Atomic spin squeezing in three level atoms

O. Civitarese, M. Reboiro, L. Rebon, and D. Tielas
Department of Physics, University of La Plata,
c.c. 67 1900, La Plata, Argentina,
Faculty of Engineering, University of Lomas de Zamora,
c.c. Km 2, (1836) Lavallol, Argentina.

The conditions under which squeezing occurs, in a system of Rb atoms and photons, are modelled by describing atom-photon interactions in three atomic levels. The time evolution of the spin population is calculated for different initial conditions. It is found that the use of coherent states does not suffice for the transfer of spin between the atoms.

Keywords: Squeezing; three level atoms; laser field; coherent states.

Se estudia la transferencia de espín entre un campo de radiación y un sistema atómico compuesto por átomos con tres niveles activos (87Rb). La evolución temporal de la población de niveles, calculada a partir de las soluciones del modelo propuesto, muestra que la utilización de estados coherentes no garantiza la óptima transferencia de espín bajo las condiciones de squeezing.

Descriptores: Compresión; átomos con tres niveles, campo de laser; estados coherentes.

PACS: 03.65.Fd; 42.50.-p; 42.50.Hz

1. Introduction

The conditions under which squeezing may take place, *e.g.* the transfer of quantum information between atomic states and laser fields, are currently under theoretical and experimental investigations [1]. For a general review see Ref. 2.

The following list of references does not intent to exhaust the material which has been published so far, rather it focus on some of the theoretical works and it may be taken as a sample of works which may serve as motivations for the present study. The subject of squeezing in quantized electromagnetic fields has received continuous attention, since the first publications appeared more than twenty years ago [3,4]. Reference 5 describes squeezing generation and revivals in a cavity-ion system in contact with a reservoir. In the work of Ref. 5, the considered system consists of a single two-level ion in a harmonic trap, at zero temperature and exposed to the action of two external lasers.

In Ref. 6 a simple scheme to measure squeezing and phase properties of a harmonic oscillator is proposed. The work of Poulsen and Mølmer [7] shows that under adequate conditions the quantum information of collective states of atoms may be transferred to a pulse of light.

Atomic squeezing under collective emission, that is for the case of large systems of radiating atoms, was studied by V. I. Yukalov and E. P. Yukalova in Ref. 8.

Entanglement and spin squeezing properties for three bosons in two modes were presented in Ref. 9. The theoretical and experimental aspects of entanglement and squeezing in a two-mode system have been presented in Ref. 10. The study of spin squeezing in non-linear spin-coherent states is found in Ref. 11, and the optimally squeezed spin states have been considered in Ref. 12.

In Ref. 13 the relations between bosonic quadrature and atomic spin squeezing have been studied. The spin transfer between photons and atoms was examined by using the Dicke Hamiltonian [14]. In this model and in the limit of a large number of atoms, a perfect transfer, *i.e.*: squeezing, was found [13].

Spin squeezing via atom-field interactions was considered within the framework of the Tavis-Cummings model [15] in Ref. 16. The work of Genes *et al.* [16] describes an ensemble of N two level atoms interacting with a quantized cavity field.

A common feature shared by the theoretical models, introduced in the references given above, is the interaction between atomic levels and photons. The general structure of the Hamiltonians which include such type of interactions belongs to the family of couplings presented in Marshalek and Klein in Ref. 17. These forms are amenable to boson expansions and/or exact boson mappings [18,19]. Boson mapping techniques allow for the generalization of the simple forms of interactions, which have been used so far, like the Dicke Hamiltonian [14] or the Tavis-Cummings Hamiltonian [15], for instance.

The commonly adopted scheme of two level atoms interacting with the radiation field was extended to consider squeezing in a three level atom situation by Z. Ficek and P. D. Drummond [20,21] and by J. Javanainen and P. L. Gould [22].

In this work we are considering:

a) three-level atoms, and

b) coherent states of photons to model the initial condition of the systems where squeezing may appear.

Concerning point (a) we shall present the algebraic details needed to construct the exact solution of a Hamiltonian de-

scribing atomic excitations of A three-level atoms induced by the exchange of photons. Relative to point (b) we shall study the dependence of the solutions upon the average number of photons in the initial state. The details of the formalism are presented in Sec. 2, and in Refs. 23 to 25.

The occurrence of squeezing in a system of A three-level atoms and photons is numerically modelled in Sec. 3, where we present and discuss the solutions for different parameters of the model and different initial conditions. The time evolution of atomic and field squeezing is shown also in Sec. 3. Conclusions are drawn in Sec. 4.

2. Formalism

The system consists of A identical three-level atoms in interaction with a radiation field [26]. The atoms and the photons are placed in a cavity. The creation (annihilation) operator for the i-th atomic level ($i = 0, 1, 2$), is denoted by $b_i^\dagger(b_i)$. The operators $b_i^\dagger$ and b_i obey boson commutation relations

The Hamiltonian of the system reads

$$H = \omega a^\dagger a + \sum_i E_i S^{ii}$$
$$+ g_1 \left(a\, S_+^{01} + a^\dagger\, S_-^{01} \right) + g_2 \left(a\, S_+^{12} + a^\dagger\, S_-^{12} \right). \quad (1)$$

The operators $S_+^{ij}, S_-^{ij}, S_{z\ i \neq j}^{ij}$ generate the $(A+1)(A+2)/2$-dimensional symmetric representation of the $su(3)$ algebra. This can be demonstrated easily, since the operators

$$S^{ij} = b_j^\dagger b_i, \quad i, j = 0, 1, 2, \quad (2)$$

commute as

$$\left[S^{ij}, S^{km} \right] = \delta_{im} S^{kj} - \delta_{jk} S^{im}, \quad (3)$$

and they are the starting operators which are used to define the atomic inversion operators

$$S_z^{ij} = \frac{1}{2}(S^{jj} - S^{ii}), \quad (4)$$

and the transition operators $S_\pm^{ij}$

$$S_+^{ij} = S^{ij}, \ \ S_-^{ij} = (S_+^{ij})^\dagger = S^{ji}, \ \ i, j = 0, 1, 2, \ i < j. \ (5)$$

In the expression of H of Eq. (1), ω is the energy of the photon, $a^\dagger(a)$ is the one photon-creation (-annihilation) operator, E_i is the energy of the $i-th$ atomic level, and g_1 and g_2 are coupling constants describing the absorption (emission) of a photon in the presence of an upward (downward) atomic excitation between levels 0 and 1 (term proportional to g_1), and between levels 1 and 2 (term proportional to g_2).

The two-photon resonance condition [26] is satisfied by fixing the energies of the atomic levels E_i at the values

$$E_2 - E_0 = 2\omega, \quad E_1 - E_0 = \omega - \Delta. \quad (6)$$

2.1. The exact solution

The operator

$$\hat{L} = a^\dagger a + 2 S_z^{02}, \quad (7)$$

commutes with the Hamiltonian of Eq. (1), which, therefore, can be diagonalized in the basis of states

$$\mid n_b n_0 n_1 n_2 \rangle = N_{n_b, n_0 n_1 n_2} a^{\dagger n_b} b_0^{\dagger n_0} b_1^{\dagger n_1} b_2^{\dagger n_2} \mid 0 \rangle, \quad (8)$$

and by enforcing the constraints

$$n_0 + n_1 + n_2 = A,$$
$$n_2 - n_0 + n_b = N, \quad (9)$$

where A is the number of atoms, and N is the sum of the number of photons, n_b, and the difference $n_2 - n_0$ between the population of the atomic states $i = 2$ and $i = 0$. $N_{n_b, n_0 n_1 n_2}$ is a normalization constant. The physical meaning of $\hat{L}$ is straightforward, since it is just the number of photons plus the number of up-wards transitions between atomic levels. In other words, it represents the population of levels 2 and 0 deduced from the number of photons and the population of level 1, as we shall see below.

The diagonalization yields the set of eigenvalues, E_α, and eigenvectors

$$\mid \Psi_\alpha \rangle = \sum_{a \equiv \{n_b, n_0, n_1, n_2\}} c_\alpha(a) \mid a \rangle. \quad (10)$$

The actual dimension of the configuration space, that is the largest possible value of N for which the exact diagonalization is still feasibly, is fixed by analyzing the stability of the wave function for increasing values of N. The adopted procedure will be discussed in Sec. 3.

By making use of the structure of the eigenvectors of H, the time-dependent expectation value of a given operator $\hat{O}$, on a given state $\phi(t)$ (which, of course, should not be an eigenstate of the Hamiltonian) can be written

$$\langle \phi(t)|\hat{O}|\phi(t) \rangle = \sum_{\alpha, \beta} D_{\alpha, \beta}(t) \langle \Psi_\alpha|\hat{O}\mid \Psi_\beta \rangle. \quad (11)$$

In this expression, $D_{\alpha, \beta}(t)$ is the matrix

$$D_{\alpha, \beta}(t) = \langle \phi(t)|\Psi_\alpha \rangle \langle \Psi_\beta|\phi(t) \rangle, \quad (12)$$

which depends on the initial condition at time $t = 0$, $|\phi(0)\rangle$, since

$$|\phi(t)\rangle = e^{-\frac{iHt}{\hbar}}|\phi(0)\rangle. \quad (13)$$

In the following, we shall write, explicitly, the results for the squeezing factor $Q(R, S)$, which relates the uncertainties of a given pair of operators, $\hat{R}$ and $\hat{S}$, namely:

$$Q(R, S) = \frac{2(\Delta \hat{R})^2}{|\langle \phi(t)|[\hat{R}, \hat{S}]|\phi(t)\rangle|}, \quad (14)$$

where $(\Delta \hat{R})^2 = \langle \phi(t)|\hat{R}^2|\phi(t)\rangle - \langle \phi(t)|\hat{R}|\phi(t)\rangle^2$, for different initial conditions $\phi(t)$.

2.2. One atom case

In this section we present the analytic results for the case $A = 1$. The diagonalization of the Hamiltonian of Eq. (1) yields the eigenvalues λ_α and eigenvectors $|\Psi_\alpha\rangle$, with $\alpha = 1, 2, 3$, namely:

$$\lambda_1(N) = \omega (n_b - 1),$$
$$|\Psi_1(N)\rangle = \mathcal{N}_1(-g_2\sqrt{n_b - 1}|a\rangle + g_1\sqrt{n_b}|c\rangle),$$
$$\lambda_2(N) = \omega (n_b - 1) - \delta - r(n_b),$$
$$|\Psi_2(N)\rangle = \mathcal{N}_2(g_1\sqrt{n_b}|a\rangle - (r(n_b) + \delta)|b\rangle$$
$$+ g_2\sqrt{n_b - 1}|c\rangle),$$
$$\lambda_3(N) = \omega (n_b - 1) - \delta + r(n_b),$$
$$|\Psi_3(N)\rangle = \mathcal{N}_3(g_1\sqrt{n_b}|a\rangle + (r(n_b) - \delta)|b\rangle$$
$$+ g_2\sqrt{n_b - 1}|c\rangle), \tag{15}$$

with

$$|a\rangle = |n_b, 1, 0, 0\rangle,$$
$$|b\rangle = |n_b - 1, 0, 1, 0\rangle,$$
$$|c\rangle = |n_b - 2, 0, 0, 1\rangle, \tag{16}$$

and

$$\delta = \frac{\Delta}{2},$$
$$r(n_b) = \sqrt{g_1^2 n_b + g_2^2 (n_b - 1) + \delta^2},$$
$$\mathcal{N}_1 = \frac{1}{\sqrt{r(n_b)^2 - \delta^2}},$$
$$\mathcal{N}_2 = \frac{1}{\sqrt{2}} \left(1 - \frac{\delta}{r(n_b)}\right)^{1/2} \frac{1}{\sqrt{r(n_b)^2 - \delta^2}},$$
$$\mathcal{N}_3 = \frac{1}{\sqrt{2}} \left(1 + \frac{\delta}{r(n_b)}\right)^{1/2} \frac{1}{\sqrt{r(n_b)^2 - \delta^2}}, \tag{17}$$

for each of the subspaces labelled by a fixed value of $N = n_b + n_2 - n_0$. For a variable number of photons each subspace of the solutions is labelled by the value N. The matrix of Eq. (12) is determined by the initial condition $|\phi(0)\rangle$.

The initial condition may be a Fock state

$$|\phi(0)\rangle = |n_b, 1, 0, 0\rangle, \tag{18}$$

and it represents the product state on n_b photons and one atom in the lower state. Then, for $N = n_b - 1$

$$D_{\alpha,\beta} = d_\alpha^* d_\beta,$$
$$d_\alpha = c_\alpha^*(n_b, 1, 0, 0) e^{-i \lambda_\alpha(N) t/\hbar}. \tag{19}$$

To compute the atomic squeezing one needs to calculate the time evolution of the ladder operator S_+^{02}, and of S_z^{02}

$$\langle S_z(n_b, t)\rangle = \langle\phi(t)|S_z^{02}|\phi(t)\rangle$$
$$= -\frac{1}{2} + \frac{1}{2} \frac{g_1^2 n_b}{r(n_b)^2} \sin^2(r(n_b)t)$$
$$+ \frac{g_1^2 g_2^2 n_b(n_b - 1)}{(r(n_b)^2 - \delta^2)^2} \Big((\cos(r(n_b)t) - \cos(\delta t))^2$$
$$+ \left(\frac{\delta}{r(n_b)} \sin(r(n_b)t) - \sin(\delta t)\right)^2 \Big)$$
$$\langle S_z^2(n_b, t)\rangle = \langle\phi(t)|(S_z^{02})^2|\phi(t)\rangle$$
$$= \frac{1}{4} \left(1 - \frac{g_1^2 n_b}{r(n_b)^2} \sin^2(r(n_b)t)\right),$$
$$\langle S_+(t)\rangle = \langle\phi(t)|S_+^{02}|\phi(t)\rangle = 0. \tag{20}$$

Therefore, for this initial condition atomic squeezing does not appear, no matter how many photons or atomic levels are included.

For the case of a coherent photon state, the calculation leads to the expression

$$D_{\alpha,\beta} = d_\alpha^* d_\beta,$$
$$d_\alpha = e^{-\frac{|z|^2}{2}} \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{k!}} c_\alpha^*(k, 1, 0, 0)$$
$$\times e^{-i \lambda_\alpha(k-1) t/\hbar}, \tag{21}$$

where $|z|^2 = n_b$ is the mean value of the number of photons in the coherent state. In this case

$$\langle S_z(t)\rangle = e^{-n_b} \sum_{k=0}^{\infty} \frac{n_b^k}{k!} \langle S_z(k, t)\rangle,$$
$$\langle S_z^2(t)\rangle = e^{-n_b} \sum_{k=0}^{\infty} \frac{n_b^k}{k!} \langle S_z^2(k, t)\rangle,$$
$$\langle S_+(t)\rangle = e^{-n_b} \sum_{k=0}^{\infty} \frac{n_b^{k+1}}{k!} f_0(k) (a(k) + ib(k)) \tag{22}$$

with

$$a(n) = (g_2^2 (n - 2) + g_1^2 (n - 1))f_1(n + 1)$$
$$+ g_1^2 (n - 1)(f_2(n - 1)f_2(n + 1)$$
$$+ f_2(n - 1)f_2(n + 1)),$$
$$b(n) = g_1^2 (n - 1)(f_1(n - 1)f_2(n + 1)$$
$$- f_2(n - 1) f_1(n + 1)), \tag{23}$$

and

$$f_0(n) = \frac{g_1 g_2 \sqrt{(n+2)(n+1)}\, e^{2i\omega t/\hbar}}{(g_1^2(n+2) + g_2^2(n+1))(g_1^2 n + g_2^2(n-1))},$$

$$f_1(n) = \cos(\delta t)\cos(r(n)t)$$
$$\qquad + \frac{\delta}{r(n)}\sin(\delta t)\sin(r(n)t) - 1,$$

$$f_2(n) = -\sin(\delta t)\cos(r(n)t)$$
$$\qquad + \frac{\delta}{r(n)}\cos(\delta t)\sin(r(n)t). \tag{24}$$

As it can be seen from Eq. (22), the contribution of states with different values of N is a necessary condition for coherence. Similar results were reported in Ref. 16.

Notice that, for this initial condition, the time evolution of the ladder operator [22] does not vanish. It means that, depending on the coupling constants of H, squeezing may appear, that is $Q(S_z, S_+) \le 1$ [16].

3. Results and discussion

The energy spacing between the atomic levels is fixed by Eq. (6), with $\Delta = 0$, thus $E_0 = -\omega$, $E_1 = 0$, $E_2 = \omega$. In all cases we have taken a coherent state in the photon sector. We have considered symmetric, $g_1 = g_2$, and non symmetric parametrizations, $g_1 \ne g_2$, of the Hamiltonian. The mean value of the number of photons in the coherent state, has been varied. The calculations have been performed for $A = 3$, $A = 6$, $A = 15$ and $A = 18$ atoms, respectively. In Figs. 1 to 3, we show the results of the present calculations for the time evolution of the atomic inversion, $\langle S_z(t)\rangle$, the atomic squeezing, $Q(S_z, S_+)$, and the field squeezing, $Q(x, p)$ [16]. The atomic initial condition consists of $n_0 = A$ atoms in the ground state (n_0), while the parameter $|z|^2$ of the photon coherent state is fixed at the value n_b. With these parameters the Hamiltonian of Eq. (1) was diagonalized and the density matrix of Eq. (12) was obtained for each subspace, A, N. Figure 1 shows the results for $A = 1$, $g_1 = g_2 = 1$ and $n_b = 10$. No signal of squeezing is seen, in spite of the use of a coherent state in the photon sector. This result may be compared with the findings of Ref. 16, where for the case of two-level atoms, and a coherent photon state squeezing was not obtained. Thus, for $g_1 = g_2$, three atomic levels, one atom, and a photon coherent state, there is no atomic squeezing. In Fig. 2 we show the results obtained with the asymmetric parametrization, $g_1 = 1$, $g_2 = 4$. This case does exhibit squeezing. The actual central value of the atomic squeezing, $Q(S_z, S_+)$, is of the order of $0.75 - 0.80$. The time evolution of the atomic inversion $\langle S_z(t)\rangle$ is consistent with the central value $\langle S_z(t)\rangle = -0.4$, while in the results shown in Fig. 2 the central value of $\langle S_z(t)\rangle$ is zero. The results shown
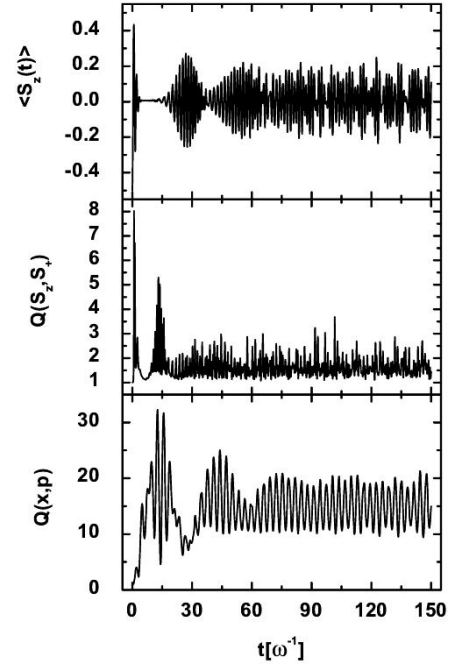
FIGURE 1. Mean value of the inversion operator, $\langle S_z(t)\rangle$, atomic squeezing, $Q(S_z, S_+)$, and field squeezing, $Q(x, p)$, as a function of time. The system consists of one atom, initially in its ground state, and a coherent photon field, with mean value of the number of photons $n_b = 10$. The coupling constants are fixed at the values $g_1 = 1$ and $g_2 = 1$.
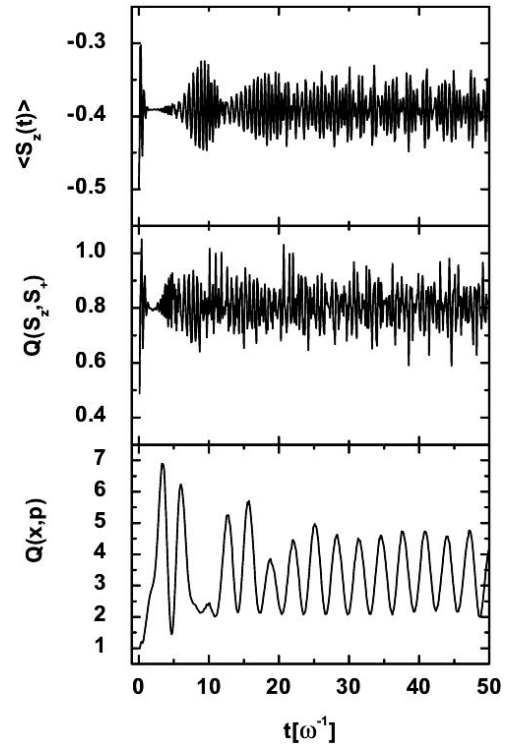
FIGURE 2. Results corresponding to $n_b = 10$, $g_1 = 1$ and $g_2 = 4$ The meaning of the quantities is explained in the caption of as Fig. 1.
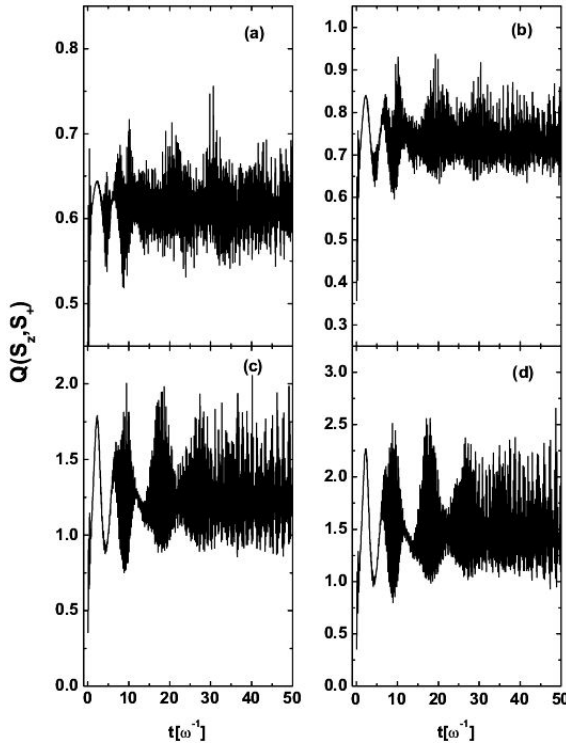
FIGURE 3. Atomic squeezing, $Q(S_z, S_+)$, as a function of time. The interaction coupling constants are fixed at the values $g_1=1$, $g_2=6$. The initial state consists of A atoms in their ground state, and of a coherent state with mean value of the number of photons $n_b = 21$. Insets (a), (b), (c) and (d), correspond to a system with $A = 3$, $A = 6$, $A = 15$ and $A = 18$ atoms, respectively.

in Fig. 2 point out to a possble critical dependence of squeezing upon the relative strength of the emission and absorption

of photons in transitions involving the ground state, n_0, and excited state n_2, independent of the mean value of the number of photons. Finally, the dependence of the atomic squeezing upon the number of atoms is shown in Fig. 3. It is seen that the atomic squeezing is washed out when the number of atoms is increased. These results indicate the strong dependence of the atomic squeezing upon the number of atoms. This result is expected from dimensional arguments.

4. Conclusions

In this work we have studied the occurrence of squeezing in systems composed by three level atoms and a radiation field. It is found that the use of initial conditions consistent of a fixed number of photons does not lead to squeezing; instead, it appears if coherent states are considered in the photon sector of the initial condition. The transfer of spin between the atoms and the photons is enhanced if the interactions between the atomic levels and the photons are parametrized in a non-symmetric form. This is achieved by taking $g_2 > g_1$ in the Hamiltonian of Eq. (1). It is also found that the use of coherent states does not, automatically, lead to squeezing unless the interactions are considered non-symmetric, in the manner explained above. Finally, for a fixed parametrization, it is found that if the number of atoms is increased the squeezing is washed-out.

Acknowledgments

1. S.R. Echanis *et al.*, *J. Opt. B : Quantum Semiclass. Opt.7*, **S 548** (2005).

2. P.D. Drummond and Z. Ficek, *Quantum Squeezing*, Springer, Berlin 2004.

3. D.F. Walls, *Nature* **306** (1983) 141.

4. C.K. Hong and L. Mandel, *Phys. Rev. Letters* **54** (1985) 323.

5. R. Rangel, R. Carvalho, and N. Zagury, ArXiv quant-ph/0412055.

6. G.T. Rubin-Linares and H. Moya-Cessa, ArXiv quant-ph/0501158.

7. U.V. Poulsen and K. Mølmer, *Phys. Rev. Letters* **87** (2001) 123601-1.

8. V.I. Yukalov and E.P. Yukalova, *Phys. Rev.* **A 70** (2004) 053828.

9. B. Zeng, D.L. Zhou, Z. Xu, and L. You, ArXiv: quant-ph/0501082.

10. V. Josse, A. Dantan, A. Bramatti, and E. Giacobino, *J. Opt. B: Quantum Semiclass. Opt.* **6** (2004) S532.

11. X. Wang, *J. Opt. B: Quantum Semiclass. Opt.* **3** (2001) 93

12. A.G. Rojo, *Phys. Rev.* **A 68** (2003) 013807.

13. X. Wang and B.C. Sanders, *Phys. Rev.* **A 68** (2003) 033821.

14. R.H. Dicke, *Phys. Rev.* **93** (1954) 99.

15. M. Tavis and F.W. Cummings, *Phys. Rev.* **170** (1968) 170; E.T. Jaynes and F.W. Cummings, *Proc. IEEE* **51** (1963) 89; E.R. Buley and F.W. Cummings, *Phys. Rev.* **134**(1964) A1454; F.W. Cummings, *Phys. Rev.* **140** (1965) A1051.

16. C. Genes, P.R. Berman, A.G. Rojo, *Phys. Rev.* **A 68** (2003) 043809.

17. A. Klein and E.R. Marshalek, *Rev. of Mod. Phys.* **63** (1991) 375.

18. O. Civitarese and M. Reboiro, *Phys. Rev.* **C 57** (1998) 3055.

19. O. Civitarese and M. Reboiro, *Phys. Rev.* **60** (1999) 034302.

20. Z. Ficek and P. Drummond, *Phys. Rev.* **43** (1991) 6247.

21. Z. Ficek and P. Drummond, *Phys. Rev.* **43** (1991) 6258.

22. J. Javanainen and P. L. Gould, *Phys. Rev.* **41** (1990) 5088.

23. O. Civitarese and M. Reboiro, *Phys. Lett* **A 357** (2006) 224

24. O. Civitarese, M. Reboiro, and L. Rebon, *Phys. Lett* **A 366** (2007) 241

25. O. Civitarese, M. R. Reboiro, L. Rebon, and D. Tielas, *Phys. Lett* **A** (2008) submitted

26. A.B. Klimov, J. Negro, R. Farias, S.M. Chumakov, *J. Opt. B: Quantum Semiclass. Opt.* **1** (1999) 562.

27. P.L. Knight, *Phys. Scr. T* **12** (1986) 51; D. Phoenix and P.L. Knight, *J. OPt. Soc. Am.* **B 4** (1990) 116; R.R. Puri and R.K. Bullough, *J. Opt. Soc. Am.* **5** (1998) 116; L.A. Lugiato, P.Galatola, and L.M. Narducci, *Opt. Commun.* **76** (1990) 276; R.R. Puri and G.S. Agarval, *Phys. Rev.* **A 37** (1988) 3879; He Ling-Sheng, *J. Opt. Soc. Am.* **B 6** (1989) 1915.