

On the WKB approximation of noncommutative quantum cosmology

E. Mena, O. Obregón, and M. Sabido

Instituto de Física de la Universidad de Guanajuato,

P.O. Box E-143, 37150 León Gto., México,

e-mail: emena@fisica.ugto.mx, octavio@fisica.ugto.mx, msabido@fisica.ugto.mx

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In this work we propose a formalism to introduce the effects of noncommutativity to classical cosmological models. The method is based on noncommutative quantum cosmology and by means of a WKB type approximation the noncommutative classical solutions can be obtained.

Keywords: non-commutativity; quantum cosmology.

En este trabajo se propone un formalismo para introducir los efectos de la no conmutatividad a modelos cosmológicos clásicos. El método esta basado en la cosmología cuántica no conmutativa en la cual por medio de la aproximación WKB se obtienen las soluciones clásicas no conmutativas.

Descriptores: No conmutatividad; cosmología cuántica.

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1. Introduction

During the early days of quantum mechanics and quantum field theory, continuous space-time and Lorentz symmetry was considered inappropriate to describe the small scale structure of the universe. It was also argued that one should introduce a fundamental length scale, limiting the precision of position measurements. The introduction of fundamental length is suggested to cure the ultraviolet divergencies occurring in quantum field theory. H. Snyder was the first to formulate these ideas mathematically [1], introducing noncommutative coordinates brings an uncertainty in the position. The success of the renormalisation made people forget about these ideas for some time. But when the quantization of gravity was considered thoroughly, it became clear that the usual concepts of space-time are inadequate and that space-time has to be quantised or noncommutative, in some way.

Quantum cosmology, is a simplified approach to the study of the very early universe, which means that the gravitational and matter variables have been reduced to a finite number of degrees of freedom (these models were extensively studied by means of Hamiltonian methods in the 1970's, for reviews see [2, 3]); for homogenous cosmological models the metric depends only on time, this permits to integrate the space dependence and obtain a model with a finite dimensional configuration space, *minisuperspace*, whose variables are the 3-metric components. One way to extract useful dynamical information is through a WKB type method. The semiclassical or WKB approximations is usually discussed in text books on nonrelativistic quantum mechanics in the context of stationary states, *i.e.*, determination of the energy eigenvalues and eigenfunctions [4]. This approximation can also be used to obtain approximate and in some cases exact solutions of the dynamical problem, *i.e.*, full Schrödinger equation, so the utility of the semiclassical approximation in obtaining exact solutions of the Schrödinger equation has not yet fully ex-

plored. The same seems to be the case for the relativistic quantum mechanics. The importance of the semiclassical approximation in the relativistic case is probably best appreciated in quantum cosmology [5], specifically, in the analysis of the Wheeler-DeWitt equation which is essentially a Klein-Gordon equation on the minisuperspace [6].

In the last few years there have been several attempts to study the possible effects of noncommutativity in the classical cosmological scenario [7, 8]. In Ref. 9 the authors avoid the difficulties of analyzing noncommutative cosmological models, if these would be derived from the full noncommutative theory of gravity. Their proposal introduces the effects of noncommutativity at the quantum level, namely quantum cosmology, by deforming the minisuperspace through a Moyal deformation of the Wheeler-DeWitt equation, similar to noncommutative quantum mechanics [10].

The aim of this work is to apply a WKB type method to noncommutative quantum cosmology, and find the noncommutative classical solutions, avoiding in this way the difficult task to solve this cosmological models in the complicated framework of noncommutative gravity [11]. We know how to introduce noncommutativity at a quantum level, by taking into account the changes that the Moyal product of functions induces on the quantum equation, and from there calculate the effects of noncommutativity at the classical level. This also has the advantage that for some noncommutative models for which the quantum solutions can not be found, the noncommutative classical solutions arise very easily from this formulation. This procedure is presented through an example, the Kantowski-Sachs cosmological.

2. The Cosmological Kantowski-Sachs Model

The Kantowski-Sachs Universe is one of the simplest anisotropic cosmological models [12]. The Kantowski-Sachs line element is

$$ds^2 = -N^2 dt^2 + e^{(2\sqrt{3}\beta)} dr^2 + e^{(-2\sqrt{3}\beta)} \times e^{(-2\sqrt{3}\Omega)} (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (1)$$

From the general relativity lagrangian we can construct the canonical momenta,

$$\Pi_\Omega = -\frac{12}{N} e^{-\sqrt{3}\beta - 2\sqrt{3}\Omega} \dot{\Omega}, \quad \Pi_\beta = \frac{12}{N} e^{-\sqrt{3}\beta - 2\sqrt{3}\Omega} \dot{\beta}, \quad (2)$$

and the corresponding Hamiltonian

$$H = \frac{N}{24} e^{-\sqrt{3}\beta - 2\sqrt{3}\Omega} \left[-\Pi_\Omega^2 + \Pi_\beta^2 + 48e^{-2\sqrt{3}\Omega} \right], \quad (3)$$

by canonical quantization we obtain the Wheeler-DeWitt (WDW) equation, using the usual identifications

$$\Pi_\Omega = -i \frac{\partial}{\partial \Omega} \quad \text{and} \quad \Pi_\beta = -i \frac{\partial}{\partial \beta}$$

we get

$$\left[\frac{\partial^2}{\partial \Omega^2} - \frac{\partial^2}{\partial \beta^2} - 48e^{-2\sqrt{3}\Omega} \right] \psi(\Omega, \beta) = 0, \quad (4)$$

in this parametrization the WDW equation has very simple form; and the solutions to this equation are given by

$$\psi = e^{\pm i\nu\sqrt{3}\beta} K_{i\nu} \left(4e^{-\sqrt{3}\Omega} \right), \quad (5)$$

where ν is the separation constant and $K_{i\nu}$ are the modified Bessel functions. We now proceed to apply the WKB method. For this we propose the wave function

$$\Psi(\beta, \Omega) = e^{i(S_1(\beta) + S_2(\Omega))}, \quad (6)$$

the WKB approximation is reached in the limit

$$\left| \frac{\partial S_1^2(\beta)}{\partial \beta^2} \right| \ll \left(\frac{\partial S_1(\beta)}{\partial \beta} \right)^2, \quad \left| \frac{\partial S_2^2(\Omega)}{\partial \Omega^2} \right| \ll \left(\frac{\partial S_2(\Omega)}{\partial \Omega} \right)^2, \quad (7)$$

yielding the Einstein-Hamilton-Jacobi (EHJ) equation

$$-\left(\frac{\partial S_2(\Omega)}{\partial \Omega} \right)^2 + \left(\frac{\partial S_1(\beta)}{\partial \beta} \right)^2 - 48e^{-2\sqrt{3}\Omega} = 0, \quad (8)$$

solving Eq.(8) gives the functions S_1, S_2 , and using the definition for the momenta

$$\Pi_\beta = \frac{dS_1(\beta)}{d\beta}, \quad \Pi_\Omega = \frac{dS_2(\Omega)}{d\Omega}, \quad (9)$$

which combined with Eq.(2) and fixing the value of $N(t) = 24e^{-\sqrt{3}\beta - 2\sqrt{3}\Omega}$ we find

$$S_1(\beta) = P_{\beta_0} \beta, \quad S_2(\Omega) = -\frac{1}{\sqrt{3}} \sqrt{P_{\beta_0}^2 - 48e^{-2\sqrt{3}\Omega}} + \frac{P_{\beta_0}}{\sqrt{3}} \operatorname{arctanh} \left[\frac{\sqrt{P_{\beta_0}^2 - 48e^{-2\sqrt{3}\Omega}}}{P_{\beta_0}} \right], \quad (10)$$

from this solutions and using (2) and (9) we obtain the classical solutions

$$\Omega(t) = \frac{1}{2\sqrt{3}} \ln \left[\frac{48}{P_{\beta_0}^2} \cosh^2 \left(2\sqrt{3}P_{\beta_0}(t - t_0) \right) \right], \quad \beta(t) = \beta_0 + 2P_{\beta_0}(t - t_0), \quad (11)$$

these solutions are the same that one gets by solving the field equations of general relativity.

3. Noncommutative Kantowski-Sachs model

In this section we construct noncommutative quantum cosmology for the Kantowski-Sachs model and calculate the noncommutative classical evolution via a WKB type approximation. To get the classical cosmological solutions would be a very difficult task in any model of noncommutative gravity [11], as a consequence of the highly nonlinear character of the field equations. One can introduce a noncommutative deformation of the minisuperspace variables [9]

$$[\Omega, \beta] = i\theta. \quad (12)$$

This noncommutativity can be formulated in terms of noncommutative minisuperspace functions with the Moyal product of functions

$$f(\Omega, \beta) \star g(\Omega, \beta) = f(\Omega, \beta) e^{i\frac{\theta}{2} \left(\overleftarrow{\partial}_\Omega \overrightarrow{\partial}_\beta - \overleftarrow{\partial}_\beta \overrightarrow{\partial}_\Omega \right)} g(\Omega, \beta). \quad (13)$$

Then the noncommutative WDW equation can be written as

$$\left(-\Pi_\Omega^2 + \Pi_\beta^2 - V(\Omega, \beta) \right) \star \Psi(\Omega, \beta) = 0, \quad (14)$$

we know from noncommutative quantum mechanics [10], that the symplectic structure is modified changing the commutator algebra. It is possible to return to the original commutative variables and usual commutation relations if we introduce the following change of variables

$$\Omega \rightarrow \Omega + \frac{\theta}{2} \Pi_\beta \quad \text{and} \quad \beta \rightarrow \beta - \frac{\theta}{2} \Pi_\Omega, \quad (15)$$

applying these ideas and using Eq.(4) we find the noncommutative Wheeler-DeWitt equation (NCWDW)

$$\left[\frac{\partial^2}{\partial \Omega^2} - \frac{\partial^2}{\partial \beta^2} - 48e^{-2\sqrt{3}(\Omega - i\frac{\theta}{2} \frac{\partial}{\partial \beta})} \right] \psi(\Omega, \beta) = 0, \quad (16)$$

assuming that we can write $\Psi(\Omega, \beta) = e^{\sqrt{3}\nu\beta} X(\Omega)$ the equation for $X(\Omega)$ is

$$\left[-\frac{d^2}{d\Omega^2} + 48e^{-3i\nu\theta} e^{-2\sqrt{3}\Omega} + 3\nu^2 \right] X(\Omega) = 0, \quad (17)$$

then the solutions of the NCWDW equation are

$$\Psi(\Omega, \beta) = e^{\pm i\sqrt{3}\nu\beta} K_{i\nu} \left(4e^{-\sqrt{3} \pm \frac{3}{2}\nu\theta} \right). \quad (18)$$

Usually the next step is to construct a ‘‘Gaussian’’ wave packet that can be normalized and do the physics with the new wave function. This is not needed for our purposes, as we will be applying the WKB method. Using Eq. (7) and

$$\Psi \simeq \exp(\pm im\beta) \exp(\pm iS(\Omega)), \quad (19)$$

we arrive at

$$S_1(\beta) = P_{\beta_0}\beta, \quad (20)$$

$$S_2(\Omega) = -\frac{1}{\sqrt{3}}\sqrt{P_{\beta_0}^2 - 48e^{-\sqrt{3}\theta P_{\beta_0}} e^{-2\sqrt{3}\Omega}} \\ + \frac{P_{\beta_0}}{\sqrt{3}} \operatorname{arctanh} \left[\frac{\sqrt{P_{\beta_0}^2 - 48e^{-\sqrt{3}\theta P_{\beta_0}} e^{-2\sqrt{3}\Omega}}}{P_{\beta_0}} \right],$$

from which we obtain the noncommutative classical solutions

$$\Omega(t) = \frac{1}{2\sqrt{3}} \ln \left[\frac{48}{P_{\beta_0}^2} \cosh^2 \left(2\sqrt{3}P_{\beta_0}(t - t_0) \right) \right] - \frac{1}{2}\theta P_{\beta_0}, \\ \beta(t) = \beta_0 + 2P_{\beta_0}(t - t_0) \\ - \frac{\theta}{2}P_{\beta_0} \tanh^2 \left[2\sqrt{3}P_{\beta_0}(t - t_0) \right], \quad (21)$$

these solutions have already been obtained in [13], in that paper the authors do a deformation of the symplectic structure at a classical level, modifying the Poisson brackets to include noncommutativity.

4. Conclusion and outlook

We found the noncommutative classical solutions for the Kantowski-Sachs model by applying a WKB type method to the noncommutative Wheeler-DeWitt equation, yielding the noncommutative generalization of the Einstein-Hamilton-Jacobi equation, from which the noncommutative classical evolution is obtained. We know from the commutative case that the classical solution obtained from the WKB type method are solutions to Einstein’s field equations, this gives confidence that the noncommutative solutions found by this method could be solutions to the noncommutative Einstein’s equations. Due to the complexity of the noncommutative theories of gravity [11], classical solutions to the noncommutative field equations are almost impossible to find, but in the approach of noncommutative quantum cosmology and by means of the WKB-type procedure, they can be constructed.

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1. H. Snyder, *Phys. Rev.* **71** (1947) 38.
2. M.P. Ryan, in: *Hamiltonian Cosmology* (Springer, Berlin, 1972).
3. M. MacCallum, in: *General Relativity: An Einstein Centenary Survey*, edited by S. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
4. D. Bohm, *Quantum Theory* (Prentice-Hall, Englewood Cliffs, NJ, 1955); Ph M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953) Vol. 2; L.I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1955); A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1961) Vol. 1.
5. D.N. Page, ‘‘Lectures on Quantum Cosmology’’, in *Gravitation: A Banff Summer Institute*, edited by R. Mann and P. Wesson (World Scientific, Singapore, 1991).
6. B.S. DeWitt, *Phys. Rev.* **160** (1967) 1113.
7. J.M. Romero and J.A. Santiago, *Mod. Phys. Lett. A* **20** (2005) 78.
8. R. Brandenberger and P.M. Ho, *Phys. Rev. D* **66** (2002) 023517; Q.G. Huang and M. Li, *Nucl. Phys. B* **713** (2005) 219; Q.G. Huang and M. Li, *JHEP* **0306** (2003) 014; H.s. Kim, G.S. Lee, H.W. Lee, and Y.S. Myung, *Phys. Rev. D* **70** (2004) 043521; H.s. Kim, G.S. Lee, and Y.S. Myung, *Mod. Phys. Lett. A* **20** (2005) 271; D.J. Liu and X.Z. Li, *Phys. Rev. D* **70** (2004) 123504.
9. H. García-Compeán, O. Obregón, and C. Ramírez, *Phys. Rev. Lett.* **88** (2002) 161301.
10. J. Gamboa, M. Loewe, and J.C. Rojas, *Phys. Rev. D* **64** 067901; M. Chaichian, M.M. Sheikh-Jabbari, and A. Tureanu., *Phys. Rev. Lett.* **86** 2716.
11. H. García-Compeán, O. Obregón, C. Ramírez, and M. Sabido, *Phys. Rev. D* **68** (2003) 044015; H. García-Compeán, O. Obregón, C. Ramírez, and M. Sabido, *Phys. Rev. D* **68** (2003) 045010; M. Bañados, O. Chandia, N. Grandi, F.A. Schaposnik, and G.A. Silva, *Phys. Rev. D* **64** (2001) 084012; H. Nishino and S. Rajpoot, *Phys. Lett. B* **532** (2002) 334; V.P. Nair, ‘‘Gravitational Fields on a Noncommutative Space’’, hep-th/0112114; S. Cacciatori, D. Klemm, L. Martucci, and D. Zanon, *Phys. Lett. B* **536** (2002) 101; S. Cacciatori, A.H. Chamseddine, D. Klemm, L. Martucci, W.A. Sabra, and D. Zanon, ‘‘Noncommutative Gravity in Two Dimensions’’, hep-th/0203038; Y. Abe and V.P. Nair, ‘‘Noncommutative Gravity: Fuzzy Sphere and Others’’, hep-th/0212270; M.A. Cardella and Daniela Zanon, ‘‘Noncommutative Deformation of Four-dimensional Einstein Gravity’’, hep-th/0212071; A.H. Chamseddine, ‘‘Invariant Actions for Noncommutative Gravity’’, hep-th/0202137; J.W. Moffat, *Phys. Lett. B* **491** (2000) 345; *Phys. Lett. B* **493** (2000) 142.
12. R. Kantowski and R. Sachs, *J. Math. Phys.* **7** (1966) 443.
13. G.D. Barbosa and N. Pinto-Neto, *Phys. Rev. D* **70** (2004) 103512, [arXiv:hep-th/0407111].