The geometric algebras of space and spacetime are derived by successively extending the real number system to include new mutually anticommuting square roots of $\pm 1$. The quantum mechanics of spin $1/2$ particles are then expressed in these geometric algebras. Classical $2$ and $4$ component spinors are represented by geometric numbers which have parity, providing new insight into the familiar bra-ket formalism of Dirac. The classical Dirac Equation is shown to be equivalent to the Dirac-Hestenes equation, so long as the issue of parity is not taken into consideration, the latter quantity being constructed in such a way that it is parity invariant.

Keywords: Bra-ket formalism; geometric algebra; spacetime algebra; Schrödinger-Pauli equation; Dirac equation; Dirac-Hestenes equation; spinor; spinor operator.

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1. Geometric Concept of Number

In his book “Number: The Language of Science” Tobias Dantzig, in describing the invention of matrices has this to say:

(It is) a theory in which a whole array of elements is regarded as a number-individual. These “filing cabinets” are added and multiplied, and a whole calculus of matrices has been established which may be regarded as a continuation of the algebra of complex numbers. These abstract beings have lately found a remarkable interpretation in the quantum theory of the atom, and in man’s other scientific fields, [2].

We now show how these “filing cabinets” take on the interpretation of geometric numbers in a geometric number system called geometric algebra. It was William Kingdon Clifford (1845-1879), himself, who first referred to his famous algebras as geometric algebras, but some authors still call these algebras Clifford algebras [3]. David Hestenes and other theoretical physicists over the last half-century have shown this comprehensive geometric interpretation imbues the equations of physics with new meaning [4]. We shall restrict our attention to developing geometric algebras of the plane, 3-dimensional space, and Minkowski spacetime and their application to quantum mechanics, but the ideas apply to higher dimensional spaces as well [5].

1.1. Geometric algebra of the plane

We extend the real number system $\mathbb{R}$ to include two new anticommuting square roots $e_1, e_2$ of $+1$, which we identify as unit vectors along the $x$ and $y$ axis of the coordinate plane $\mathbb{R}^2$. Thus,

$$e_1^2 = e_2^2 = 1, \quad \text{and} \quad e_{12} := e_1 e_2 = -e_2 e_1.$$  

We assume that the associative and distributive laws of multiplication of real numbers remain valid in our geometrically extended number system, and give the new quantity $i := e_{12}$ the geometric interpretation of a directed plane segment or bivector. Note that

$$i^2 = e_1 e_2 e_1 e_2 = -e_1^2 e_2^2 = -1$$

so that $i$ has the same algebraic property as the imaginary unit of the complex numbers $\mathbb{C}$. Indeed, every unit bivector of the $n$-dimensional Euclidean space $\mathbb{R}^n$ shares this property and is the generator of rotations in the vector plane of that bivector.

The Euler identity

$$e^{i\theta} = \cos \theta + i \sin \theta$$

for $\theta \in \mathbb{R}$ and the unit bivector $i = e_{12}$, and the hyperbolic Euler identity

$$e^{e_1 \phi} = \cosh \phi + e_1 \sinh \phi$$

for $\phi \in \mathbb{R}$ and the unit vector $e_1$, are easily established as special cases of the general algebraic definition of the exponential function

$$e^X \equiv \sum_{n=0}^{\infty} \frac{X^n}{n!} = \cosh X + \sinh X.$$  

In [5, Chp. 2] and [6], the author shows that the hyperbolic number plane, regarded as the extension of the real number system to include a new square root $e_1 := \sqrt{-1}$ of $+1$, has much in common with the more famous complex number plane. Since it took centuries for the complex numbers to be recognised as bonified numbers, I suppose that it is not surprising that the sister hyperbolic numbers are discriminated against even to this day. After all, it was Leopold Kronecker (1823-1891) who said, “God made the integers, all the rest is the work of man”.

The standard basis over the real numbers of the geometric algebra $\mathbb{G}_2 := \mathbb{G}(\mathbb{R}^2)$ is

$$\mathbb{G}_2 = \text{span}\{1, e_1, e_2, e_{12}\}.$$  

(1)
With respect to this basis a geometric number $g \in \mathbb{G}_2$ can be written
\[ g = \alpha + v = (a + bi) + (ce_1 + de_2) \tag{2} \]
where $\alpha = a + bi = re^{i\theta}$ behaves formally like a complex number and $v = ce_1 + de_2 \in \mathbb{R}^2$ is a vector in the Euclidean plane of the bivector $i = e_{12}$. It can also be easily checked that $v\alpha = \overline{v}\alpha$, where $\overline{\alpha} = a - \beta i$ is the (complex) conjugate of $\alpha$. It is this property that gives $\alpha$ the geometric significance as the generators of rotations in the plane of $e_{12}$, [5, p.55].

Another basis of $\mathbb{G}_2$, the spectral basis, gives the algebra of real $2 \times 2$ matrices the interpretation of representing geometric numbers in $\mathbb{G}_2$. We write
\[ \mathbb{G}_2 = \text{span}\{ \frac{1}{e_1}u_+ (1 \cdot e_1) \} \]
\[ = \text{span}\{ u_+ e_1u_+ e_1u_- \} \tag{3} \]
or $\mathbb{G}_2 = \text{span}\{ u_+, e_1u_+, e_1u_-, u_- \}$, where the idempotents $u_+ := (1/2)(1 \pm e_2)$. The relationship between the standard basis (1) and the spectral basis (3) is directly expressed by
\[
\begin{pmatrix}
1 \\
0 \\
e_1 \\
e_2
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & -1 \\
o & -1 & 0
\end{pmatrix}
\begin{pmatrix}
u_+ \\
e_1u_+ \\
e_1u_- \\
u_-
\end{pmatrix}
\]
Noting $e_1u_+ = u_e_1$, it follows that
\[ (1 \cdot e_1)u_+(1 \cdot e_1) = u_+ + e_1u_+e_1 = u_+ + u_- = 1. \]
Using this last relationship, and the fact that
\[ u_+^2 = u_+ = e_2u_+, \quad \text{and} \quad u_+ e_1u_+ = 0 = u_+ e_2u_+, \]
with similar relationships for $u_- g u_+, u_+ g u_- \quad \text{and} \quad u_- g u_-$, we calculate for the geometric number $g$ given in (2),
\[ g = (1 \cdot e_1)u_+(1 \cdot e_1)g(1 \cdot e_1)u_+(1 \cdot e_1) \]
\[ = (1 \cdot e_1)u_+(\frac{\alpha + v}{e_1(\alpha + v)} e_1 \cdot (\alpha + v)e_1)u_+(1 \cdot e_1) \]
\[ = (1 \cdot e_1)u_+ (\frac{a + d}{e_1} \cdot c - b c + b \cdot a - d \cdot e_1) \]
\[ = (a + d)u_+ + (c - b)e_1u_- + (c + b)e_1u_+ + (a - d)u_- \tag{5} \]
We say that
\[ [g] = \begin{pmatrix} a + d & c - b \\
c + b & a - d \end{pmatrix} \]
is the matrix of the geometric number
\[ g = (a + be_{12}) + (ce_1 + de_2) \in \mathbb{G}_2 \]
with respect to the spectral basis (3). From (4) it follows that
\[ g = (1 \cdot e_1)u_+[g] \begin{pmatrix} 1 \\
e_1 \end{pmatrix}. \tag{6} \]

The interesting thing about equation (6) is that it can be turned around and solved directly for the matrix of $[g]$ of $g$. To accomplish this we define the $e_1$-conjugate of $g \in \mathbb{G}_2$ by
\[ g^{e_1} := e_1g e_1. \]
Multiplying Eq. (6) on the left and right by
\[ u_+ \begin{pmatrix} 1 \\
e_1 \end{pmatrix} \]
and
\[ \begin{pmatrix} 1 \\
e_1 \end{pmatrix} u_+, \]
respectively, gives
\[ u_+ \begin{pmatrix} 1 \\
e_1 \end{pmatrix} g(1 \cdot e_1)u_+ \]
\[ = u_+ \begin{pmatrix} 1 \\
e_1 \end{pmatrix}u_+[g] \begin{pmatrix} 1 \\
e_1 \end{pmatrix} u_+ = u_+[g]. \]

Adding the last two equations gives the desired result
\[ [g] = u_+ \begin{pmatrix} g \\
e_1 \end{pmatrix} \begin{pmatrix} 1 \\
e_1 \end{pmatrix}u_+ + u_- \begin{pmatrix} g^{e_1} \\
e_1 \end{pmatrix} \begin{pmatrix} 1 \\
e_1 \end{pmatrix} u_- \tag{7} \]
Let
\[ g_k = (1 \cdot e_1)u_+[g_k] \begin{pmatrix} 1 \\
e_1 \end{pmatrix} \]
where $[g_k]$ is the matrix of the geometric number $g_k \in \mathbb{G}_2$ for $k = 1, 2$ as given in (2), (4) and (6). The algebra of real $2 \times 2$ matrices $\text{Mat}_{2\times2}(\mathbb{R})$ is algebraically isomorphic to the geometric algebra $\mathbb{G}_2$ because $[g_1 + g_2] = [g_1] + [g_2]$ and $[g_1g_2] = [g_1][g_2]$ for all $2 \times 2$ matrices $[g_1], [g_2] \in \text{Mat}_{2\times2}(\mathbb{R})$ and their corresponding geometric numbers $g_1, g_2 \in \mathbb{G}_2$. For example, using (6) and $u_+ e_1u_+ = 0$,
\[ g_1g_2 = (1 \cdot e_1)u_+[g_1] \begin{pmatrix} 1 \\
e_1 \end{pmatrix} (1 \cdot e_1)u_+[g_2] \begin{pmatrix} 1 \\
e_1 \end{pmatrix} \]
\[ = (1 \cdot e_1)[g_1] \begin{pmatrix} u_+ e_1u_+ [g_2] [g_2] u_+ \begin{pmatrix} 1 \\
e_1 \end{pmatrix} \]
\[ = (1 \cdot e_1)[g_1][g_2] \begin{pmatrix} 1 \\
e_1 \end{pmatrix} = (1 \cdot e_1)u_+[g_1][g_2] \begin{pmatrix} 1 \\
e_1 \end{pmatrix} \]
which shows that $[g_1g_2] = [g_1][g_2]$.

Extending the real number system $\mathbb{R}$ to include the new anticommuting square roots $e_1, e_2$ of $+1$ is well defined since the resulting geometric number system $\mathbb{G}_3$ is algebraically isomorphic to the $2 \times 2$ matrix algebra $\text{Mat}_{2\times2}(\mathbb{R})$. Indeed, each geometric algebra $\mathbb{G}_{p,q}$ is either algebraically isomorphic to a real or complex square matrix algebra, or matrix subalgebra, of the appropriate dimension. [5, p.205]. Next, we consider the case of the geometric algebra $\mathbb{G}_3$ of space.
1.2. Geometric algebra of space

The geometric algebra $\mathbb{G}_2$ of the $xy$-plane in $\mathbb{R}^2$ can be extended to the geometric algebra $\mathbb{G}_3$ of the space $\mathbb{R}^3$ simply by extending $\mathbb{G}_2$ to include $e_3$, a new square root of $+1$, which anticommutes with both $e_1$ and $e_2$. Naturally, $e_3$ has the geometric interpretation of a unit vector along the $z$-axis. The standard basis of $\mathbb{G}_3$, with respect to the coordinate frame $\{e_1, e_2, e_3\}$ of the Euclidean space $\mathbb{R}^3$, is

$$\mathbb{G}_3 := \mathbb{G}(\mathbb{R}^3) = \text{span}\{1, e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}\}. \quad (8)$$

The elements $e_{12}, e_{13}, e_{23}$ are unit bivectors in the $xy$-, $xz$-, and $yz$-planes and generate rotations in those planes, and $i := e_{123} := e_1e_2e_3$ is the unit trivector or oriented volume element of $\mathbb{R}^3$. The unit trivector $i$ also has the property that $i^2 = -1$, as follows from

$$i^2 = e_1(e_2e_3)e_1(e_2e_3) = e_1^2(e_2e_3)^2 = -1.$$

Any geometric number $g \in \mathbb{G}_3$ can be written in the form

$$g = \sum_{k=0}^{3} \alpha_k e_k \quad (9)$$

where $e_0 := 1$ and $\alpha_k = a_k + ib_k$ for $a_k, b_k \in \mathbb{R}$ and where $0 \leq k \leq 3$. The conjugation known as the reverse $g^\dagger$ of the geometric number $g$, is defined by reversing the orders of all the products of the vectors that make up $g$, giving

$$g^\dagger = \sum_{k=0}^{3} \overline{\alpha}_k e_k. \quad (10)$$

In particular, writing $g = s + v + B + T$, the sum of a real number $s \in \mathbb{G}_3^0$, a vector $v \in \mathbb{G}_3^1$, a bivector $B \in \mathbb{G}_3^2$ and a trivector $T \in \mathbb{G}_3^3$, $g^\dagger = s + v - B - T$ as can be easily checked.

Another conjugation widely used in geometric algebra is the grade inversion, obtained by replacing each vector in a product by its negative. It corresponds to an inversion in the origin, otherwise known as a parity inversion. For the geometric number $g$ given in (9), the grade inversion is

$$g^- := \overline{\alpha}_0 - \sum_{k=1}^{3} \overline{\alpha}_k e_k. \quad (11)$$

When $g$ is written in the form $g = s + v + B + T$, the grade inversion takes the form $g^- = s - v + B - T$.

Alternatively, we can obtain the geometric algebra $\mathbb{G}_3$ by extending or complexifying $\mathbb{G}_2$ to include a commuting imaginary $i$, which we subsequently identify as the unit trivector. We write

$$\mathbb{G}_3 := \mathbb{G}_2(i) := \text{span}\{1, e_1, e_2, e_{21}i, e_{12}, -e_2i, e_1i, i\}, \quad (12)$$

and then define $e_3 := e_{21}i$, $e_{13} = -e_2i$ and $e_{23} = e_1i$.

Since the unit trivector $i$ commutes with all the elements of $\mathbb{G}_3$, we can simply complexify the spectral basis (3) of $\mathbb{G}_2$ to obtain the spectral basis of $\mathbb{G}_3$,

$$\mathbb{G}_3 = \text{span}\{1, e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}\}$$

$$= \text{span}\{u_+, e_1u_+, e_2u_+, e_3u_+\}, \quad (13)$$

where in this case $u_+ := (1/2)(1 + e_3)$. We can then directly apply the analogous formulas (4), (5), (6), and (7) to elements $g \in \mathbb{G}_3$ merely by allowing the values of $a, b, c, d$ to be of the form $s + it$ where $s, t \in \mathbb{R}$ and $i = e_{123}$. The famous Pauli matrices of the coordinate frame $\{e_1, e_2, e_3\}$ are simply obtained using the spectral basis (13), with $u_+ = (1/2)(1 \pm e_3)$, getting

$$[e_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad [e_2] = \begin{pmatrix} 1 & -i \\ i & 0 \end{pmatrix}$$

$$[e_3] = -i[e_1][e_2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (14)$$

In this representation, the imaginary unit $i$ acquires the geometric interpretation of the oriented volume element in $\mathbb{G}_3$, as follows from

$$[e_{123}] = [e_1][e_2][e_3] = i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (15)$$

In terms of the geometric product, all other products in the geometric algebra are defined. For example, given vectors $a, b \in \mathbb{G}_3^1 \equiv \mathbb{R}^3$,

$$ab = a \cdot b + a \wedge b \in \mathbb{G}_3^{0+2}, \quad (16)$$

where $a \cdot b := (1/2)(ab + ba) \in \mathbb{R}$ is the symmetric inner product and $a \wedge b := (1/2)(ab - ba)$ is the antisymmetric outer product of the vectors $a$ and $b$, respectively. The outer product satisfies $a \wedge b = i(a \times b)$, expressing the duality relationship between the standard Gibbs-Heaviside cross product $a \times b$ and the outer product $a \wedge b \in \mathbb{G}_3^2$. In what follows the outer product $a \wedge b \wedge c$ of three vectors is also used. Similar to (16), we write

$$a(b \wedge c) = a \cdot (b \wedge c) + a \wedge (b \wedge c), \quad (17)$$

where

$$a \cdot (b \wedge c) := \frac{1}{2}(a(b \wedge c) - (b \wedge c)a)$$

$$= -a \times (b \times c) \in \mathbb{G}_3^1,$$

and

$$a \wedge (b \wedge c) := \frac{1}{2}(a(b \wedge c) + (b \wedge c)a)$$

$$= a \cdot (b \times c)i \in \mathbb{G}_3^2.$$
A much more detailed treatment of \( G_3 \) is given in [5,Chp.3], and in Ref. 7 I explore the close relationship that exists between geometric algebras and their matrix counterparts. Geometric algebra is developed in Ref. 5 as the geometric conclusion of the real number system, providing a new foundation for much of mathematics and physics.

1.3. Properties of idempotents and nilpotents

An idempotent \( p \in G_3 \) has the defining property \( p^2 = p \). Of course \( \pm 1,0 \) are idempotents, and so is \( p = (1/2)(1 + \sqrt{2}e_1 + i e_2) \). For a general geometric number \( g = \omega + \mathbf{m} + i \mathbf{n} \in G_3 \) to be an idempotent, we must have

\[
g^2 = \omega^2 + \mathbf{m}^2 - \mathbf{n}^2 + 2i \mathbf{m} \cdot \mathbf{n} = \omega + \mathbf{m} + i \mathbf{n},
\]

or equivalently,

\[
\omega^2 + \mathbf{m}^2 - \mathbf{n}^2 + 2i \mathbf{m} \cdot \mathbf{n} = \omega
\]

for \( \mathbf{m}, \mathbf{n} \in G_3^1 \). It follows immediately that any idempotent \( A \) has the important canonical form

\[
A = \frac{1}{2}(1 + \mathbf{m} + i \mathbf{n}), \quad \text{where}
\]

\[
\mathbf{m}^2 - \mathbf{n}^2 = 1, \quad \text{and} \quad \mathbf{m} \cdot \mathbf{n} = 0,
\]

where \( \mathbf{m}, \mathbf{n} \in G_3^1 \). Spinors have the defining property \( (\alpha) = \text{real scalar part of the enclosed expression} \)

\[
(\alpha) = (\beta) + (\gamma) \Rightarrow (\alpha) = (\beta) + (\gamma)
\]

for all \( g_1, g_2 \in G_3 \) is the operation of reverse defined in (10).

Another important property that easily follows is

\[
\hat{b}_+ \mathbf{m} \hat{b}_+ = \frac{1}{2} \hat{b}_+ \mathbf{m} (1 + \hat{b})
\]

for \( \mathbf{m}, \mathbf{n} \in G_3^1 \). It follows immediately that any idempotent \( A \) has the important canonical form

\[
A = \frac{1}{2}(1 + \mathbf{m} + i \mathbf{n}) = \mathbf{m} \cdot (1 + \mathbf{m}^{-1} + i \mathbf{m}^{-1} \mathbf{n})
\]

where \( \mathbf{m} \cdot \mathbf{n} = 0 \) and \( \mathbf{m} \cdot \mathbf{m} = 1 \). We use the notation \( \mathbf{m} \cdot \mathbf{n} = 0 \) and \( \mathbf{m} \cdot \mathbf{m} = 1 \) for the mutually annihilating idempotents \( \hat{a}_+ \) and \( \hat{a}_- \).

For the idempotents \( \hat{a}_+ \), note that

\[
\hat{a}_+ \hat{a}_+ = \hat{a}_+ \quad \text{and} \quad \hat{a}_\hat{a}_- = -\hat{a}_-
\]

Because of these properties, we would like to identify \( \hat{a}_+ \) as the two pure eigenprojector spin states of an electron. Unfortunately, as is discussed in the next section, this is not quite correct.

Given a second idempotent \( \hat{b}_+ \), we calculate

\[
2(\hat{a}_+ \hat{b}_+)_0 = \frac{1}{2}((1 + \hat{a})(1 + \hat{b}))_0
\]

\[
= \frac{1}{2}(1 + \hat{a} \cdot \hat{b}),
\]

where \( (\cdot)_0 \) denotes real scalar part of the enclosed expression, and using \( \mathbf{a} \cdot \mathbf{b} = (\hat{a} \hat{b})_0 \) that we found in (16). A closely related calculation gives almost the same answer

\[
(\hat{a}_+ \hat{b}_+)(\hat{a}_+ \hat{b}_+) = (\hat{b}_+ \hat{a}_+)(\hat{a}_+ \hat{b}_+) = \hat{b}_+ \hat{a}_+ \hat{b}_+
\]

\[
= \frac{1}{4}(1 + \hat{a})(1 + \hat{b}) = \frac{1}{4}(1 + \hat{a} + \hat{b} + \hat{a} \hat{b})
\]

\[
= \frac{1}{4}(1 + \hat{a} + \hat{b} - \hat{a} \hat{b} + 2\hat{a} \cdot \hat{b}) = \frac{1}{2}(1 + \hat{a} \cdot \hat{b}) \hat{b}_+,
\]

but multiplied by the idempotent \( \hat{b}_+ \). In summary,

\[
(\hat{a}_+ \hat{b}_+)(\hat{a}_+ \hat{b}_+) \hat{b}_+ = \hat{b}_+ \hat{a}_+ \hat{b}_+ = \text{Prob}(\hat{a}_+ \hat{b}_+) \hat{b}_+ = \text{Prob}(\hat{a}_+ \hat{b}_+) \hat{b}_+ = \text{Prob}(\hat{a}_+ \hat{b}_+) \hat{b}_+ = \text{Prob}(\hat{a}_+ \hat{b}_+) \hat{b}_+
\]

\[
= \frac{1}{2}(\mathbf{m} - \mathbf{b} m + 2\mathbf{b} \cdot \hat{a}) = (\mathbf{m} \cdot \mathbf{n} \cdot \mathbf{b}_+).
\]

2. Quantum mechanics of spin

The observable corresponding to the spin of an electron in a coordinate frame \( \{e_1, e_2, e_3\} \) is a unit vector \( \mathbf{m} = m_1 e_1 + m_2 e_2 + m_3 e_3 \) in the Euclidean space \( \mathbb{R}^3 \) of that frame. The Hermitian matrix of the observable \( \mathbf{m} \) is

\[
[\mathbf{m}] = \begin{pmatrix}
m_3 & m_1 - im_2 \\
m_1 + im_2 & -m_3
\end{pmatrix} = \begin{pmatrix}
m_3 & -m_1 \\
-m_1 & -m_3
\end{pmatrix},
\]

as given in (13) and (14), with \( u_\pm = (1/2)(1 \pm e_3) \) and \( m_\pm = m_1 \pm im_2 \), where

\[
\hat{m} = (1 e_1) u_+ [\mathbf{m}] \begin{pmatrix}
1 \\
e_1
\end{pmatrix}
\]

\[
= m_3 u_+ + m_- e_1 u_- + m_+ e_1 u_+ - m_3 u_-
\]

(25)

By a geometric spinor \( |\alpha\rangle \) with respect to \( u_+ = (1/2)(1 + e_3) \), we mean

\[
|\alpha\rangle := \sqrt{2} (\alpha_0 + \alpha_1 e_1) u_+ \Leftrightarrow |\alpha\rangle
\]

\[
:= |\alpha\rangle = \sqrt{2} u_+ (\alpha_0 + \alpha_1 e_1)
\]

(26)

where \( \alpha_0, \alpha_1 \in \mathbb{C}^3 \). The spinor \( |\alpha\rangle \) is said to be non-degenerate if in addition \( \alpha_0 \alpha_0 + \alpha_1 \alpha_1 \neq 0 \). Spinors have
the important superposition property that if $|\alpha\rangle$ and $|\beta\rangle$ are spinors with respect to $u_+$ then
\[|\omega\rangle := \lambda_1|\alpha\rangle + \lambda_2|\beta\rangle \quad \text{for} \quad \lambda_1, \lambda_2 \in \mathbb{G}_3^{0+3} \tag{27}\]
is also a spinor with respect to $u_+$. This is precisely the property that the spinor equation (40) is satisfied by all of the above quantities can be expressed in terms of $\alpha\rangle$. By taking the reverse of the spinor equation (40), we get
\[|\alpha\rangle = \sqrt{2\alpha_0}A = \sqrt{2\alpha_0}u_+ = \sqrt{2\alpha_0}2u_+ \tag{33}\]
for $A = (1 + (\alpha_1/\alpha_0)e_1)u_+$. It is easily checked that $A$ is an idempotent, so that $A$ can be expressed in the canonical form (20),
\[A = \frac{1}{2} \left( 1 + \frac{\alpha_1}{\alpha_0} e_1 - i \frac{\alpha_1}{\alpha_0} e_2 + e_3 \right) = \frac{1}{2} (1 + \mathbf{m} + i \mathbf{n}) = \mathbf{m} \frac{1}{2} (1 + \hat{a}) = \mathbf{m}_{u_+}, \tag{34}\]
since $\hat{a} = (\mathbf{m} + i \mathbf{m}/m^2) = e_3$. Another closely related canonical form for the idempotent $A$, with the help of (23), is
\[A = \mathbf{m}_{u_+} = \mathbf{m}_{u_+} \mathbf{m}_{u_+} \mathbf{m}_{u_+} = \mathbf{m}^2 \hat{a}_{u_+}. \tag{35}\]
where we have redefined the unit vector $\hat{a} := \mathbf{m}_e\mathbf{m}$.

All of the above quantities can be expressed in terms of $\alpha_0$ and $\alpha_1$. We find that
\[\mathbf{m} = \frac{1}{2\alpha_0\alpha_0} \left( (\alpha_1 \bar{\alpha}_0 + \alpha_0 \bar{\alpha}_1) e_1 + i(\alpha_0 \bar{\alpha}_1 - \alpha_1 \bar{\alpha}_0) e_2 \right) + e_3, \tag{36}\]
\[\mathbf{n} = \frac{1}{2\alpha_0\alpha_0} \left( i(\alpha_0 \bar{\alpha}_1 - \alpha_1 \bar{\alpha}_0) e_1 - (\alpha_0 \bar{\alpha}_1 + \alpha_1 \bar{\alpha}_0) e_2 \right), \tag{37}\]
and the unit vector $\hat{a} = \mathbf{m}_e\mathbf{m} = a_1e_1 + a_2e_2 + a_3e_3$ is specified by
\[a_1 = \frac{\alpha_0 \alpha_1 + \alpha_0 \bar{\alpha}_1}{\alpha_0 \bar{\alpha}_0 + \alpha_1 \bar{\alpha}_1}, \quad a_2 = i(\alpha_0 \bar{\alpha}_1 - \alpha_1 \bar{\alpha}_0)/\alpha_0 \bar{\alpha}_0 + \alpha_1 \bar{\alpha}_1, \quad a_3 = \frac{\alpha_0 \bar{\alpha}_0 - \alpha_1 \bar{\alpha}_1}{\alpha_0 \bar{\alpha}_0 + \alpha_1 \bar{\alpha}_1}. \tag{38}\]

For a normalized spinor $|\alpha\rangle$, the spincor $|\hat{a}_+\rangle$ takes the canonical form
\[|\alpha\rangle = \sqrt{2\alpha_0}u_+ = \sqrt{2\alpha_0}2u_+ \tag{40}\]
The matrix $[\mathbf{a}_{u_+}]$ of $\hat{a}_{u_+}$ is the density matrix of the pure state $|\alpha\rangle$. By taking the reverse of the spinor equation (40), we get
\[|\alpha\rangle = \sqrt{2\alpha_0}e_1 + \sqrt{2\alpha_0}e_2 + \sqrt{2\alpha_0}e_3 \tag{41}\]
and find that
\[\frac{1}{2} |\alpha\rangle \langle\alpha| = \alpha_0 \mathbf{m}_{u_+}^2 \mathbf{m}_e \mathbf{m} = \hat{a}_{u_+}. \tag{42}\]

It is interesting to note that a Cartan spinor is specified by
\[|\alpha\rangle e_1 |\alpha\rangle^* = |\alpha\rangle e_1 |\alpha\rangle - \frac{1}{\alpha_0} \mathbf{m}(e_1 + i e_2)\mathbf{m} = \sqrt{2\alpha_0} e_3 \hat{a}_{u_+} e_3 u_+ \tag{43}\]
and represents a null complex vector, [8,p.41].

There are several other operations on spinors which are important. By multiplying the spinor equation (40) on the left and right by $e_3$, we find that
\[|\alpha\rangle e_3 := e_3 |\alpha\rangle e_3 = \sqrt{2\alpha_0} e_3 \mathbf{m} e_3 u_+ \equiv \sqrt{2\alpha_0} m^2 e_3 \hat{a}_{u_+} e_3 u_+. \tag{43}\]

Multiplying the spinor equation (40) on the left and right by $e_1$, we get
\[|\alpha\rangle e_1 := e_1 |\alpha\rangle e_1 = \sqrt{2\alpha_0} e_1 \mathbf{m} e_1 u_+ \equiv \sqrt{2\alpha_0} m^2 e_1 \hat{a}_{u_+} e_1 u_. \tag{44}\]

The spinor $|\alpha\rangle e_3$ is called the $e_3$-conjugate of $|\alpha\rangle$, and the spinor $|\alpha\rangle e_1$ is called the $e_1$-conjugate of $|\alpha\rangle$. Using (11) and (40) the grade inversion $|\alpha\rangle^-$ of the spinor $|\alpha\rangle \in \mathbb{G}_3$ is
\[|\alpha\rangle^- := \sqrt{2} \left( |\alpha\rangle + |\alpha\rangle e_1 \right)^{-} u_+ = \sqrt{2} \left( \bar{\alpha}_0 - \bar{\alpha}_1 e_1 \right) u_+ \tag{45}\]
Given normalized spinors \(|\alpha\rangle, |\beta\rangle\), we calculate with the help of (40) and (23),
\[
\langle\alpha|\beta\rangle = 2\pi\beta_0u_+ \mathbf{m}u_+ = 2\pi\beta_0\left(\mathbf{m} \cdot \mathbf{n} + i(\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}_3\right)u_+ = 0
\]
iff \(\mathbf{m} \cdot \mathbf{n} = 0\) and \((\mathbf{m} \times \mathbf{n}) \cdot \mathbf{e}_3 = 0\). Expressed in terms of the associated spinor directions \(\mathbf{m}, \mathbf{n}\), the inner product \(\langle\alpha|\beta\rangle\) is quite different than the component expression (29). A great simplification occurs when we calculate \(\langle\alpha|\beta\rangle|\alpha\rangle\), finding with the help of (30) and (39),
\[
\frac{\langle\alpha|\beta\rangle \langle\alpha|\beta\rangle}{\langle\alpha|\beta\rangle} = \frac{1}{2}\left\{\langle\alpha|\beta\rangle\right\}^{\dagger}\left\{\langle\alpha|\beta\rangle\right\}_{0+3} = \frac{1}{2}\langle\beta|\beta\rangle\langle\alpha|\beta\rangle_{0+3} = \frac{1}{2}(1 + \hat{\alpha} \cdot \hat{b}).
\]

Given an observable \(\hat{a}\), suppose that an electron has been prepared in the pure spin state \(|\alpha\rangle\). Then the probability of finding the electron in the pure spin state \(|\beta\rangle\) is
\[
Prob(|\alpha\rangle, |\beta\rangle) := \langle\alpha|\beta\rangle|\alpha\rangle = \frac{1}{2}(1 + \hat{\alpha} \cdot \hat{b}) = 2(\hat{\alpha}_+ \hat{b}_+)_{0},
\]
just as we have already seen in (21) and (22). Immediately after the measurement is carried out, the electron will be in the pure state \(|\beta\rangle\). Identity (46) is independent of the basis spinor \(u_+\) that was used in the definition (26) of the spinors \(|\alpha\rangle\) and \(|\beta\rangle\). Clearly, the spinor states \(|\alpha\rangle\) and \(|\beta\rangle\) are orthogonal iff \(\hat{a} = -\hat{b}\).

Given a spinor state \(|\alpha\rangle\), the average value of the observable \(\hat{b}\) in that state is defined by \(\langle\alpha|\hat{b}|\alpha\rangle\). Using (22), (23), (39), (40), we calculate
\[
\langle\alpha|\hat{b}|\alpha\rangle = \sqrt{2} \alpha_0 \hat{b} \mathbf{m} u_+ = \sqrt{2} \alpha_0 (\hat{b} \cdot \mathbf{m} + \mathbf{b} \wedge \mathbf{m}) u_+,
\]
and
\[
\langle\alpha|\hat{b}|\alpha\rangle = 2\langle u_+ (\hat{m} \mathbf{m}) u_+\rangle_{0+3} = (\hat{m} \mathbf{m}) \cdot \mathbf{e}_3 = \hat{b} \cdot (\mathbf{e}_3 \mathbf{m}) = \hat{b} \cdot \hat{a}.
\]

Using (48) and (49), we also calculate
\[
\langle0|\hat{b}|0\rangle = b_3, \quad \text{and} \quad \langle1|\hat{b}|1\rangle = -b_3,
\]
and
\[
\langle0|\hat{b}|0\rangle = b_+ = b_1 + ib_2, \quad \text{and} \quad \langle1|\hat{b}|1\rangle = b_- = b_1 - ib_2.
\]

Often an electron in the spinor state \(|0\rangle = \sqrt{2}u_+\) is said to be up, and an electron in the spinor state \(|1\rangle = \sqrt{2}e_1 u_+\) is said to be down. Using (50), the average value of the observable \(\hat{b}\) in the state \(|0\rangle = b_3\), and the average value of \(\hat{b}\) in the state \(|1\rangle\) is \(-b_3\).

We can also carry out the above calculation using spinor operators. Following [10,p.63], given the spinor \(|\alpha\rangle\), the spinor operator \(\psi\) of \(|\alpha\rangle\) is defined by
\[
\psi := \frac{1}{\sqrt{2}} \left( |\alpha\rangle + |\alpha\rangle^\dagger \right) = m(\alpha_0 u_+ - \overline{\alpha}_0 u_-) \in SU(2),
\]
where the inversion \(|\alpha\rangle^\dagger\) has been defined in (45). The spinor operator is an even multivector in \(\mathbb{G}_3\) and hence invariant under a parity swap. It retains the normalization property that \(\psi \psi^\dagger = \det |\psi| = 1\), and allows us to calculate the associated direction \(\hat{a}\) of \(|\alpha\rangle\) as a rotation of the unit vector \(e_3\). With the help of (22), (38) and (52), we find that
\[
\psi e_3 \psi^\dagger = m(\alpha_0 u_+ - \overline{\alpha}_0 u_-) e_3 (\overline{\alpha}_0 u_+ - \alpha_0 u_-) m = m(\alpha_0 u_+ - \overline{\alpha}_0 u_-) (\overline{\alpha}_0 u_+ - \alpha_0 u_-) = \alpha_0 \overline{\alpha}_0 m^2 \hat{m}(u_+ - u_-) = \hat{a}.
\]

The relationship (54) should be compared with (42).

### 3. Geometric algebra of spacetime

The geometric algebra \(\mathbb{G}_3\) of the coordinate frame \(\{e_1, e_2, e_3\}\) in \(\mathbb{R}^3\) can be extended to the geometric algebra \(\mathbb{G}_{1,3}\) of the pseudoeuclidean space \(\mathbb{R}^{1,3}\) of Minkowski spacetime, by extending \(\mathbb{G}_3\) by an additional square root \(\gamma_0\) of +1 which anticommutes with each of the space vectors \(e_1, e_2, e_3\). So \(\gamma_0\) satisfies \(\gamma_0^2 = 1\) and \(\gamma_0 e_k = -e_k \gamma_0\) for \(k = 1, 2, 3\). Equivalently, the spacelike vectors
\[
\gamma_k := e_k \gamma_0 \iff e_k = \gamma_k \gamma_0 = \gamma_0 \gamma_k,
\]
together with the timelike vector \(\gamma_0\), define the spacetime algebra
\[
\mathbb{G}_{1,3} = \text{gen}\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}
\]
of the orthonormal spacetime frame \( \{ \gamma_\mu \mid 0 \leq \mu \leq 3 \} \) in \( \mathbb{R}^{1,3} \). In summary, the spacetime vectors obey the rules

\[
\gamma_0^2 = 1, \quad \gamma_k^2 = -1, \quad \gamma_\mu \gamma_\nu = -\gamma_\nu \gamma_\mu
\]

for \( \mu \neq \nu, \mu, \nu = 0, 1, 2, 3 \), and \( k = 1, 2, 3 \). Note also that pseudoscalar

\[
\gamma_{0123} = \gamma_{10}\gamma_{20}\gamma_{30} = e_{123}
\]

of \( \mathbb{G}_{1,3} \) is the same as the pseudoscalar of the rest frame \( \{ e_1, e_2, e_3 \} \) of \( \mathbb{G}_3 \), and it anticommutes with each of the spacetime vectors \( \gamma_\mu \) for \( \mu = 0, 1, 2, 3 \).

The forgoing shows that the geometric algebra \( \mathbb{G}_3 \) is naturally isomorphic to the even subalgebra \( \mathbb{G}^+_{1,3} \) of the spacetime algebra \( \mathbb{G}_{1,3} \) generated by the bivectors \( \gamma_\mu \gamma_\nu \) of \( \mathbb{G}_{1,3} \), [9]. Any element \( g \in \mathbb{G}_{1,3} \) can be expressed in the form \( g = G_1 + G_2 \gamma_0 \) where \( G_1, G_2 \in \mathbb{G}_3 \). In particular, a unit vector \( \mathbf{m} = m_1 e_1 + m_2 e_2 + m_3 e_3 \) in the geometric algebra \( \mathbb{G}_3 \) of the rest frame \( \{ e_1, e_2, e_3 \} \) becomes the unit spacetime bivector

\[
\mathbf{m} = (m_1 \gamma_1 + m_2 \gamma_2 + m_3 \gamma_3)\gamma_0 \in \mathbb{G}_{1,3},
\]

with the spacelike vector \( m_1 \gamma_1 + m_2 \gamma_2 + m_3 \gamma_3 \in \mathbb{G}_{1,3} \) as a factor.

In the above, we have carefully distinguished the rest frame \( \{ e_1, e_2, e_3 \} \) in the spacetime algebra \( \mathbb{G}_{1,3} \). Any other rest frame \( \{ e'_1, e'_2, e'_3 \} \) can be obtained by an ordinary space rotation of the rest frame \( \{ e_1, e_2, e_3 \} \) followed by a Lorentz boost. In the spacetime algebra \( \mathbb{G}_{1,3} \), this is equivalent to defining a new frame of spacetime vectors \( \{ \gamma'_\mu \mid 0 \leq \mu \leq 3 \} \subset \mathbb{G}_{1,3} \), and the corresponding rest frame \( \{ e'_k = \gamma'_k \gamma_0 \mid k = 1, 2, 3 \} \) of a Euclidean space \( \mathbb{R}^3 \) moving with respect to the Euclidean space \( \mathbb{R}^3 \) defined by the rest frame \( \{ e_1, e_2, e_3 \} \). The way we introduced the geometric algebras \( \mathbb{G}_3 \) and \( \mathbb{G}_{1,3} \) may appear novel, but they perfectly reflect all the common relativistic concepts [5,Chp.11].

The well-known Dirac matrices can be obtained as a subalgebra of the 4\( \times \)4 matrix algebra \( \text{Mat}_C(4) \) over the complex numbers. We first define the idempotent

\[
\gamma_{12} := \frac{1}{4} (1 + \gamma_0)(1 + i\gamma_{12})
\]

\[
= \frac{1}{4} (1 + i\gamma_{12})(1 + \gamma_0),
\]

where the unit imaginary \( i = \sqrt{-1} \) is assumed to commute with all elements of \( \mathbb{G}_{1,3} \). Whereas it would be nice to identify this unit imaginary \( i \) with the pseudoscalar element \( \gamma_{0123} = e_{123} \) as we did in \( \mathbb{G}_3 \), this is no longer possible since \( \gamma_{0123} \) anticommutes with the spacetime vectors \( \gamma_\mu \) as previously mentioned.

Noting that

\[
\gamma_{12} = \gamma_1 \gamma_0 \gamma_0 \gamma_2 = e_2 e_1 = e_{21}
\]

and similarly \( \gamma_{31} = e_{13} \), it follows that

\[
e_{13}u_{++} = u_{++} e_{13}, \quad e_3 u_{++} = u_{--} e_3, \quad e_1 u_{++} = u_{--} e_1,
\]

where

\[
u_{++} := \frac{1}{4} (1 + \gamma_0)(1 - i\gamma_{12}),
\]

\[
u_{--} := \frac{1}{4} (1 - \gamma_0)(1 + i\gamma_{12}),
\]

\[
u_{+-} := \frac{1}{4} (1 - \gamma_0)(1 - i\gamma_{12}).
\]

The idempotents \( u_{++}, u_{--}, u_{+-}, u_{-+} \) are mutually annihilating in the sense that the product of any two of them is zero, and partition unity

\[
u_{++} + \nu_{--} + \nu_{+-} + \nu_{-+} = 1.
\]

By the spectral basis of the Dirac algebra \( \mathbb{G}_{1,3} \) we mean the elements of the matrix

\[
\left( \begin{array}{c}
1 \\
e_{13} \\
e_3 \\
e_1
\end{array} \right) u_{++} (1 - e_{13} e_3 e_1)
\]

\[
= \left( \begin{array}{cccc}
u_{++} & -e_{13} u_{+-} & e_3 u_{++} & e_1 u_{--}
\end{array} \right).
\]

Any geometric number \( g \in \mathbb{G}_{1,3} \) can be written in the form

\[
g = (1 e_{13} e_3 e_1) u_{++} |g|
\]

where \( |g| \) is the Dirac matrix corresponding to the geometric number \( g \). In particular,

\[
|\gamma_0| = \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{array} \right),
\]

\[
|\gamma_1| = \left( \begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{array} \right),
\]

and

\[
|\gamma_2| = \left( \begin{array}{cccc}
0 & 0 & 0 & i \\
0 & 0 & -i & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right),
\]

\[
|\gamma_3| = \left( \begin{array}{cccc}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{array} \right).
\]
It is interesting to see what the representation is of the basis vectors of $G_3$. We find that for $k = 1, 2, 3$,
\[
[e_k]_4 = [\gamma_k][\gamma_0] = \begin{pmatrix} 0 & e_k \\ e_k & 0 \end{pmatrix} \quad \text{and} \quad [e_{123}]_4 = i \begin{pmatrix} 0 & e_k \\ e_k & 0 \end{pmatrix},
\]
where the outer subscripts denote the order of the matrices and, in particular, $[0]_2$, $[1]_2$ are the $2 \times 2$ zero and unit matrices, respectively. The last relationship shows that the $i$ occurring in the Pauli matrix representation (15), which represents the oriented unit of volume, is different than the $i = \sqrt{-1}$ which occurs in the complex matrix representation (60) of the of Dirac algebra.

A Dirac spinor is a 4-component column matrix $[\varphi]_4$.

\[
[\varphi]_4 := \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} \quad \text{for} \quad \varphi_k \in \mathbb{C}. \quad (61)
\]

Following [10,p.143], we use the Dirac spinor to construct the (matrix) spinor operator

\[
[\psi] = \begin{pmatrix} \psi_1 & -\overline{\psi}_2 & \psi_3 & -\overline{\psi}_4 \\ \psi_2 & \overline{\psi}_1 & -\overline{\psi}_3 & \psi_4 \\ \psi_3 & \overline{\psi}_4 & \psi_1 & -\overline{\psi}_2 \\ \psi_4 & \overline{\psi}_3 & -\overline{\psi}_2 & \psi_1 \end{pmatrix}. \quad (62)
\]

Unlike the Dirac spinor $[\varphi]_4$, the spinor operator $[\psi]$ is invertible iff $\det[\psi] \neq 0$. We find that

\[
\det[\psi] = r^2 + 4a^2 \geq 0, \quad (63)
\]

where

\[
r = |\varphi_1|^2 + |\varphi_2|^2 - |\varphi_3|^2 - |\varphi_4|^2 \quad \text{and} \quad a = \Im(\overline{\psi}_1\varphi_3 + \overline{\psi}_2\varphi_4).
\]

Where-as the spinor operator $[\psi]$ obviously contains the same information as the Dirac spinor $[\varphi]_4$, we will shortly see that it acquires the geometric interpretation of an even multivector in $G^+_3$.

Noting that

\[
u_{++} = \frac{1}{4}(1 + \gamma_0)(\gamma_{21} + i\gamma_{12}\gamma_{21}) = \frac{1}{2}(1 + \gamma_0)(i - \gamma_{12}) = iu_{++} = \gamma_{21}u_{++},
\]

it follows that $u_{++}\varphi_k = u_{++}\Re(\varphi_k) + u_{++}i\gamma_{12}\Im(\varphi_k)i = u_{++}\Re(\varphi_k) + 2\Im(\varphi_k)i$ and hence

\[
u_{++}[\psi] = [\psi],
\]

where each of the elements $\alpha_k$ in $[\psi]_4$ is defined by $\alpha_k = \varphi_k|_{i\rightarrow\gamma_{21}}$. Using (59), the geometric number $\psi \in G^+_{1,3}$ corresponding to the matrix spinor operator $[\psi]$ is

\[
\psi = (1 \ e_{13} \ e_3 \ e_1)u_{++}[\psi]_\alpha \begin{pmatrix} 1 \\ -e_{13} \\ e_3 \\ e_1 \end{pmatrix}
\]

\[
= \alpha_1 + \alpha_2 e_{13} + \alpha_3 e_3 + \alpha_4 e_1 \in G^+_{1,3}. \quad (64)
\]

To establish the last equality above, with the help of (57), note that

\[
\psi = \psi u_{++} + \psi u_{+-} + \psi u_{-+} + \psi u_{--},
\]

and then successively show that

\[
\psi u_{++} = (\alpha_1 + \alpha_2 e_{13} + \alpha_3 e_3 + \alpha_4 e_1)u_{++},
\]

for each of different sign combinations $++, ++$, $-, -$, $-, +$. As an even geometric number in $G^+_{1,3}$, $\psi$ generates Lorentz boosts in addition to ordinary rotations in the Minkowski space $\mathbb{R}^{1,3}$. The amazing property of (64) is that where as we started by formally introducing the complex number $i = \sqrt{-1}$, in order to represent the Dirac gamma matrices, we have ended up with a spinor operator $\psi \in G^+_{1,3}$ in which the role of $i = \sqrt{-1}$ is taken over by the $\gamma_{21} = e_{12} \in G_3$. This is the key idea in the Hestenes representation of the Dirac equation [18].

Although we have formally eliminated the need for the artificial $i = \sqrt{-1}$ in the Dirac spinor by replacing it with the bivector $\gamma_{21} = e_{12}$ in the spinor operator, such a quantity occurring in a real geometric algebra is unacceptable. The imaginary $i = \sqrt{-1}$ has the effect of complexifying the geometric algebra $G_{1,3}$ to the complex geometric algebra $G_{4}(C) \cong M(4,C)$ [10,p.217]. We might, instead, search for a larger real geometric algebra containing the geometric algebra $G_{1,3}$, just as we enlarged the geometric algebra $G_3$ of space to the geometric algebra $G_{1,3}$ of spacetime. The real geometric algebra $G_{2,3}$, obtained from the geometric algebra $G_{1,3}$ by assuming the existence of a new timelike vector $\gamma_0$ which anticommutes with all the spacetime vectors $\gamma_k \in G_{1,3}$, has the required properties. The unit pseudoscalar $i' := \gamma_0\gamma_1\gamma_2\gamma_3 \in G_{2,3}$ has all the desired algebraic properties of the imaginary $i = \sqrt{-1}$, commuting with all the elements of $G_{1,3}$ and having square $-1$. All of the previous arguments in the complex algebra $G_{4}(C)$ remain valid in $G_{2,3}$, but now $i'$ is the oriented element of volume of the 5-dimensional pseudoeuclidean space $\mathbb{R}^{2,3}$.

However, the real geometric algebras of the pseudoeuclidean spaces $\mathbb{R}^{4,1}$ and $\mathbb{R}^{0,5}$ also fit the bill, as is seen in Table I in [10, p.217]. In [11, p.236], Hestenes argues that the geometric algebra $G_4$ of $\mathbb{R}^{4,1}$ might be an even better choice because the even subalgebra $G^+_{4,1}$ is isomorphic to $G_{1,3}$ in exactly the same way that $G^+_{1,3}$ is isomorphic to $G_3$. In the paper, “Vector Analysis of Spinors” [12], the author explores the interpretation of a geometric Pauli spinor on the Riemann sphere, ideas which he then generalizes in another paper to geometric Dirac spinors in the complex geometric algebra $G_{2}(C)$ [13]. These real and complex geometric algebras open up many new possibilities for the interpretation of relativistic quantum mechanics.
4. Tensor products of geometric numbers

Tensor products of geometric numbers, and their corresponding geometric algebras, are necessary for describing multi-particle systems used, for example, in the construction of a quantum computer, or even in the much simpler “double slit” experiments. The resulting quantum entanglement, which implies the “spooky action-at-a-distance” that even Einstein couldn’t stomach, is discussed in Sec. 5.

By the tensor product $G_3 \otimes G_3$ of the geometric algebra $G_3$ with itself, we essentially mean two distinguishable copies of the same geometric algebra $G_3$. We can express this as

$$G_3 \otimes G_3 = \{ g_1 \otimes g_2 \mid g_1, g_2 \in G_3 \}. \quad (65)$$

In addition, the subspace consisting of everything which commutes with the entire algebra, called the center of $G_3$, may be identified with the complex numbers

$$\mathbb{C} \equiv \text{span}\{1, i = e_{123}\}.$$  

The tensor products of $g_1, g_2, g_3, g_4 \in G_3$ and $\alpha, \beta \in \mathbb{C}$ satisfy the rules

$$\alpha(g_1 \otimes g_2) = (\alpha g_1) \otimes g_2 = g_1 \otimes (\alpha g_2), \quad (66)$$

$$(\alpha g_1 + \beta g_2) \otimes g_3 = \alpha(g_1 \otimes g_3) + \beta(g_2 \otimes g_3), \quad (67)$$

and

$$g_1 \otimes (\alpha g_2 + \beta g_3) = \alpha(g_1 \otimes g_2) + \beta(g_1 \otimes g_3). \quad (68)$$

Either the standard basis (8) or the spectral basis (13) can be tensored with itself to get the respective standard basis or spectral basis of the tensor algebra $G_3 \otimes G_3$ over $\mathbb{C}$. By the product of tensors $g_1 \otimes g_2$ and $g_3 \otimes g_4$, we mean

$$\left( g_1 \otimes g_2 \right) \left( g_3 \otimes g_4 \right) = (g_1 g_3) \otimes (g_2 g_4). \quad (69)$$

The tensor product (65) satisfies the same algebraic rules (67) and (68) as the geometric product itself. This suggests extending the tensor product to commuting copies of the geometric algebra $G_{1,3}$ of spacetime. In Ref. [14,15] the authors identify the tensor product of $N$ copies of $G_3$ with commuting copies of $G_{1,3}^+$ in the geometric algebra $G_{N,3}$ to represent the spinor operators of an $N$-qubit quantum computer. As explained in Ref. 14, in order to get the usual tensor products of complex matrices that is used for multi-distinguishable spin $(1/2)$ particles, one must introduce a commuting idempotent to project down to only a single set of complex numbers.

5. Non-entangled and entangled states

If $|\alpha\rangle = \sqrt{2}(\alpha_0 + \alpha_1)u_+$ and $|\beta\rangle = \sqrt{2}(\beta_0 + \beta_1)u_+$ are normalized states, then the product state is given by the tensor product

$$|\alpha\rangle \otimes |\beta\rangle = \left( \sqrt{2}(\alpha_0 + \alpha_1 e_1)u_+ \right) \otimes \left( \sqrt{2}(\beta_0 + \beta_1 e_1)u_+ \right) \equiv \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle. \quad (70)$$

where we are using the qubit notation

$$|00\rangle := 2u_+ \otimes u_+, \quad |01\rangle := 2u_+ \otimes e_1 u_+$$

and, similarly, for the definitions of $|10\rangle$ and $|11\rangle$. Taking the reverse of the product state $|\alpha\rangle \otimes |\beta\rangle$ gives

$$\left( |\alpha\rangle \otimes |\beta\rangle \right)^\dagger = |\alpha\rangle \otimes \langle \beta|$$

and

$$\left( |\alpha\rangle \otimes \langle \beta| \right) \left( |\alpha\rangle \otimes |\beta\rangle \right) = \langle \alpha|\alpha\rangle \langle \beta|\beta\rangle.$$  

The product state $|\alpha\rangle \otimes |\beta\rangle$ represents the state of two unentangled electrons, where each electron is prepared independently, and can be measured independently. A direct consequence of (48) is that for any values $s, t \in [-1, 1]$ chosen, directions $\hat{m}$ and $\hat{n}$ can always be found resulting in the average values

$$\langle \alpha|\hat{m}|\alpha\rangle = s, \quad \langle \beta|\hat{n}|\beta\rangle = t.$$

The states $|S_\pm\rangle = |01\rangle \pm |10\rangle$ are entangled states. In an entangled state if one of the electrons is measured to be up along a given direction, then the other will definitely be down without the need for measurement. For any direction $\hat{m}$, the averages of the states $|S_\pm\rangle$ measured in the directions $\hat{m} \otimes |1\rangle$ and $1 \otimes \hat{m}$ is 0, meaning that measurements of the first and second electron along this direction are equally likely to be $\pm 1$. With the help of (49), we easily calculate

$$\langle \hat{m} \otimes |1\rangle |S_\pm\rangle = \langle \hat{m}|u_+ \otimes |1\rangle \pm \langle \hat{m}|1\rangle |0\rangle$$

$$= \left( m_3|0\rangle + m_+|1\rangle \right) \otimes |1\rangle$$

$$\pm \left( -m_3|1\rangle + m_-|0\rangle \right) \otimes |0\rangle$$

$$= \left( m_3|01\rangle + m_+|11\rangle \right)$$

$$\pm \left( -m_3|10\rangle + m_-|00\rangle \right),$$

from which we find that

$$\langle S_\pm|\hat{m} \otimes |1\rangle |S_\pm\rangle = \langle \langle 01 \rangle \pm \langle 10 \rangle \rangle$$

$$= \left( m_3|01\rangle + m_+|11\rangle \mp m_3|10\rangle \pm m_-|00\rangle \right)$$

$$= m_3 - m_3 = 0,$$

and a similar calculation shows that $\langle S_\pm |1 \otimes \hat{m}|S_\pm\rangle = 0$. For a further insightful discussion of these results, see [1].
6. The Schrödinger, Pauli, and Dirac Equations

Spin and quantum entanglement have been studied by many authors. The Susskind Lectures [1] give a fascinating account, presenting clearly the issues that separate quantum and classical mechanics. Another elementary presentation is given in [16]. Of course, no account of quantum mechanics can be complete without at least a cursory presentation of the fundamental Schrödinger and Dirac equations, the foundation upon which all of quantum mechanics is built.

The Schrödinger equation is obtained from the classical expression for the total energy of a particle at the position \( x = xe_1 + ye_2 + ze_3 \),

\[
H = \frac{1}{2}mv^2 + V = \frac{1}{2m}p^2 + V, \tag{71}
\]

where \( p = px e_1 + py e_2 + pz e_3 \) is the momentum vector of the particle and \( V \) is the potential, by way of the substitution

\[
p \to -ih\nabla, \tag{72}
\]

where \( \nabla := \nabla_x \) is the standard gradient with respect to the position \( x \in \mathbb{R}^3 \), \( h = h/2\pi \) and \( h \) is Planck’s constant. The Schrödinger equation thus becomes

\[
\frac{i}{\hbar}\frac{\partial \varphi}{\partial t} = \hat{H}\varphi = \left( -\frac{\hbar}{2m}\nabla^2 + V \right)\varphi, \tag{73}
\]

for the Hamiltonian \( \hat{H} = -\hbar^2/2m\nabla^2 + V \), where \( \varphi := \varphi(x,t) \in \mathbb{C} \).

When the potential \( V \) is independent of time, there will be a complete set of stationary states

\[
\varphi_n(x,t) = e^{-iE_nt/\hbar}, \tag{74}
\]

where the spatial wave function \( \varphi_n(x) \) satisfies the time independent Schrödinger equation

\[
\left( -\frac{\hbar}{2m}\nabla^2 + V \right)\varphi_n(x) = E_n\varphi_n(x), \tag{74}
\]

in terms of which the time dependent Schrödinger equation has the general solution

\[
\varphi(x,t) = \sum_n c_n \varphi_n(x)e^{-iE_nt/\hbar}, \tag{75}
\]

[16,p.122]. According to the Born interpretation of the wave function, the probability of finding the particle in the infinitesimal volume

\[
|d^3x| = dx dy dz \quad \text{is} \quad |\varphi(x,t)|^2|d^3x|. \]

The Schrödinger equation (73) is generalized to include spin in the Schrödinger-Pauli equation [10,p.51]. Expressed in terms of our spinor (26) with respect to \( u_+ \),

\[
\frac{i}{\hbar}\frac{\partial |\alpha\rangle}{\partial t} = \frac{1}{2mc} \left( i\hbar\nabla + \frac{e}{c}A \right)^2 |\alpha\rangle + eV |\alpha\rangle \tag{76}
\]

for the electric field potential \( V \) and the magnetic field \( B = -i(\nabla \times A) = \nabla \times A \) of a particle of mass \( m \) with electric charge \( e \), and where \( i = e_1 \) is the unit pseudoscalar of \( \mathbb{G}_3 \). Expanding Eq. (76), we find that

\[
\frac{i}{\hbar}\frac{\partial |\alpha\rangle}{\partial t} = \frac{1}{2mc} \left( \frac{\hbar^2}{c^2} \nabla^2 + \frac{e^2}{c^2}A^2 \right) |\alpha\rangle + eV |\alpha\rangle
+ \frac{e}{c} \left( \nabla A + A \nabla \right) |\alpha\rangle + eV |\alpha\rangle
+ \frac{1}{2mc} \left( \frac{\hbar^2}{c^2} \nabla^2 + \frac{e^2}{c^2}A^2 + \frac{e}{c} \left( \nabla \cdot A + 2A \cdot \nabla \right) \right) |\alpha\rangle
+ \frac{e}{c} \left( \nabla \cdot A + 2A \cdot \nabla \right) |\alpha\rangle
+ \frac{e}{c} |\alpha\rangle.
\]

This is equivalent to the classical 2 \times 2 matrix form of this equation,

\[
\frac{i}{\hbar}\frac{\partial |\alpha\rangle}{\partial t} = \frac{1}{2mc} \left( i\hbar\nabla + \frac{e}{c}A \right)^2 |\alpha\rangle + eV |\alpha\rangle, \tag{77}
\]

obtained by taking the matrix of both sides of (76) and extracting the first column, where \( |\alpha\rangle \) are the Pauli matrices given in (14) and \( |\varphi\rangle_2 \) is the 2-component spinor (32). The matrix Eq. (77) can be expanded in the same way that we expanded Eq. (76). See reference [17, eqn(A.4)]. The reader is referred to the Appendix for more technical details.

Recall that the spinor operator (52) is \( \psi = (1/\sqrt{2}) (|\alpha\rangle + |\alpha\rangle^\dagger) \). If \( |\alpha\rangle \) satisfies the Schrödinger-Pauli equation (76), then \( |\alpha\rangle^\dagger \) will satisfy the parity inversion of this equation,

\[
-\frac{i}{\hbar}\frac{\partial |\alpha\rangle^\dagger}{\partial t} = \frac{1}{2mc} \left( i\hbar\nabla - \frac{e}{c}A \right)^2 |\alpha\rangle^\dagger + eV |\alpha\rangle^\dagger. \tag{78}
\]

It then follows that the operator spinor \( \psi \) will satisfy the equation obtained by taking \( (1/\sqrt{2}) \) times the sum of equations (76) and (78), giving the parity invariant Schrödinger-Pauli-Hestenes equation

\[
\frac{i}{\hbar}\frac{\partial \psi}{\partial t} = \frac{1}{2mc} \left( i\hbar\nabla + \frac{e}{c}A \right)^2 \psi_{u_+}
+ \frac{1}{2mc} \left( i\hbar\nabla - \frac{e}{c}A \right)^2 \psi_{u_-} + eV \psi
= \frac{1}{2mc} \left( -\hbar^2\nabla^2 + \frac{e}{c}A^2 \right) \psi
+ \frac{i}{\hbar} \left( \nabla \cdot A + 2A \cdot \nabla \right) \psi_{e_3}
- \frac{e}{2mc^2} B \psi + eV \psi,
\]

[17, eqn(1.8)].

The Schrödinger equation (73) applies to all quantum phenomena except magnetism and relativity. The magnetism and spin of an electron are taken care of in the Schrödinger-Pauli equation (76), or equivalently (77). In order to extend quantum mechanics to special relativity, we start with the expression for relativistic energy (\( E^2/c^2 - p^2 = m^2c^2 \)). Following [19, (5.5)] and [10, p.136], we express the Dirac equation in terms of the Dirac spinor operator (64), getting what is known as the Dirac-Hestenes equation

\[
\hbar \frac{\partial \psi}{\partial t} - eA \psi = m \psi, \tag{79}
\]
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where $\partial = \sum_{\mu=0}^{3} \gamma^{\mu} (\partial/\partial x^{\mu})$, and an interaction with an electromagnetic field is included by way of the spacetime potential $A = \sum_{\mu=0}^{3} \gamma_{\mu} A^{\mu}$.

Just like for the Schrödinger-Pauli equation (76), the classical Dirac equation can be retrieved from its spinor operator form (79). First, multiply both sides of the Eq. (76) on the right by the primitive idempotent $u_{++}$, using that $u_{++} \gamma_{0} = u_{++}$ and $u_{++} \gamma_{21} = i u_{++}$, and using the spectral basis relations (58), (59), and (60). Finally, extracting the first column on each side of the resulting matrix equation, we obtain

$$\frac{1}{\sqrt{2}} [\langle \beta \rangle^{c_1}] = [(\beta_0 + \beta_1 e_1) u_{-}]$$

$$= \begin{pmatrix} 0 & \beta_1 \\ \beta_0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & \beta_1 \\ 0 & \beta_0 \end{pmatrix}$$

so the right column spinor $\begin{pmatrix} \beta_1 \\ \beta_0 \end{pmatrix}$ is represented by the matrix $\begin{pmatrix} 0 & \beta_1 \\ \beta_0 & 0 \end{pmatrix}$. Unlike with ordinary 2-component spinors (32), the full matrix $[g]$ of a geometric number $g \in G_3$ can be recovered from the corresponding matrices of left and right column spinors,

$$g = g u_{+} + g u_{-} = \frac{1}{\sqrt{2}} (|\alpha\rangle + |\beta\rangle^{c_1})$$

Taking the matrix representation of (28), we find that

$$\langle \alpha | \langle \beta \rangle \rangle = 2 [u_{+}] [\langle \overline{\alpha_0} + \overline{\alpha_1} e_1 \rangle [\langle \overline{\beta_0} + \overline{\beta_1} e_1 \rangle [u_{+}]$$

or

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{\alpha_0} \\ \overline{\alpha_1} \end{pmatrix} \begin{pmatrix} \beta_0 & \beta_1 \\ \beta_0 & \beta_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{\alpha_0} \beta_0 + \overline{\alpha_1} \beta_1 \\ 0 \end{pmatrix}.$$ 

from which it follows that

$$\langle \alpha | \langle \beta \rangle \rangle = \begin{pmatrix} \langle \alpha | \langle \beta \rangle \rangle \end{pmatrix}_{0+3} = \text{Tr} \begin{pmatrix} \overline{\alpha_0} \beta_0 + \overline{\alpha_1} \beta_1 \\ 0 \\ 0 \end{pmatrix} = \overline{\alpha_0} \beta_0 + \overline{\alpha_1} \beta_1.$$

The matrix of the spinor operator (52) is given by

$$[\psi] = \frac{1}{\sqrt{2}} ([|\alpha\rangle] + [|\alpha\rangle^{-}])$$

$$= \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \overline{\alpha_1} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0 - \overline{\alpha_1} \\ \alpha_1 \end{pmatrix}.$$ 

Recalling (40), $|\alpha\rangle = \sqrt{2} \alpha_0 m u_{+}$, the calculation (42), is equivalent to the matrix calculation

$$\frac{1}{2} [\langle \alpha | \langle \alpha \rangle \rangle] = \alpha_0 \overline{\alpha_0} [m]_{u_{+}} [m] = [\hat{a}_+]$$

where the matrices $[m]$, $[u_{+}]$ and $[\hat{a}_+]$ are specified by

$$[m] = \begin{pmatrix} 1 \\ \frac{\overline{\alpha}_1}{\alpha_0} \end{pmatrix},$$

$$[u_{+}] = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$[\hat{a}_+] = \frac{1}{2} \begin{pmatrix} 1 + a_3 & a_- \\ a_+ & 1 - a_3 \end{pmatrix}.$$ 

Appendix

In this Appendix, we explore in greater detail how our presentation of quantum mechanics is related to the more usual approaches.

Multiplying the spectral basis (13) on the right by $u_{+} = (1/2)(1 + e_3)$ gives the left spinor basis

$$\begin{pmatrix} 1 \\ e_1 \end{pmatrix} u_{+} (1 + e_1) u_{+} = \begin{pmatrix} u_{+} \\ e_1 u_{+} \end{pmatrix} = \begin{pmatrix} u_{+} \\ 0 \end{pmatrix}$$

Similarly, multiplying the spectral basis (13) on the right by $u_{-} = (1/2)(1 - e_3)$ gives the right spinor basis

$$\begin{pmatrix} 1 \\ e_1 \end{pmatrix} u_{+} (1 + e_1) u_{+} = \begin{pmatrix} u_{+} \\ e_1 u_{+} \end{pmatrix} = \begin{pmatrix} 0 \\ e_1 u_{-} \end{pmatrix}.$$ 

Taking the matrix of each side of Eq. (26), we get

$$\frac{1}{\sqrt{2}} [\langle \alpha \rangle] = [(\alpha_0 + \alpha_1 e_1) u_{+}]$$

$$= \begin{pmatrix} \alpha_0 & \alpha_1 \\ \alpha_0 & \alpha_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_0 & 0 \\ \alpha_1 & 0 \end{pmatrix}$$

so the left column spinor $\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$ is represented by the matrix $\begin{pmatrix} \alpha_0 & 0 \\ \alpha_1 & 0 \end{pmatrix}$. The factor of $(1/\sqrt{2})$ is inserted as a matter of convenience in the way that we defined the Hermitian inner product (29). Similarly, taking the matrix of each side of the $e_1$-conjugate (44), we get

$$\frac{1}{\sqrt{2}} [\langle \beta \rangle^{c_1}] = [(\beta_0 + \beta_1 e_1) u_{-}]$$

$$= \begin{pmatrix} 0 & \beta_1 \\ \beta_0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & \beta_1 \\ 0 & \beta_0 \end{pmatrix}$$

where the matrices $[m]$, $[u_{+}]$ and $[\hat{a}_+]$ are specified by

$$[m] = \begin{pmatrix} 1 \\ \frac{\overline{\alpha}_1}{\alpha_0} \end{pmatrix},$$

$$[u_{+}] = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$[\hat{a}_+] = \frac{1}{2} \begin{pmatrix} 1 + a_3 & a_- \\ a_+ & 1 - a_3 \end{pmatrix}.$$ 

Analogously to (82) for the spectral basis (13) of the Pauli algebra, we can extract any column of the spectral basis (58) of the Dirac algebra. For example, to extract the third column we multiply (58) on the right by \( u \) of the Dirac algebra. For example, to extract the third column we multiply (58) on the right by \( u \) of the Dirac algebra. For example, to extract the third column we multiply (58) on the right by \( u \) of the Dirac algebra. For example, to extract the third column we multiply (58) on the right by \( u \) of the Dirac algebra.

\[
\begin{pmatrix}
u_{++} & -e_{13}u_{++} & e_{3}u_{++} & e_{1}u_{--} \\
e_{13}u_{++} & u_{--} & e_{1}u_{++} & -e_{3}u_{--} \\
e_{3}u_{++} & e_{2}u_{++} & u_{--} & -e_{13}u_{--} \\
e_{1}u_{++} & -e_{3}u_{++} & e_{13}u_{++} & u_{--}
\end{pmatrix}
\]

Multiplying the spinor operator (64) on the right by \( u_{++} \) we obtain

\[
\psi u_{++} = \varphi_1 u_{++} + \varphi_2 e_{13} u_{++} + \varphi_3 e_{3} u_{++} + \varphi_4 e_{1} u_{++},
\]

whose spinor matrix

\[
[\psi u_{++}] = \begin{pmatrix}
\varphi_1 & 0 & 0 & 0 \\
\varphi_2 & 0 & 0 & 0 \\
\varphi_3 & 0 & 0 & 0 \\
\varphi_4 & 0 & 0 & 0
\end{pmatrix} \equiv \begin{pmatrix}
\alpha_1 & 0 & 0 & 0 \\
\alpha_2 & 0 & 0 & 0 \\
\alpha_3 & 0 & 0 & 0 \\
\alpha_4 & 0 & 0 & 0
\end{pmatrix} \text{ Mod}(u_{++})
\]

corresponds to the Dirac spinor (61).

The classical Dirac equation (80) is equivalent to the Dirac-Hestenes equation (79) multiplied on the right by \( u_{++} \),

\[
(i\hbar \partial \psi - eA \psi - m\psi) u_{++} = 0. \tag{83}
\]

Taking the spacetime inversion of this equation, by replacing all spacetime vectors by their negatives, gives the parity related equation

\[
(i\hbar \partial \psi - eA \psi + m\psi) u_{--} = 0.
\]

Two more equations are obtained, which are parity related to (83), by taking the complex conjugate of both of these last two equations, giving

\[
(i\hbar \partial \psi + eA \psi + m\psi) u_{--} = 0,
\]

and

\[
(i\hbar \partial \psi + eA \psi - m\psi) u_{--} = 0.
\]

But the sum of these four parity related equations is exactly the parity invariant Dirac-Hestenes equation (79).

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