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Alternative method for determining the Feynman propagator of a relativistic quantum mechanical problem

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The authors, together with A. del Campo, developed an alternative method for determining the Feynman propagator [1] for a non-relativistic problem. One started with the time dependent Schroedinger equation for the problem. Carried out a Laplace transform with respect to time to get the equation for the energy dependent Green function and derived it explicitly. We then carried out the inverse Laplace transform in the energy to get Feynman propagator. In this paper we carry out the same programme for a relativistic problem associated with the one dimensional Dirac equation of a free particle and the Dirac oscillator proposed by Moshinsky and Szczepaniak [2] twenty years ago.

Keywords: Feynman propagator; Dirac oscillator; relativistic Green functions.

Los autores desarrollaron, junto con A. del Campo, un método alternativo para determinar el propagador de Feynman para un problema norelativista. El método consiste en la evaluación de la transformada de Laplace con respecto al tiempo para derivar la ecuación para la función de Green dependiente de energía. Se obtiene el propagador de Feynman usando la transformada inversa de Laplace de la función de Green. En este artículo llevamos a cabo el mismo programa para un problema relativista asociado a la ecuación de Dirac en una dimensión de una partícula libre y el oscilador de Dirac propuesto hace veinte años por Moshinsky and Szczepaniak [2].

Descriptores: Propagador de Feynman; oscilador de Dirac; funciones de Green relativistas.

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1. Introduction

The Feynman propagator $K(x,t,x',t')$ is the operator that takes the wave function from the point x',t', to x,t and in one dimension is given by the expression [3]

$$\bar{\psi}(x,t) = \int \bar{\psi}(x',t')K(x,t,x',t')dx' \tag{1}$$

Feynman invented a procedure by which $K(x,t,x',t')$ could be evaluated summing the actions between the time dependent paths that relate x to x'.

In this paper we shall use the procedure indicated in the abstract for a time dependent relativistic equation in one dimension of the form [2]

$$i\hbar\frac{\partial\bar{\psi}}{\partial t} = [\alpha c(p + im\omega x\beta) + mc^2\beta]\bar{\psi} \tag{2}$$

where the wave function $\bar{\psi}$ has two components

$$\bar{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{3}$$

p, x are respectively the momenta and coordinate, t is the time, m the mass of the particle, c the velocity of light and α, β the 2×2 matrices

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

Before proceeding with the program outlined in the abstract we firs want to understand what are the problems described by Eq. (2). For this purpose we make the usual substitution

$$\bar{\psi} \rightarrow \bar{\psi}\exp(-iEt/\hbar) \tag{5}$$

where E is the energy and write it in components as

$$(E - mc^2)\psi_1 = c(p - im\omega x)\psi_2 \tag{6}$$

$$(E + mc^2)\psi_2 = c(p + im\omega x)\psi_1 \tag{7}$$

Substituting ψ_2 of (7) in (6) and dividing everything by $2mc^2$ we get

$$\frac{(E^2 - m^2c^4)}{2mc^2}\psi_1 = \left[\left(\frac{p^2}{2m} + \frac{1}{2}\omega^2 x^2\right) + \frac{\hbar\omega}{2}\right]\psi_1 \tag{8}$$

which is the Dirac oscillator equation [2] but also the free particle relativistic equation if $\omega = 0$.

2. The equation for the Green function of our problem

From here on we suppress the t' in our propagator writing it as $K(x, x', t)$. We shall call $\bar{G}(x, x', s)$ the Laplace transform of the propagator *i.e.*

$$\bar{G}(x, x', s) \equiv \int_0^\infty e^{-st} K(x, x', t) dt \tag{9}$$

From the appearance in (1) it is clear that $K(x, x', t)$ is a 2×2 matrix and it must satisfy Eq. (2) so the Laplace transform of it also vanishes and we thus have

$$\int_0^\infty e^{-st} \left\{ i\hbar \frac{\partial}{\partial t} - \alpha c(p + im\omega x\beta) - mc^2\beta \right\} \times K(x, x', t) dt = 0 \tag{10}$$

As the integral is respect to time it does not affect the operators p and x while the firs term in time becomes

$$i\hbar \int_0^\infty e^{-st} \left[\frac{\partial}{\partial t} K(x, x', t) \right] dt = i\hbar \int_0^\infty \frac{\partial}{\partial t} \left[e^{-st} K(x, x', t) \right] dt + i\hbar s \bar{G}(x, x', s) \tag{11}$$

The firs integral on the right hand side of (11) is

$$\left[e^{-st} K(x, x', t) \right]_{t=0}^{t=\infty} = -K(x, x', 0) = -\delta(x - x') \tag{12}$$

as required from (1) at $t = 0$, while the second term in the right hand side of (11) comes from (9). To obtain $K(x, x', t)$ we use the inverse Laplace transform [4]

$$K(x, x'.t) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \bar{G}(x, x', s) e^{st} ds \tag{13}$$

where the integration takes place along a line in the complex plane s parallel to the imaginary axis and at a distance λ from it so that all singularities of $\bar{G}(x, x', s)$ in the s plane are on the left of it.

To have a more transparent notation rather than the s plane we shall consider an energy variable E proportional to it through the relation [1]

$$E = i\hbar s \quad \text{or} \quad s = -i(E/\hbar) \tag{14}$$

and defin $G(x, x', E)$ by

$$G(x, x', E) = \bar{G}(x, x', -iE/\hbar) = \int_0^\infty K(x, x', t) e^{i(E/\hbar)t} dt \tag{15}$$

which has the property of being symmetric under exchange of x and x'

$$G(x, x', E) = G(x', x, E) \tag{16}$$

From (9) and (10) $G(x, x', E)$ satisfie the equation

$$\left[\left(E - mc^2\beta \right) - \alpha c(p + im\omega x\beta) \right] \times G(x, x', E) - i\hbar\delta(x - x') = 0 \tag{17}$$

and as from (4) α, β are 2×2 matrices so is $G(x, x', E)$ *i.e.*

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \tag{18}$$

The Eq. (18) in components can be written as

$$(E - mc^2)G_{11} - c(p + im\omega x)G_{21} = i\hbar\delta(x - x') \tag{19}$$

$$(E + mc^2)G_{21} - c(p - im\omega x)G_{11} = 0 \tag{20}$$

$$(E - mc^2)G_{12} - c(p + im\omega x)G_{22} = 0 \tag{21}$$

$$(E + mc^2)G_{22} - c(p - im\omega x)G_{12} = i\hbar\delta(x - x') \tag{22}$$

From equations (20), (21) we have that

$$G_{21} = (E + mc^2)^{-1} c(p - im\omega x)G_{11} \tag{23}$$

$$G_{12} = (E - mc^2)^{-1} c(p + im\omega x)G_{22} \tag{24}$$

and substituting these values in (19), (22) and eliminating the denominators we have the equations

$$\begin{aligned} &\left[(E^2 - m^2c^4) - c^2(p^2 + m^2\omega^2x^2) - mc^2\hbar\omega \right] G_{11} \\ &= i\hbar(E + mc^2)\delta(x - x'), \\ &\left[(E^2 - m^2c^4) - c^2(p^2 + m^2\omega^2x^2) + mc^2\hbar\omega \right] G_{22} \\ &= i\hbar(E - mc^2)\delta(x - x') \end{aligned} \tag{25}$$

Dividing the Eq. (25) by $2mc^2$ and definin

$$\epsilon_\pm = \frac{E^2 - m^2c^4}{2mc^2} \pm \frac{\hbar\omega}{2} \tag{26}$$

we can write them as

$$\epsilon_\pm G_{ii} = \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2 \right) G_{ii} + \frac{i\hbar}{2mc^2} (E \pm mc^2) \delta(x - x') \tag{27}$$

where when $i = 1$ we have the upper sign $+$ and when $i = 2$ we have the lower sign $-$.

Once we solve for the $G_{ii}, i{=}1,2$ in Eq. (27) the G_{12}, G_{21} can be obtained from them using Eqs. (23), (24) and we have all the components of the Green function $G(x,x',E)$.

The propagator $K(x,x',t)$ is also $2{\times}2$ matrix given by the inverse Laplace transform (13) where instead of an integration with respect to s is convenient to express it as integration with respect to the energy E related as in (14) $s=-i(E/\hbar)$ that give us then the expression

$$K_{ij}(x,x',t)=\frac{1}{2\pi\hbar}\int\limits_{i\hbar\lambda-\infty}^{i\hbar\lambda+\infty} e^{-iEt/\hbar}G_{ij}(x,x',E)dE \quad (28)$$

where the integration takes place in a line in the complex energy plane parallel to the real axis and a distance $\hbar\lambda$ from it with all the singularities of $G(x,x',E)$ below this line.

In the next sections we shall consider firs the integral (28) for the free particle *i.e.* when $\omega=0$ and then in the following section the general case of the Dirac oscillator *i.e.* $\omega\neq 0$.

3. The propagator of the free relativistic Dirac particle

As mentioned in the previous section for the free relativistic particle the Green function is given by (27) when $\omega=0$ and thus its equation is

$$\epsilon G_{ii}(x){=}-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}G_{ii}{+}\frac{i\hbar}{2mc^2}(E{\pm}mc^2)\delta(x-x') \quad (29)$$

where there are no $\pm$ sign for ϵ as ω is 0 in (26), but i continues to have the values 1,2 associated with the $\pm$ sign on the right hand of (29). Multiplying (29) by $(-2m/\hbar^2)$ we get

$$\left(\frac{d^2}{dx^2}+\frac{2m\epsilon}{\hbar^2}\right)G_{ii}=\frac{i}{\hbar c^2}(E\pm mc^2)\delta(x-x') \quad (30)$$

When $x\neq x'$ the equation reduces to

$$\left(\frac{d^2}{dx^2}+k^2\right)G_{ii}=0 \quad \text{where} \quad k^2=\frac{2m\epsilon}{\hbar^2} \quad (31)$$

Thus two independent solutions of (31) are

$$u_E(x)=e^{\pm ikx} \quad (32)$$

and using the result (19) of Ref. 1, we can represent the Green function of our problem as an ordered product of the two solutions in (32), together with the reciprocal of their Wronskian and the additional factor appearing in the r.h.s. of (30) to get

$$G_{ii}(x,x',E) = -\frac{2m}{\hbar^2}\frac{1}{2ik}\frac{i\hbar}{2mc^2}(E\pm mc^2)\exp\left[ik|x-x'|\right] \quad (33)$$

as the inverse of the Wronskian of the two independent solutions $\exp(\pm ikx)$ of (32) is $(2ik)^{-1}$.

As $\epsilon=(E^2-m^2c^4)(2mc^2)^{-1}$ we see from the definitio of k^2 in (31) that

$$k=\sqrt{\frac{2m\epsilon}{\hbar^2}}=\frac{\sqrt{E^2-m^2c^4}}{\hbar c} \quad (34)$$

Using then the Eq. (28) for the propagator $K_{ii}(x,x',t)$ but expressing the $G_{ii}(x,x',E)$ in the integrand only in terms of E we get

$$K_{ii}(x,x',t)=\frac{i}{4\pi\hbar c}\int\limits_{i\hbar\lambda-\infty}^{i\hbar\lambda+\infty}\frac{e^{-iEt/\hbar}}{\sqrt{E^2-m^2c^4}} \times(E\pm mc^2)\exp\left[i\left(\frac{\sqrt{E^2-m^2c^4}}{\hbar c}\right)|x-x'|\right]dE \quad (35)$$

This integral looks very complex both for the square root and the possibility $\pm$ sign in front of it. It is possible though to eliminate this difficult by introducing the variables

$$\xi=(mc^2)^{-1}(E+\sqrt{E^2-m^2c^4})$$
$$\xi^{-1}=(mc^2)^{-1}(E-\sqrt{E^2-m^2c^4}) \quad (36)$$

as $\xi\xi^{-1}=1$. This implies that

$$E=\frac{1}{2}mc^2(\xi+\xi^{-1}),$$
$$\sqrt{E^2-m^2c^4}=\frac{1}{2}mc^2(\xi-\xi^{-1}) \quad (37)$$

from which we obtain

$$dE=\frac{1}{2}mc^2(1-\xi^{-2})d\xi \quad (38)$$

and we can express Eq. (35) in the form

$$K_{ii}(x,x',t)=\frac{mc}{2\pi\hbar}\int\limits_C\frac{(\xi^2-1+2\xi)}{\xi} \times\exp\left[(u\xi-v\xi^{-1})\right]d\xi \quad (39)$$

where

$$u\equiv\frac{1}{2}mc^2\left[\frac{|x-x'|}{\hbar c}-\frac{t}{\hbar}\right],$$
$$v\equiv\frac{1}{2}mc^2\left[\frac{|x-x'|}{\hbar c}+\frac{t}{\hbar}\right]. \quad (40)$$

The contour C corresponds to the variable ξ and if $E{>}mc^2, \xi{=}(mc^2)^{-1}(E+\sqrt{E^2-m^2c^4})$ is real but if $E{<}mc^2, \xi{=}(mc^2)^{-1}(E+i\sqrt{m^2c^4-E^2})$ and it is a circle in the complex plane of ξ of radius mc^2. The contour C in the ξ plane is given in the Fig. 1.

The integrand in (39) has a term $(1/\xi)$ that diverges only at $\xi=0$, but ξ^{-1} appears in the exponential and gives ∞

at $\xi = 0$ so the term $\xi^{-1}\exp(v\xi^{-1})$ goes to 0 when $\xi = 0$. Thus the circle in Fig. 1 can be reduced to the point $\xi = 0$ and as the functions are analytic in the upper half plane we just have to consider separately the integrals

$$\int_0^\infty \text{ and } \int_{-\infty}^0$$

associated with integration of the positive and negative energies of the problem.

Consider firs the positive energy part that can be written as

$$K_{ii}(x, x', t) = \frac{1}{2}\frac{mc}{\pi\hbar}\int_0^\infty (\xi + 2 - \xi^{-1})$$
$$\times \exp i(u\xi - v\xi^{-1})d\xi \tag{41}$$

as the $(\xi^2 - 1)$ terms in numerator and denominator cancel.

On the other hand in formula (3,471.9) of p. 340 and 8.407.1 of p. 952 of Ref. 5 we have that

$$\int_0^\infty \xi^{\nu-1}\exp(u\xi - v\xi^{-1})d\xi$$
$$= \pi i\left(-\frac{v}{u}\right)^{\nu/2} e^{i\pi\nu/2} H_\nu^{(1)}\left(2\sqrt{uv}\right) \tag{42}$$

where the requirement $Re(u) < 0$ is fulfille if events (ct, x), $(0, x')$ are causally connected, *i.e.* if their difference is a time-like vector. Taking into account the definition of u and v in (41) and the general expression (41) for $K_{ii}(x, x', t)$ we have that

$$K_{ii}(x, x', t)$$
$$= \frac{imc}{2\hbar}\left\{\left[\frac{|x - x'| + ct}{|x - x'| - ct}\right] H_2^{(1)}\left[\frac{imc}{\hbar}\sqrt{(x - x')^2 - c^2t^2}\right]\right.$$
$$\pm 2i\sqrt{\frac{ct + |x - x'|}{ct - |x - x'|}} H_1^{(1)}\left[\frac{imc}{\hbar}\sqrt{(x - x')^2 - c^2t^2}\right]$$
$$\left. + H_0^{(1)}\left[\frac{imc}{\hbar}\sqrt{(x - x')^2 - c^2t^2}\right]\right\} \tag{43}$$

where $H_\nu^{(1)}(z)$ is the out going Hankel function and the indices $i = 1, 2$ are related with the $\pm$ sign.

As the Eq. (43) depends only on $\sqrt{uv}$ and $(u/v)^{\nu/2}$ clearly the sign of u, v does not change anything and we get the same expression for the negative energy part.

Thus we have the full solution of $K_{ii}(x, x', t)$ and we can get $K_{12}(x, x', t)$, $K_{21}(x, x', t)$ from (23) (24). For instance, $K_{21}(x, x', t)$ is computed as follows. Eq. (23) with $\omega = 0$ together with the explicit form of G_{11} (33) leads to

$$G_{21} = -\frac{1}{2c}\exp ik|x - x'|. \tag{44}$$

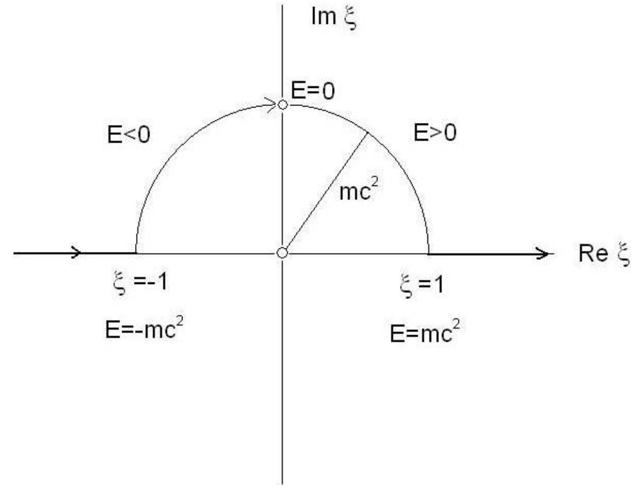

FIGURE 1. Integration C in the ξ plane for the free particle relativistic problem.

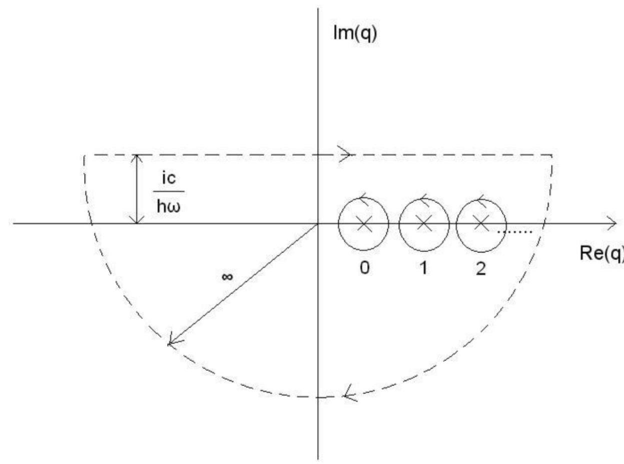

FIGURE 2. Integration contour for the determination of the propagator of the Dirac oscillator.

Using (28) and variables (36) we fin the corresponding entry of the propagator to be

$$K_{21}(x, x', t) = -\frac{mc}{8\pi\hbar}\int_{-\infty}^\infty (1 - \xi^{-2})e^{u\xi - v\xi^{-1}}d\xi \tag{45}$$

which yields, together with the identity $H_{-1}^{(1)} = -H_1^{(1)}$, a positive energy part in the form

$$K_{21}(x, x', t) = \frac{mc}{8\hbar}\left[\sqrt{\frac{ct + |x - x'|}{ct - |x - x'|}} + \sqrt{\frac{ct - |x - x'|}{ct + |x - x'|}}\right]$$
$$\times H_1^{(1)}\left[\frac{imc}{\hbar}\sqrt{(x - x')^2 - c^2t^2}\right] \tag{46}$$

while for the negative energy part we proceed as before, *i.e.* by studying the parity of the integrand in formula (42). The K_{12} entry can be obtained in a similar way from K_{22} and we have all K_{ij} for the free particle of the Dirac equation.

4. The propagator of the relativistic Dirac oscillator

The Green function of the Dirac oscillator is given by (27) with $\omega \neq 0$ and if $x \neq x'$ so $\delta(x-x')=0$, we have

$$\left\{\epsilon_\pm - \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}m\omega^2x^2\right]\right\}G_{ii}=0 \qquad (47)$$

which is identical to Eq. (31) of Ref. 1 just replacing E by $\epsilon\pm$ so that all the non-relativistic analysis is applicable.

We start by definin

$$z=\sqrt{\frac{2m\omega}{\hbar}}x,\ q=\frac{\epsilon_\pm}{\hbar\omega}-\frac{1}{2} \qquad (48)$$

where q is associated to the $+$ sign for G_{11} and the $-$ sign for G_{22} in (47) and it takes the form

$$\left[\frac{d^2}{dz^2}-\frac{z^2}{4}+q+\frac{1}{2}\right]u_E^\pm(x)=0 \qquad (49)$$

where two independent solutions are those of the parabolic cylinder

$$u_E^\pm(x)=D_q(\pm z) \qquad (50)$$

We have now to take for the presence of the term $\delta(x-x')$ in Eq. (27) for which we integrate this equation with respect to x in the interval $x'-\lambda\leq x\leq x'-\lambda$ with $\lambda\rightarrow 0$. Taking into account that on the left hand side of (27) we have the term $(-\hbar^2/2m)d^2/dx^2$ whose integration give us $(-\hbar^2/2m)d/dx$ and multiplying the integrated Eq. (27) by $(-2m/\hbar^2)$ we see that the process gives

$$\left(\frac{dG_{ii}}{dx}\right)_{x=x'+0}-\left(\frac{dG_{ii}}{dx}\right)_{x-x'-0} = -\frac{2m}{\hbar^2}\frac{i\hbar}{2mc^2}(E\pm mc^2) \qquad (51)$$

With the exception of the factor $(E\pm mc^2)(2mc^2)^{-1}$ Eqs (49) and (51) are identical to the non-relativistic (12) and (13) of [1] for the oscillator when we replace in them $V(x)=(1/2)m\omega^2x^2$ and the notations E by $\epsilon\pm$ and p by q. We can then make use of the non-relativistic Green function (36) of [1] substituting in it E by $\epsilon\pm$ and adding the factor mentioned at the beginning of the paragraph and get

$$G_{ii}(x,x',E)=\left(\frac{E\pm mc^2}{2mc^2}\right)\sqrt{\frac{2m}{\pi\hbar\omega}}\Gamma\left(\frac{1}{2}-\frac{\epsilon_\pm}{\hbar\omega}\right) \times D_{\frac{\epsilon_\pm}{\hbar\omega}-\frac{1}{2}}\left(\sqrt{\frac{2m\omega}{\hbar}}x_>\right)D_{\frac{\epsilon_\pm}{\hbar\omega}-\frac{1}{2}}\left(-\sqrt{\frac{2m\omega}{\hbar}}x_<\right) \qquad (52)$$

where i=1, 2 correspond respectively to the $+,-$ sign on the right hand side. Once we have $G_{11}(x,x',E)$ and $G_{22}(x,x',E)$ we can obtain $G_{21}(x,x',E)$ and $G_{21}(x,x',E)$ through the Eqs. (23), (24) .

The $x_>, x_<$ stand for $x_>$=max(x,x'), $x_<=\min(x,x')$.

Thus we can determine the full Green function $G_{ij}(x,x',E)$ of the Dirac oscillator problem but we still need to fin $K(x,x',t)$ which as we indicate in Eq. (2) of [1] and (42) above can be written as

$$K_{11}=\frac{1}{2\pi\hbar}\int_{i\hbar\lambda-\infty}^{i\hbar\lambda+\infty}e^{-iEt/\hbar}\sqrt{\frac{2m}{\pi\hbar\omega}}\left(\frac{E+mc^2}{2mc^2}\right) \times\Gamma\left(\frac{m^2c^4-E^2}{2mc^2\hbar\omega}\right)D_{\frac{E^2-m^2c^4}{\hbar\omega 2mc^2}}\left(\sqrt{\frac{2m\omega}{\hbar}}x\right) \times D_{\frac{E^2-m^2c^4}{\hbar\omega 2mc^2}}\left(-\sqrt{\frac{2m\omega}{\hbar}}x'\right)dE \qquad (53)$$

where we restrict ourselves to $K_{11}(x,x',E)$ in which we just have ϵ_+ which we will denote by ϵ that from (26) has the value

$$\epsilon=\frac{E^2-m^2c^2}{2mc^2}+\frac{\hbar\omega}{2} \qquad (54)$$

so this index of the parabolic cylinder function is the one indicated (53) as we assume $x>x'$ so $x_>=x$ and $x_<=x'$.

The integral in (53) in the E plane is parallel to the real axis and at positive distance $\hbar\lambda$ from it. Because of the term $\exp(-iEt/\hbar)$ we can close the contour by a lower semicircle of infinit radius and the integration takes place over the contour C in Fig. 2.

The gamma function $\Gamma(z)$ can be expressed (formula 8.314 of p. 435 Ref. 5) as

$$\Gamma(z)=\int_1^\infty e^tt^{z-1}dt+\sum_{n=0}^\infty\frac{(-1)^n}{n!(z+n)} \qquad (55)$$

where the integral in (55) as well as the parabolic cylinder function

$$D_z\left(\pm\sqrt{\frac{2m\omega}{\hbar}}x\right)$$

are analytic function of z in the whole complex plane.

Because of the analytic properties mentioned $K_{11}(x,x',t)$ can be written as

$$K_{11}=\frac{1}{2\pi\hbar}\int_C dEe^{-iEt/\hbar}\left(\frac{E+mc^2}{2mc^2}\right)\sqrt{\frac{2m}{\pi\hbar\omega}}\left[\sum_{n=0}^\infty\frac{(-1)^n}{n!}\frac{1}{\frac{1}{2}-\frac{\epsilon}{\hbar\omega}+n}D_{\frac{\epsilon}{\hbar\omega}-\frac{1}{2}} \times\left(\sqrt{\frac{2m\omega}{\hbar}}x\right)D_{\frac{\epsilon}{\hbar\omega}-\frac{1}{2}}\left(-\sqrt{\frac{2m\omega}{\hbar}}x\right)\right]dE \qquad (56)$$

where C is the contour of Fig. 2 that can be reduced to the circles surrounding the values where

$$\frac{\epsilon}{\hbar\omega} - \frac{1}{2} = n \tag{57}$$

The index of parabolic cylinder functions is then n, but we have other terms in (56) as functions of E and to be able to evaluate (56) in terms of residues we need to determine E using (54) and (57). Substituting E of (54) in (57) we have

$$\frac{1}{(\frac{1}{2} - \frac{\epsilon}{\hbar\omega} + n)} = \frac{2mc^2\hbar\omega}{E_n^2 - E^2} = \frac{2mc^2\hbar\omega}{2E_n}\left(\frac{1}{E_n - E} + \frac{1}{E_n + E}\right) \tag{58}$$

where $E_n \equiv \sqrt{\hbar\omega 2mc^2 n + m^2c^4}$. If we make the substitution (58) in (56) we have to evaluate the residues in $E = \pm E_n$ which give us a factor $2\pi i$ and thus we get

$$K_{11}(x,x',t) = -i\omega\sqrt{\frac{2m}{\pi\hbar\omega}}\sum_{n=0}^{\infty}\left\{e^{-iE_n t/\hbar}\left(\frac{E_n + mc^2}{2mc^2}\right)\frac{(-1)^n}{n!}\frac{mc^2}{E_n}D_n\left(\sqrt{\frac{2m\omega}{\hbar}}x\right)D_n\left(-\sqrt{\frac{2m\omega}{\hbar}}x'\right)\right\}$$
$$+ i\omega\sqrt{\frac{2m\omega}{\pi\hbar\omega}}\sum_{n=0}^{\infty}\left\{e^{iE_n t/\hbar}\left(\frac{-E_n + mc^2}{2mc^2}\right)\frac{(-1)^n}{n!}\frac{mc^2}{E_n}D_n\left(\sqrt{\frac{2m\omega}{\hbar}}x\right)D_n\left(-\sqrt{\frac{2m\omega}{\hbar}}x'\right)\right\} \tag{59}$$

In a similar way we can get $K_{22}(x,x',t)$ and from (23, 24) we obtain $K_{12}(x,x',t), K_{21}(x,x',t)$. The results reduce to the non relativistic ones of (53) in Ref. 1 if we assume $c \to \infty$. Let us compute, for instance, K_{21}. Using (23), (24), the component of Green's function reads

$$G_{21}(x,x',t) = c\,(p - im\omega x)\,\frac{1}{2mc^2}\sqrt{\frac{2m}{\hbar\pi\omega}}\Gamma\left(\frac{1}{2} - \frac{\epsilon_+}{\hbar\omega}\right)D_{\frac{\epsilon_+}{\hbar\omega}-\frac{1}{2}}\left(\sqrt{\frac{2m\omega}{\hbar}}x_>\right)D_{\frac{\epsilon_+}{\hbar\omega}-\frac{1}{2}}\left(-\sqrt{\frac{2m\omega}{\hbar}}x_<\right) \tag{60}$$

while the inverse Laplace transform is written as

$$K_{21} = \left[(2\pi\hbar)^3 mc^2\omega\right]^{-\frac{1}{2}}(p - im\omega x)\int_{-\infty}^{\infty} dE e^{-iEt/\hbar}\Gamma\left(\frac{m^2c^4 - E^2}{2mc^2\hbar\omega}\right)D_{\frac{E^2 - m^2c^4}{2mc^2\hbar\omega}}\left(\sqrt{\frac{2m\omega}{\hbar}}x_>\right)$$
$$\times D_{\frac{E^2 - m^2c^4}{2mc^2\hbar\omega}}\left(-\sqrt{\frac{2m\omega}{\hbar}}x_<\right). \tag{61}$$

The integral can be evaluated through the calculation of residues as before, giving

$$K_{21}(x,x',t) = i\sqrt{\frac{m\omega}{2\pi\hbar}}c(p - im\omega x)\left\{\sum_{n=0}^{\infty}\left[E^{-iE_n}\frac{(-)^n}{n!E_n}\times D_n\left(\sqrt{\frac{2m\omega}{\hbar}}x_>\right)D_n\left(-\sqrt{\frac{2m\omega}{\hbar}}x_<\right)\right]\right.$$
$$\left. - \sum_{n=0}^{\infty}\left[E^{iE_n}\frac{(-)^n}{n!E_n}D_n\left(\sqrt{\frac{2m\omega}{\hbar}}x_>\right)D_n\left(-\sqrt{\frac{2m\omega}{\hbar}}x_<\right)\right]\right\} \tag{62}$$

which exhibits contributions from both positive and negative energies. Thus we can obtain the full propagator for the Dirac oscillator. It should be mentioned that the presence of infinit sums in our expressions is not an obstacle for their applicability. In fact, the authors have worked out previously an example involving the evolution of a wave packet in Ref. 6, where the convergence of these sums is proven.

In the present paper and in Ref. 1 we dealt only with one dimensional problems both non-relativistic and relativistic to make clear the main ideas of our developments.

We plan to extend our analysis to more dimensions so the procedure outlined in the abstract of this paper can cover the determination of all kinds of Feynman propagators.

5. Discussion on applications

In the years following the formulation of the Dirac oscillator [2], a number of publications appeared on the many-body generalization of this system. The applicability to the spectroscopy of quarkonia and non-strange baryons (where a relativistic approach is reasonable) was readily given. For a full description see [7] and references cited therein. It was clear though that the Dirac oscillator potential could be used only as a model interaction in the context of nuclear and subnuclear physics. However, the possibility of an effective realization appears now at the level of condensed matter as we shall indicate.

For several years, the band structure of electrons in materials such as graphene (two dimensional graphite) or boron nitride, has been of interest due to its Dirac-like structure [8, 9] with both effective speed $c \sim 10^6$ m/s [10] and effective mass. The latter appears in the case of binary crystals (graphene electrons are effectively massless) and for the case of boron nitride its value is proportional to the energy difference of excited electrons in boron and nitrogen. Recently, a renewed interest in these studies has led to experimental results [11] and novel theoretical treatments in the context of the Dirac theory [12]. In some idealization, $2+1$ dimensional electrons in the lowest conduction band behave as free Dirac particles and the presence of external interactions has been introduced through deformations of the hexagonal lattice describing the material, small external fields electron-electron interactions, etc.

In the case of lattice deformations with applications to nanotubes, we follow [13], where effective Hamiltonians contemplating interactions with phonons have been obtained near the Brillouin points located at opposite corners of the reciprocal hexagon. They have the form of the free Dirac Hamiltonian plus an effective potential of a quite general form (see eqns. (3.1,3.9,3.10) in Ref. 13). Such potential is given in terms of in-plane displacements as functions of the lattice points. One can notice that a suitable choice of such displacements leads to a $2 + 1$ dimensional Dirac oscillator (without mass term for graphene). Though it can be argued that setting a specifi deformation fiel could be an experimentally difficul task, deformations induced by external inhomogeneous temperature baths can be considered as a possibility. The latter is the subject of theoretical work in progress.

Finally. we would like to point out that the determination of the Feynman propagator seems to be a natural step in the study of these systems.

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