Darboux-deformed barriers and resonances in quantum mechanics

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Scattering states in the continuum are used as Darboux transformation functions to deform square barrier potentials. The results include complex as well as real new potentials. It is shown that an appropriate superposition of Breit-Wigner distributions connects the transmission coefficient of one dimensional short range potentials.

Keywords: Darboux transformations; Gamow vectors; Breit-Wigner ditribution.

Se usan estados de dispersión como funciones de transformación en el método de Darboux para obtener nuevos potenciales (reales o complejos) a partir de barreras cuadradas. Se muestra que una superposición apropiada de distribuciones de Breit-Wigner permite construir una buena aproximación del coeficiente de transmisión.

Descriptores: Transformaciones de Darboux; vectores de Gamow; distribución de Breit-Wigner.

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1. Introduction

The study of decaying systems was fundamental for establishing quantum mechanics as the theory of the microscopic world. Indeed, in his paper of alpha decay [1], Gamow imposed an *outgoing wave boundary condition* on the solutions of the Schrödinger equation for a radial symmetric potential (see also [2, 3]) and found that the involved eigenvalues are complex $\epsilon = E - i\Gamma/2$, $\Gamma > 0$. Thus, Gamow functions Ψ_G represent quasi-stationary (resonant) states (E and Γ^{-1} are the energy and lifetime, respectively). Resonant states u are formally scattering states in the continuum and usually refer to metastable behavior [4,5] (see also [6]). They behave asymptotically as purely outgoing waves (Gamow-Siegert functions) and are associated with poles of the S-matrix in the 4th quadrant of the complex ϵ-plane [7, 8]. The fact that Ψ_G diverges at large distances is usually stressed to motivate the study of the rigged Hilbert space [9], the mathematical structure of which lies on the spectral theorem of Dirac, Maurin, Gelfand and Vilenkin [10] (see also [11]). The usefulness of Gamow functions was realized also by Zel'dovich in 1961 [12], who proposed a procedure to define an appropriate norm, followed by Hokkyo [13], Berggren [14] and Romo [15]. Hence, the Gamow functions can be normalized to form a complete set of states but in a space which is not the Hilbert space.

In this paper we use a relaxed definition of the physically interpretable Gamow-Siegert functions in the context of Darboux-deformations of a given potential (see e.g. the review papers in Ref. 16). The Darboux-Gamow transformation has been previously applied to the harmonic oscillator [17, 18] and Coulomb-like potentials [19, 20]. In both cases, after the transformation, the discrete spectrum is preserved and, occasionally, it is enlarged by a single complex eigenvalue associated to normalized eigenfunctions (see also Ref. 21). Similar results have been obtained for short range attractive potentials [22, 23]. Indeed, the construction of Schrödinger solutions for complex eigenvalues is sound by itself (see, *e.g.* [24–27]).

This work presents a graphical method to obtain Gamow energies $\epsilon = \epsilon_R + i\epsilon_I$ associated with arbitrary shape short range potentials. In particular, we shall focus on the square barrier potential. As it will be seen, the Darboux transformation generates a complex potential while the further iteration of the procedure carries out a real barrier. The main result is that the supersymmetric partner (Darboux transformed) potentials are constructed via the scattering states, a procedure which has been neglected in almost all the literature on the subject (see though [20, 22, 23, 28]).

The next section introduces general expressions to generate short range 1-dimensional potentials by using Gamow vectors as transformation functions. In Sec. 3 we present a graphical method to obtain the Gamow energies in terms of a finite sum of Lorentz-Breit-Wigner distributions converging to the transmission coefficient of the involved potential. The method is applied to the square barrier potential. Finally, Sec. 4 contains some concluding remarks.

2. Darboux-Gamow transformations

For short range potentials in one dimension the resonances can be studied by means of the transmission coefficient in the regime of scattering energies. The real parts of the Gamow energies ϵ_R, represent the positions of the resonances and the absolute value of the imaginary part $|\epsilon_I|$, is proportional to the width $2|\epsilon_I| = \Gamma$ of the involved peaks. Following Taylor [29], we shall work on resonances for which the width is much less than the spacing between them, *i.e.* $\Gamma \ll \Delta E$. According to our definition, we look for solutions u of the Schrödinger equation ($\hbar/2m = 1$)

$$-u'' + Vu = \epsilon u \tag{1}$$

with $\mathbb{C} \ni \epsilon = k^2$, and the outgoing boundary condition:

$$\lim_{x\to\pm\infty} (u' \mp iku) = 0. \tag{2}$$

Now, let us introduce a function $\beta \equiv \beta(x,\epsilon)$ as follows

$$\beta \equiv -\frac{d}{dx}\ln u. \tag{3}$$

After this logarithmic transformation, the Schrödinger equation (1) is transformed to the Riccati one

$$-\beta' + \beta^2 + \epsilon = V, \tag{4}$$

which conveniently can be decomposed as

$$-\beta_R' + \beta_R^2 - \beta_I^2 + \epsilon_R = V, \tag{5}$$

$$-\beta_I' + 2\beta_R\beta_I + \epsilon_I = 0. \tag{6}$$

If $\beta_I = 0$, equation (6) leads to $\epsilon_I = 0$. In the sequel, we shall assume $\mathrm{Im}\beta \neq 0$ so that $\epsilon_R \neq 0$ and $\epsilon_I \neq 0$. The advantage of introducing the complex function β is that the potential V in (1,4,5) is automatically intertwined with a new one $\widetilde{V}$, by means of the Darboux transformation

$$\widetilde{V} = V + 2\beta'. \tag{7}$$

The solutions y of the corresponding Schrödinger equation

$$-y'' + \widetilde{V}y = \mathcal{E}y \tag{8}$$

are also easily obtained

$$y = \frac{\mathrm{W}(u,\Phi)}{u}, \tag{9}$$

where $\mathrm{W}(*,*)$ stands for the Wronskian of the involved functions and Φ is eigensolution of (1) with eigenvalue $\mathcal{E}$. For a general potential, if $\mathcal{E}_n$ is in the discrete spectrum of V, it is straightforward to show that y_n is a square integrable solution of (8) as Φ_n is in $L^2(a,b)$ and $\epsilon \notin \{\mathcal{E}_n\}/\mathcal{E}_0$ [17–20, 22]. In general, the members of the new set $\{y_n\}$ do not form an orthogonal set of vectors although they are normalizable [18, 19]. If $\mathcal{E}$ is in the scattering energies of V, then y will have the same global properties as Φ [19]. The new potential $\widetilde{V}$ is complex since $\beta \in \mathbb{C}$. Hence, the term $\mathrm{Im}\widetilde{V} \neq 0$ could modify the number of particles (see *e.g.* [30]). The iteration of the procedure is easy once a finite denumerable set of solutions u has been given [31, 32]:

$$V_2 = \widetilde{V} + 2\beta'(\alpha) \tag{10}$$

with $\beta(\alpha) \equiv \beta(x,\epsilon,\alpha)$ is a new function fulfilling

$$-\beta'(\alpha) + \beta^2(\alpha) + \alpha = \widetilde{V}, \quad \alpha \in \mathbb{C}. \tag{11}$$

The new potential V_2 is, in general, complex. However, if $\alpha = \bar{\epsilon}$, it becomes real [17, 19]:

$$V_2 = V - 2\mathrm{Im}\left(\frac{\epsilon}{\beta}\right)' \tag{12}$$

and the new solutions Ψ, such that $H_2\Psi = \mathcal{E}\Psi$, read

$$\Psi = (\epsilon - \mathcal{E})\Phi - y\mathrm{Im}\left(\frac{\epsilon}{\beta}\right). \tag{13}$$

3. Gamow Vectors for the square barrier

Let us consider equation (1) with the potential

$$V(x) = \begin{cases} 0, & x < -b/2 \\ V_0, & -b/2 \le x \le b/2 \\ 0, & b/2 < x \end{cases} \tag{14}$$

where $V_0 > 0, \quad b > 0$. For a particle coming from the left the general solution is

$$\Phi(x) = \begin{cases} Ae^{ikx} + A\frac{i(q^2-k^2)}{2\Delta(k,q)}(\sin qb)\, e^{-ik(x+b)}, & x < -b/2 \\ A\frac{ke^{\frac{-ikb}{2}}}{\Delta(k,q)}\Big(\Big[i\left(k\cos\frac{qb}{2} - iq\sin\frac{qb}{2}\right)\sin qx\Big] & \\ \quad + \Big[\left(q\cos\frac{qb}{2} - ik\sin\frac{qb}{2}\right)\cos qx\Big]\Big), & -b/2 \le x \le b/2 \\ A\frac{kq}{\Delta(k,q)}e^{ik(x-b)}, & b/2 < x \end{cases} \tag{15}$$

with A=constant, $k^2 = E, \quad q^2 = k^2 - V_0$, and

$$\Delta(k,q) = \left(k\cos\tfrac{qb}{2} - iq\sin\tfrac{qb}{2}\right)\left(q\cos\tfrac{qb}{2} - ik\sin\tfrac{qb}{2}\right). \tag{16}$$

Similar expressions hold for particles coming from the right.

The transmission coefficient T reads

$$T = \left|\frac{kq}{\Delta(k,q)}\right|^2. \tag{17}$$

It is clear that, for $\epsilon = E > 0$ and V_0 fixed, the coefficient

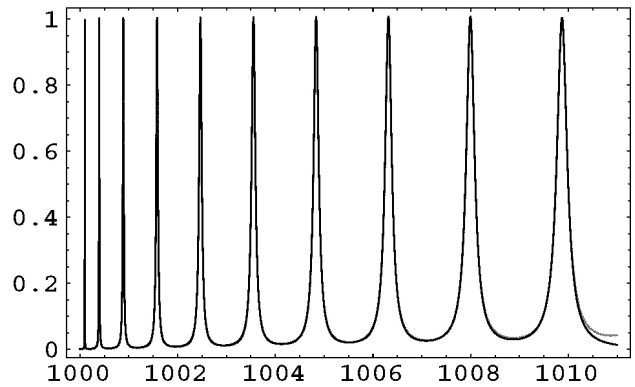

FIGURE 1. The functions T (solid curve) and ω_N (dotted curve) for $V = 1000$ and $b = 10$.

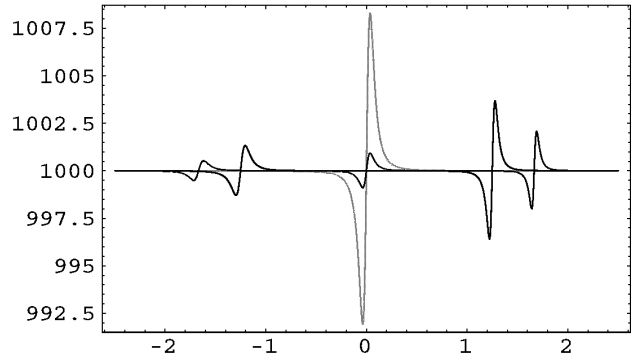

FIGURE 2. The top of the twice Darboux-deformed square barriers for $V_0 = 1000$, $b = 5$ with the first (gray curve), second (dotted curve) and third (solid curve) resonances.

T has a series of local maxima and minima. The maxima are reached for the values of E such that Δ is minimum. These energies are known as transparencies because the particles travel trough the barrier zone as if they were free of interactions [33]. If $0 < \epsilon = E < V_0$, then q is pure imaginary, there is not solution of $\Delta = 0$, so $0 < T < 1$, and the tunneling effect plays the principal role. Hence, neither the simple transparencies nor the tunneling waves can be interpreted as Gamow vectors. It is required a certain delay (phase shift) in the traveling of the involved wave functions. In other words, the transmission T should present a series of narrow peaks in order to include resonance energies (see Fig. 1). Such an effect is obtained by the adequate combination of the strength V_0 and the width b of the potential. In particular, we take $V_0 b^2/4 \gg 1$ [22].

Let us remark that coefficient A in (15) is arbitrary. Thereby, $A = \Delta(k, q)$ clearly provides the outgoing boundary condition of Φ for the points k such that $\Delta = 0$. Since the transmission coefficient T in (17) is exactly the same for these new parameters, the resonance picture is clear: the points $\epsilon_R = E$, for which $T = 1$, are the resonaces while the width $\Gamma = 2|\epsilon_I|$ of the peaks define the inverse of the lifetime (for $\epsilon_I < 0$). Each of the peaks correspond to a lorentz (Breit-Wigner) distribution centered at $\epsilon_R = E$:

$$\omega(\epsilon_R, E) = \frac{(\Gamma/2)^2}{(\epsilon_R - E)^2 + (\Gamma/2)^2}. \tag{18}$$

As it is well known, the superposition of a denumerable set of these distributions (each one centered at one resonance $E_n, \quad n = 1, 2, ...$) entails an approximation of the coefficient T such that the larger the number N of involved resonances, the higher will be the precision of the approximation:

$$T \approx \omega_N(\epsilon_R) = \sum_{n=1}^{N} \omega(\epsilon_R, E_n). \tag{19}$$

The Fig. 1 depicts the behaviour of the sum $\omega_N(\epsilon_R)$ as contrasted with the transmission coefficient T for a barrier of strength $V_0 = 1000$ and width $b = 10$. For these values of the parameters our method successfully connect ten Breit-Wigner distribution resonances with the first ten peaks of T. The precision is lost for large values of the energy ($E \gg V_0$), for which Γ is not narrow anymore as compared with ΔE. Finally, Fig. 2 shows the behavior of the twice transformed barrier as constructed by means of Eq. (12). The appearance of 'hair' over the top of these barriers induces stronger resonant phenomena. In general, the new scattering states Ψ will have a delay longer than for the initial ones.

4. Concluding remarks

We have found resonances above the top of a square barrier by using a superposition of (Lorentz) Breit-Wigner distributions centered at ϵ_R, each one of width $\Gamma = 2|\epsilon_I|$. The point $\epsilon = \epsilon_R + i\epsilon_I$ is in the fourth quadrant of the complex plane if $\epsilon_I < 0$, and defines outgoing solutions of the involved Schrödinger equation (compare with Ref. 34). These solutions transformed the initial barrier into a complex potential by means of the Darboux-deformations. The iteration of the method produced barriers with 'hair'. Both of these new kinds of potentials admit resonant states for which the global analytical behavior of the initial resonances are preserved. All these results can be seen as a supplement of previous works, including the cases of discrete initial spectra [17–21] and 'scaled interwined' potentials [35] (see also [26, 27]). Our method can be easily extended to 'soft' potentials (work in this direction is in preparation). Finally, we want to stress that, as far as we known, the use of scattering states in the continuum as transformation functions starts to be exhaustively studied in the literature. We hope that the present work has shed some light onto this subject.

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