\delta^j{}_k\right) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2\delta^i{}_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -4\left(\delta^i{}_k\delta^j{}_l - \delta^i{}_l\delta^j{}_k\right) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4\partial_k & 2 & 0 \\
0 & -\frac{2R}{n}\partial_i & 0 & 0 & -\frac{4n\delta^i{}_k}{R} & 0 & 0 & 0 & -4\partial_i & 0 & 0 & 4\delta^i{}_k \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4\delta^i{}_k & 0 & 0
\end{array}
\delta^2(x-y). \tag{B.7}
$$

In this manner, the Dirac brackets for the excited modes are given by

$$
\begin{aligned}
\{A^{(n)}_i(x), \Pi^j_{(n)}(y)\}_D &= \left(\delta^j{}_i - \frac{\partial_i\partial_j}{\nabla^2}\right)\delta^2(x-y), \\
\{A^{(n)}_3(x), \Pi^i_{(n)}(y)\}_D &= \frac{n}{R}\partial_i\left(\frac{\delta^2(x-y)}{\nabla^2}\right) \\
\{A^{(n)}_3(x), \Pi^3_{(n)}(y)\}_D &= \delta^2(x-y), \\
\{B^{ij}_{(n)}(x), \Pi^{(n)}_{kl}(y)\}_D &= 0, \\
\{B^{ij}_{(n)}(x), \Pi^k_{(n)}(y)\}_D &= 2\left(\delta^k{}_j\partial_i - \delta^k{}_i\partial_j\right)\delta^2(x-y), \\
\{B^{ij}_{(n)}(x), B^{0k}_{(n)}(x)\}_D &= -2\left(\delta^k{}_j\partial_i - \delta^k{}_i\partial_j\right)\delta^2(x-y), \\
\{B^{i3}_{(n)}(x), B^{03}_{(n)}(x)\}_D &= -\partial_i\delta^2(x-y), \\
\{B^{i3}_{(n)}(x), \Pi^j_{(n)}(y)\}_D &= \frac{n}{R}\delta^i{}_j\delta^2(x-y), \\
\{B^{i3}_{(n)}(x), \Pi^3_{(n)}(y)\}_D &= \partial_i\delta^2(x-y), \\
\{A^{(n)}_i(x), B^{0j}_{(n)}(x)\}_D &= -\left(\delta^j{}_i - \frac{\partial_j\partial_i}{\nabla^2}\right)\delta^2(x-y), \\
\{A^{(n)}_3(x), B^{03}_{(n)}(x)\}_D &= \delta^2(x-y).
\end{aligned} \tag{B.8}
$$

## Acknowledgments

This work was supported by CONACyT under Grant No. CB-2014-01/ 240781. We would like to thank R. Cartas-Fuentevilla for discussion on the subject and reading of the manuscript.

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