The Kodama state for topological quantum field theory beyond instantons

R. Cartas-Fuentevilla and J.F. Tlapanco-Limón
Instituto de Física, Universidad Autónoma de Puebla,
Apartado postal J-48 72570 Puebla Pue., México,
e-mail: rcartas@sirio.ifuap.buap.mx, tlapanco@sirio.ifuap.buap.mx
Recibido el 1 de mayo de 2006; aceptado el 1 de noviembre de 2006

We constructed a symplectic structure that preserves the symmetries including the topological invariance for topological Yang-Mills theory, and it is shown that the Kodama (Chern-Simons) state traditionally associated with a topological phase of unbroken diffeomorphism invariance for instantons, exists actually for the complete topological sector of the Yang-Mills theory.

Keywords: Teorías topológicas; estado de Kodama; instantones.

Se construye una estructura simpléctica que preserva las simetrías, incluida la invariancia topológica, de la teoría de Yang-Mills topológica, se muestra que el estado de Kodama (Chern-Simons) generalmente asociado a instantones existe per se para el sector topológico de la teoría, sin la necesidad de imponer condiciones de auto-dualidad.

Descriptores: Topological theories; Kodama state; instantons.

PACS: 02.40.-k; 11.15.-q; 03.70.+k

1. Introduction

Yang-Mills theory in four dimensions admits the so called Chern-Simons wavefunction as an exact zero energy eigenfunction of the Schrödinger equation [1]. This solution presents deep problems, since such a state is neither normalizable, nor invariant under CPT and negative helicity states not only have negative energy but also negative norm. Therefore the Chern-Simons state is not admissible as the ground state of the quantum Yang-Mills theory [2]. Despite all these properties, it is important to understand what this intriguing state describe.

The self-duality condition on the fields associated with instantons play a important role producing the deformation of the original action into a topological action, picking up thus a topological phase among various ground states of that theory. That state turn out to be the only quantum state for the topological quantum field theory obtained [5, 6]. Therefore, it is natural to associate the Chern-Simons (or Kodama) state with the sector corresponding to instantons.

We shall show that the Chern-Simons state is associated actually with the whole of the topological sector of the Yang-Mills theory (TYM), without using the self-duality conditions for instantons, provided that we start from the appropriate topological action, given in Yang-Mills case by the second Chern class.

2. TYM theory

We can construct a YM theory from a topological invariant, the second Chern class,

\[ S_{TYM}(A) = \beta \int_{M} \text{Tr} (F \wedge F), \]

where \( \beta \) is a parameter, \( A \) is the gauge connection and \( F = dA + A \wedge A \) its curvature; \( d \) and \( \wedge \) correspond to the exterior derivative and the wedge product on \( M \), which we assume as the four-dimensional Minkowski spacetime.

The topological Yang-Mills action (1) is the subject of the present study forgotten at all the self-duality condition.

3. The symplectic structure for TYM theory

We can construct from the action (1) a symplectic structure that preserves all relevant symmetries of the theory given by [8, 10]

\[ \omega = \int_{\Sigma} 4 \beta \text{Tr} \delta \tilde{F}^{\mu
\nu} \delta A_{\mu} d\Sigma_{\mu}. \]

where \( \Sigma \) is a Cauchy hypersurface, and \( \delta \) corresponds to the exterior derivative on the phase space \( \tilde{Z} \) of the theory [9].

It is important remark that \( \omega \) retains all the symmetries of the topological action. \( \omega \) is an gauge invariant symplectic structure, is a topological invariant and independent on the choice of \( \Sigma \) because it is covariantly conserved [10].
4. Classical and Quantum Hamiltonian for TYM theory and the Chern-Simons state

Our simplectic structure gives us the canonical variables and the (symmetric and gauge-invariant) energy-momentum tensor for TYM theory [9, 10].

If we consider that \( d\Sigma \mu \) is a time-like vector field in Eq. (3) we can obtain in particular the following (non-covariant) description of the phase space,

\[
\omega = \int \Sigma \frac{4}{\beta} \text{Tr} (\delta \tilde{F}^{0i} \wedge \delta A_i),
\]

in order to make contact with [2, 5]. Equation (4) shows explicitly that the canonical variables for TYM theory are given by \((2\beta)\tilde{F}^{0i}\) and \(A_i\).

In other hand the energy-momentum tensor for TYM theory is given by

\[
T^{\mu\nu} = -\text{Tr} \frac{4}{\beta} (F^{(\mu \alpha)} \tilde{F}^{0i} \wedge \delta A_i),
\]

which is classically zero, as expected for a topological action.

Using the component \(T_{00}\) and the classical-quantum correspondence

\[
(2\beta)\tilde{F}^{0i} \rightarrow i \frac{\delta}{\delta A_i},
\]

we find the quantum Hamiltonian

\[
H_Q = \int d\Sigma \text{Tr} F^{0i} (i \frac{\delta}{\delta A_i} - \beta \epsilon_{ijk} F_{jk}).
\]

In the temporal gauge \( A_0 = 0 \), it is easy to show that \([F_{0i}, \delta/\delta A_i] = 0\), and thus we have no ordering ambiguity in the quantum Hamiltonian (7). Thus, any wave function \(\psi\) satisfying

\[
(i \frac{\delta}{\delta A_i} - \beta \epsilon_{ijk} F_{jk})\psi = 0,
\]

will correspond to a state of zero energy for the Hamiltonian (7). Therefore, the solution for Eq. (8) is given by

\[
\psi(A) = e^{-4\pi i \beta I(A)},
\]

where \(I\) is the Chern-Simons functional

\[
I = \frac{1}{4\pi} \int d^3 x \text{Tr} (\epsilon^{ijk} (A_i \partial_j A_k + \frac{2}{3} A_i A_j A_k)).
\]

Equation (9) is essentially the Chern-Simons wave function [3, 4].

We conclude then that the Chern-Simons state corresponds strictly to a (topological) state of the TYM theory, without invoking the self-duality condition on the fields. Thus, such a state is associated with the topological phase of unbroken diffeomorphism invariance of the complete topological sector.

The Hamiltonian (7) is purely a combination of constraints, with \(F_{0i}\), the dual of the canonical momentum, playing the role of a Lagrange multiplier field [10].

5. Concluding remarks

The Chern-Simons state exists for the complete topological sector of the theory, and in order to establish its existence, neither the self-dual condition nor the Yang-Mills equations are required. The Bianchi identity becomes the generator of gauge symmetries at quantum level. As a particular case, the topological phases of Yang-Mills theory can be obtained invoking self-duality.

Acknowledgments

This work was supported by the Sistema Nacional de Investigadores and Conacyt (México).

2. E. Witten, gr-qc/0306083.