Quantum control with periodic sequences of non resonant pulses

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The resonant quantum control techniques are vulnerable to cumulative errors specially if the manipulation operations involve many individual steps. It is shown that non trivial operations can be induced by non resonant fields challenging too simplified control models.

Keywords: Quantum control; time-dependent Hamiltonians; periodical systems.

El control cuántico resonante es sensible a la acumulación de errores, especialmente si el proceso involucra muchas operaciones individuales. Se muestra que, a través de campos externos no resonantes, se pueden inducir mecanismos no triviales de control que se contraponen a los modelos simplificados convencionales.

Descriptores: Control cuántico; Hamiltonianos dependientes del tiempo; sistemas periódicos.

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1. Introduction

The dynamics of quantum systems, represented by time dependent Hamiltonians $H(t)$, is encoded in the family of unitary operators $U(t,t_0)$ describing the evolution of the system in the time interval $[t_0,t]$ ($\hbar=1$):

$$\frac{d}{dt}U(t,t_0) = -iH(t)U(t,t_0), \qquad U(t_0,t_0)=\mathbf{1}. \tag{1}$$

The essential problem in the control theory is the generation of arbitrary unitary operations causing a desired evolution of the quantum system. The main mathematical dilemma is to determine the proper Hamiltonian $H(t)$ which generates a given $U(t,t_0)$ according to (1).

These evolution operators must have exponential representations in terms of Hermitian operators $H_{\rm eff}(t,t_0)$, called the *effective Hamiltonians* of the system in $[t_0,t]$, defined by the Baker-Campbell-Hausdorff (BCH) formulae [1–3]:

$$U(t,t_0) = e^{-i(t-t_0)H_{\rm eff}(t,t_0)}. \tag{2}$$

The construction of the effective Hamiltonians (2) for a given quantum system is, in general, a quite complicated task, even in the case of finite dimensional systems. The theoretical group techniques may, in some cases, provide the desired $U(t,t_0)$ as a product of a long sequence of elementary steps [4–6]; however, such formal solutions are not always the most convenient from the practical point of view.

One of the typical tools of quantum control is to induce the dynamical operations by means of soft, patiently repeated pulses. Suppose, *e.g.*, an initially stationary system (represented by H_0) is being manipulated by an external field. If the strength ε of this field is small enough, and if the probability of causing radiative transitions is negligible, the proper description is then semiclassical, *i.e.*, the stationary system H_0 has quantum mechanical properties while the manipulation part $V(t)$ typically depends on certain classical parameters or fields:

$$H(t) = H_0 + \varepsilon V(t). \tag{3}$$

One of the most interesting exact solutions of (1-3) generalizing the *rotating wave approximation* arises if

$$V(t) = e^{-i\Omega t} V e^{i\Omega t}, \tag{4}$$

with V a time independent operator and $[\Omega, H_0]=0$. The corresponding evolution $U(t)=U(t,0)$ becomes :

$$U(t) = e^{-i\Omega t} e^{-i[H_0-\Omega+\varepsilon V]t} \tag{5}$$

(we put $t_0=0$), where the first factor is the unitary transformation to the "rotating frame" and the second one can be interpreted as the evolution operator in the Ω-frame. By taking $\Omega=H_0$ one obtains the simplest exact description of the resonant process (the generalized Rabi rotations).

2. The Resonant control

A frequently used mechanism in control theory is to generate the evolution operations by taking full advantage of the sensitivity of the system to pulses of some particular frequencies (the resonant control technique). In this case, the external field is generally of the form:

$$V(t) = \frac{1}{2}\left[e^{i\omega t}V + e^{-i\omega t}V^\dagger\right]. \tag{6}$$

Such harmonic, monochromatic field is *resonant* to H_0, in the conventional sense, if $\omega=\omega_0$, with $\omega_0 = E_l - E_m$ the difference of any pair of H_0 eigenvalues. Notice that

$U_0(T_0) = e^{-iH_0T_0}$, $T_0 = 2\pi/\omega_0$, will be, in this case, proportional to $\mathbf{1}$ in the spectral subspaces $\mathcal{H}_l$, $\mathcal{H}_m$ corresponding to this pair of energy levels. In turn, the evolution operator $U(T_0)$ will then be proportional to $e^{-i\varepsilon VT_0}$: a small "rotation" in $\mathcal{H}_l \oplus \mathcal{H}_m$ which becomes cumulative if the operation is applied over many oscillation periods of the external field. A perturbative description is therefore not proper in this case, even if ε is small. To the contrary, this cumulative effect do not occur in the non resonant case $\omega \neq \omega_0$: the external field will only produce a slight *precession* of the system around its basic evolution $U_0(t) = e^{-iH_0t}$ [7,8].

2.1. The Rabi rotations

In order to provide a clear picture of this situation, consider the simplest case of a two-level system (a *qubit*) with corresponding states and energies $|1\rangle$, $|2\rangle$ and E_1, E_2 respectively, perturbed by the monochromatic, harmonic field (6) in the dipole approximation:

$$H(t) = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| + \frac{\varepsilon}{2}\left[e^{i\omega t}|1\rangle\langle 2| + e^{-i\omega t}|2\rangle\langle 1|\right]. \tag{7}$$

The state of the system, at an arbitrary time t, is

$$|\phi(t)\rangle = a_1(t)|1\rangle + a_2(t)|2\rangle, \tag{8}$$

where $a_1(t)$, $a_2(t)$ are the respective probability amplitudes of finding the system in the states $|1\rangle$, $|2\rangle$ (under the initial condition $a_1(0) = 1$, $a_2(0) = 0$) at that moment:

$$|a_2(t)|^2 = \frac{\varepsilon^2}{\omega_1^2}\sin^2\frac{\omega_1}{2}t, \qquad |a_1(t)|^2 = 1 - |a_2(t)|^2, \tag{9}$$

with $\omega_1 = \sqrt{\varepsilon^2 + (\omega_0 - \omega)^2}$ [9, 10]. In the resonant case ($\omega = \omega_0$), (8) reduces to

$$|\phi(t)\rangle = \cos\frac{\varepsilon}{2}t|1\rangle + \sin\frac{\varepsilon}{2}t|2\rangle, \tag{10}$$

which represents a rotation of the state of the system between $|1\rangle$, $|2\rangle$, with an angular frequency $\varepsilon/2$. If the field is weak enough, the process is slow and free of radiative jumps. This phenomenon is known as "Rabi rotation" [9]. When the process is continued for a complete period of time, the evolution operator $U(T_0)$ coincides with $e^{-i\varepsilon VT_0}$, indicating that the rotations can be accumulated by applying the operation a number of times. The system can then be softly manipulated: once in its ground state $|1\rangle$, it slowly collects information from the exterior and makes a transition to the excited state $|2\rangle$; next, it returns to $|1\rangle$ and so on.

To the contrary, if the frequency of the driving field is non resonant, the basic evolution operator $U_0(t)$ produces a violent draft of the system along its natural evolution orbit and the precession effects are hardly visible. Even so, the transitions $|1\rangle \leftrightarrow |2\rangle$ are not completely suppressed in this frequency regime. In fact, the maximum probability (which is reached within a time $t = \pi/\omega_1$) of such a transition is, in this case, ε^2/ω_1^2 and it is non zero for $\omega \neq \omega_0$. Figure 1 shows this maximum probability as a function of ω for different values of ε. There is a marked peak in the resonance which is sharper as the strength of the field tends to zero. The mean width of the curve is 2ε, telling the transitions occur just in the resonance regime if the intensity of the field is small enough.

2.2. A geometric picture

An elementary illustration of the phenomenon previously described [11–13] corresponds to the Hopf π-map $S^3 \to S^2$ [14–16]. The states $|\phi(t)\rangle$ of the system can be associated with the points of a unitary two-sphere S^2 (the Poincaré sphere) with its center at the origin of a frame defined by the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, provided that orthogonal states correspond to antipodal points [11]. The Hamiltonian (7), expressed in terms of spin Pauli matrices (take for simplicity $E_1 = -\omega_0/2$, $E_2 = \omega_0/2$), has the form:

$$H(t) = \frac{\omega_0}{2}\sigma_3 + \frac{\varepsilon}{2}e^{-i\frac{\omega}{2}t\sigma_3}\sigma_1 e^{i\frac{\omega}{2}t\sigma_3}. \tag{11}$$

(compare with (4)). If $\varepsilon = 0$, the evolution operator

$$U(t) = e^{-i\frac{\omega_0}{2}t\sigma_3}$$

causes rotations of a state $|\phi(t)\rangle$ upon the sphere surface with an angular frequency $\omega_0/2$ around $\mathbf{e}_3$ (the natural evolution trajectory, see Fig. 2). When the external field is applied, Eq. (5) reads

$$U(t) = e^{-i\frac{\omega}{2}t\sigma_3}e^{-i\frac{1}{2}t[(\omega_0-\omega)\sigma_3+\varepsilon\sigma_1]}. \tag{12}$$

In the rotating frame, $|\phi(t)\rangle$ describes a circular trajectory around $\mathbf{e}_3' = (\varepsilon/|\omega_0 - \omega|)\mathbf{e}_1 + \mathbf{e}_3$ with an angular frequency $\omega_1/2$ (see Fig. 2). If $\varepsilon \ll |\omega_0 - \omega|$, this trajectory practically coincides with the basic one (the angle between $\mathbf{e}_3$ and $\mathbf{e}_3'$ is $\alpha = \arctan\varepsilon/|\omega_0 - \omega|$), meaning that the whole evolution process is confined in a narrow belt of width $\delta\theta = 2\alpha$ around the orbit defined by H_0. In the inertial frame $\mathbf{e}_3'$ rotates around $\mathbf{e}_3$ with an angular frequency ω and $|\phi(t)\rangle$ precesses around the natural evolution orbit keeping its position on the sphere confined again to the same belt (see Fig. 2).

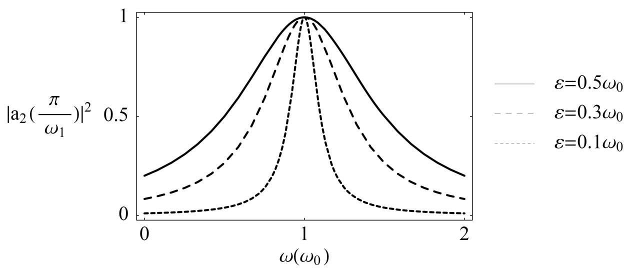

FIGURE 1. The maximum probability of finding the system in the excited state has a peak which is sharper as $\varepsilon \to 0$.

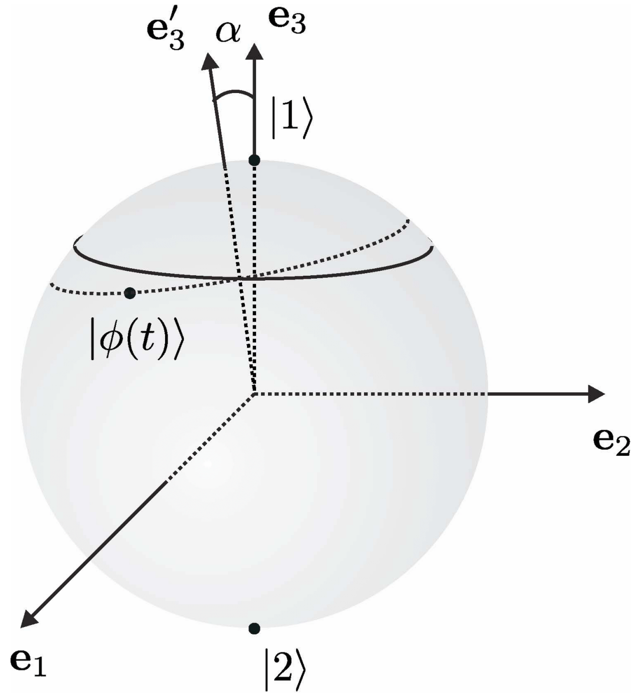

FIGURE 2. If $\varepsilon = 0$, $|\phi(t)\rangle$ describes a circular trajectory orthogonal to $\mathbf{e}_3$ (solid curve). When $\varepsilon \neq 0$, in the rotating frame, $|\phi(t)\rangle$ circulates around $\mathbf{e}_3'$ (dotted curve). This vector rotates around $\mathbf{e}_3$ from the point of view of the inertial frame, meaning the evolution is precessing around the unperturbed H_0-orbit. In both cases, the trajectory is confined to a sharp belt of width 2α.

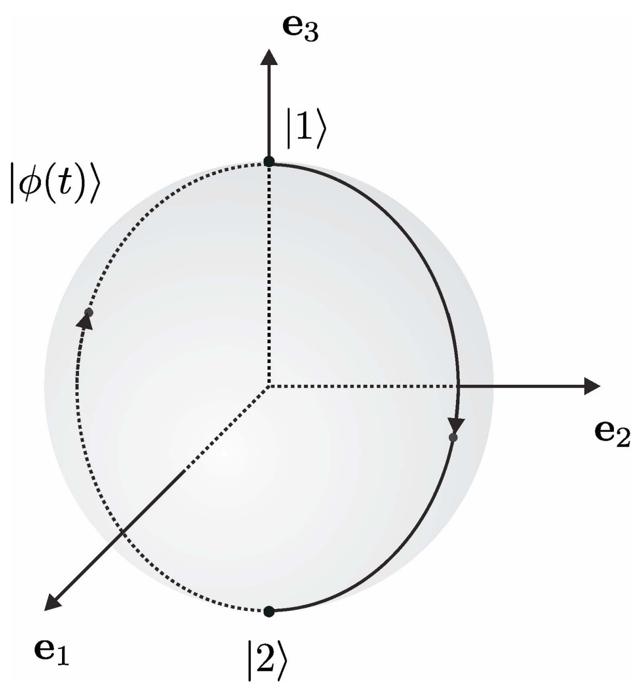

FIGURE 3. When the external field is resonant, the system performs Rabi rotations.

However, in the exactly resonant case $\Omega = H_0$,

$$U(T_0) = e^{-i\frac{\varepsilon}{2}\sigma_1 T_0}$$

and $|\phi(t)\rangle$ rotates around $\mathbf{e}_1$ with an angular frequency $\varepsilon/2$. After one period T_0 the polar position θ changes by $\delta\theta = \pi\varepsilon/\omega_0$, and even when ε is small, the process can be superposed by a number of times until $\delta\theta$ reaches an arbitrary value (the Rabi rotations between $|1\rangle, |2\rangle$, see Fig. 3).

The difference between the evolution of Fig. 2 and that of Fig. 3 is usually taken as an argument that if $\varepsilon/|\omega_0 - \omega| \ll 1$, then the resonant control operations upon one qubit (Fig. 3) can be selectively performed with negligible non resonant consequences for the other ones (Fig. 2). However, the control problem is far from being completely solved.

3. The selectivity breaking in the step by step operations

A very important amendment appears for systems composed by a great number of qubits. In this case, the most efficient control techniques involves operations developed step by step, always with a monochromatic field of frequency proper to manipulate just one part of the system (a qubit or a pair of levels) [4–6, 17]. Consistently with the mechanism of Fig. 2, it is assumed that the non resonant qubits will ignore this field. This is indeed true for the proper parameters ω, ε, but always under the condition that the external field is applied in a continuous way. It turns out however that, for the interrupted sequences of pulses, the final response may be, in spite of all, unexpected [18].

To illustrate this phenomenon suppose one has a pair of noninteracting qubits Q_ω, Q_{ω_0} with characteristic frequencies ω, ω_0 respectively. Suppose also that a unitary operation is carried out on Q_ω by means of a harmonic field of strength ε and frequency ω acting during a time T_1. Then, this field is turned off (presumably while other fields are being applied on Q_{ω_0}) for a time τ_1 before it is turned on again, etc. The evolution operators in a complete period of time ($T = T_1 + \tau_1$), are

$$U_{\omega_0}(T) = e^{-i\frac{\omega_0}{2}\tau_1\sigma_3} e^{-i\frac{\omega}{2}T_1\sigma_3} e^{-i\frac{\varepsilon}{2}T_1[(\omega_0-\omega)\sigma_3+\varepsilon\sigma_1]}$$

$$= e^{-i\frac{1}{2}[\omega_0\tau_1+\omega T_1]\sigma_3} e^{-iT_1 G}, \qquad (13)$$

$$U_\omega(T) = e^{-i\frac{\omega}{2}(T_1+\tau_1)\sigma_3} e^{-i\frac{\varepsilon}{2}T_1\sigma_1}. \qquad (14)$$

Let now T_1 and τ_1 be such that e^{-iT_1G} and $e^{-i(1/2)[\omega_0\tau_1+\omega T_1]\sigma_3}$ generate rotations of π-rad on Q_{ω_0} states around $\mathbf{e}_3'$ and $\mathbf{e}_3$ respectively, while $e^{-i(\omega/2)(T_1+\tau_1)\sigma_3}$ generates also rotations of π-rad on Q_ω states around $\mathbf{e}_3$, then

$$\omega_1 T_1 = (2n+1)\pi, \qquad (15)$$

$$\omega_0\tau_1 + \omega T_1 = (2m+1)\pi, \qquad (16)$$

$$\omega\,(T_1+\tau_1) = (2l+1)\pi, \qquad (17)$$

where $n, m, l \in \mathbb{Z}^+$. If the sequence is applied many times, something extraordinary happens: Q_{ω_0} (the non resonant qubit!) experiences an effect similar to the Rabi rotations (see Fig. 4).

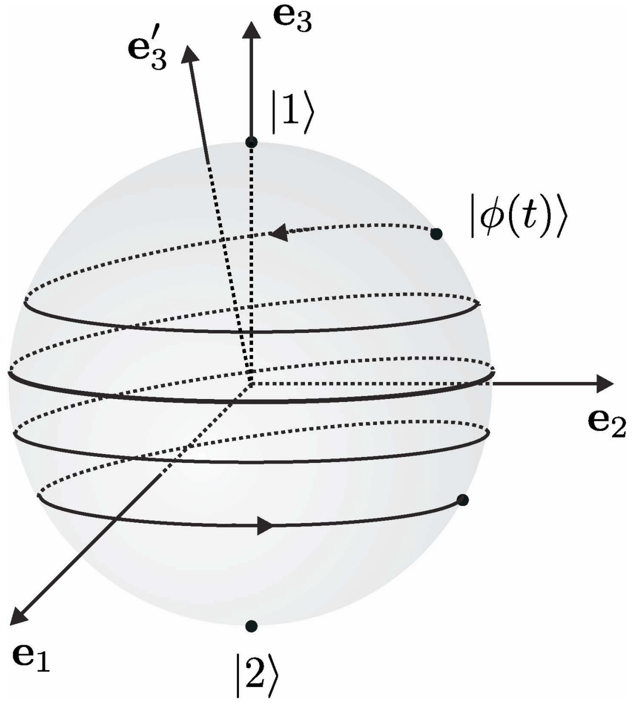

FIGURE 4. A sequence of interrupted pulses applied in the proper time intervals, may cause the escape of the system out of the confinement zone. In the most dramatic case, the system may experiment an effect similar to the Rabi rotations (see [18]).

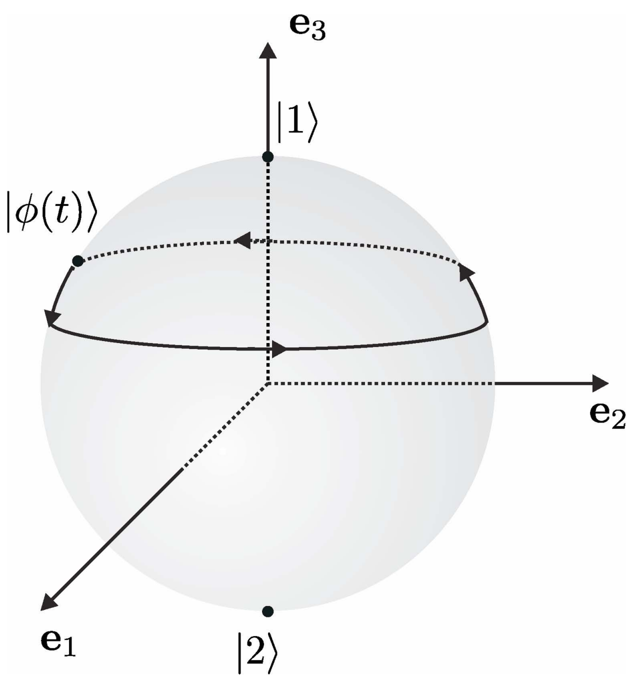

FIGURE 5. If a sequence of resonant pulses are applied in an interrupted way in the proper time intervals, the Rabi rotations may be completely annihilated (following [19]).

In fact, the effective rotation of the state of Q_{ω_0} after a time T is

$$\delta\theta = 2\arctan\frac{\varepsilon}{|\omega_0-\omega|}$$

in the plane generated by $\mathbf{e}_3$ and $\mathbf{e}'_3$. After applying several times this "on" and "off" sequence, Q_{ω_0} will have escaped from the confinement zone. The evolution of the system will be comparable to the resonant "stationary" one, changing the polar position θ of its state at will. What is happening meanwhile to Q_ω? It turns out that the alphabet of the "on-off" pulses is, in the same way, determinant. If the sequence is applied an even number of times, the Rabi rotations will be completely annihilated (see Fig. 5).

Now a question arises: is it possible that these phenomena, namely, the *revival* of the non resonant effects and the total annihilation of the Rabi rotations, simultaneously happen? The answer is YES!. From (15-17) it follows that

$$\frac{2n+1}{\omega_1}+\frac{2(m-l)}{\omega_0-\omega}=\frac{2l+1}{\omega}. \tag{18}$$

The question is now: for which values of n, l, m this expression is held for all values of ε? There are different families of cases [19]:

(i) For $0<\omega_0<\omega$, (18) implies $n<m<l$ and $(2l+1)/(2m+1)\omega_0<\omega<(2l+1)/(2(m-n))\omega_0$. The right hand end of this interval corresponds to $\varepsilon\to 0$ while the left hand one to the case $\varepsilon\to\infty$.

(ii) For $0<\omega<\omega_0$, (18) states that $l<m, l-m<n$ and $(2l+1)/(2(n+m+1))\omega_0<\omega<(2l+1/2m+1)\omega_0$, where now the left hand end corresponds to the case $\varepsilon\to 0$ and the right hand one to $\varepsilon\to\infty$.

Time intervals T_1 and τ_1 can then be evaluated from these ranges of frequencies, for a chosen triplet (n,l,m). It is worthwhile to mention that, in both cases, the non resonant cumulative rotations on Q_{ω_0} can reach an angular speed $\omega'\sim\varepsilon/\pi$, which is comparable to that in the Rabi rotations (namely $\varepsilon/2$). It can be shown that this effects could have a resonant reinterpretation when $\varepsilon\to 0$; in any other case they have a pure non resonant nature [18].

It seems to be that this phenomena become specially significant only for an unlikely coincidence in the parameters in these periodical on-off sequences, however, there are some evidences that when applying arbitrary sequences of interrupted pulses, the state of the system will perform an erratic movement indicating that it is out of control [18]. This is a very important challenge, *e.g.*, in designing a quantum computer, since after some number of operations, the system will not be able to store information. Yet, not all the news are bad, since there is an evidence that the off resonant effects not always destabilize the system: if orderly applied, they might be also used to maintain the stability [18], permitting then to manipulate the system in a more precise way.

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