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Extra-dimensional noncommutative field theory model

J.C. López-Domínguez and O. Obregón
Instituto de Física de la Universidad de Guanajuato,
P.O. Box E-143, 37150 León Gto., México,
e-mail: jlopez@fisica.ugto.mx, octavio@fisica.ugto.mx

C. Ramírez and J.J. Toscano
Facultad de Ciencias Físico Matemáticas, Universidad Autónoma de Puebla,
P.O. Box 1364, 72000 Puebla, México,
e-mail: cramirez@fcfm.buap.mx, jtoscano@fcfm.buap.mx

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The influence of higher dimensions in noncommutative field theories is considered. For this purpose, we analyze the bosonic sector of a recently proposed 6 dimensional SU(3) orbifold model for the electroweak interactions. The corresponding noncommutative theory is constructed by means of the Seiberg-Witten map in 6D. We find in the reduced bosonic interactions in 4D theory, couplings which are new with respect to other known 4D noncommutative formulations of the Standard Model using the Seiberg-Witten map.

Keywords: non-commutativity; extra dimensions.

Se considera el efecto de la presencia de dimensiones extra en las teorías de campo no conmutativas. Para este propósito, se analiza el sector bosónico del recientemente propuesto modelo de orbifoldio SU(3) para las interacciones electro-débiles. La teoría no conmuntativa correspondiente se contruye utilizando el mapa de Seiberg-Witten en 6D. Encontramos en las interacciones bosónicas reducidas en la teoría de 4D acoplamientos que son nuevos con respecto a otras formulaciones no conmutativas conocidas en 4D del modelo estándar que utilizan el mapa de Seiberg-Witten.

Descriptores: No conmutatividad; dimensiones extra.

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1. Introduction

A renewed interest in theories in 6D has recently emerged [1]. An anomaly free gauged supergravity in $D = 6$, the Salam-Sezgin model [2], has been considered. This model is compactified on a 2-sphere and in four dimensions gives a $SU(2) \times U(1)$ gauge theory [3]. In particular, it has been argued that these theories with 3-Branes could point out towards solving the cosmological constant problem [4]. Also, in Ref. 5 it is shown that chaotic inflation consistent with constraints coming from the amplitude of the cosmic microwave anisotropies can be naturally realized.

In the search for a unified theory of elementary particles, the incorporation of the Higgs field in the standard model (SM) of electroweak interactions has motivated various proposals in 6D [6]. These are 6D pure gauge theories, in which after dimensional reduction the Higgs field naturally arises. Recently new proposals have been made, considering orbifold compactifications; in Ref. 7, a $U(3) \times U(3)$ model was considered. In these works the mass term of the Higgs potential is generated radiatively, with a finite value without the need of supersymmetry. Further, a $SU(3)$ model was developed in Refs. 8 and 9 with one Higgs doublet and a predicted W-boson mass. In this case the weak angle has a nonrealistic value, although it can be improved by an extended gauge group as in Ref. 7 or by the introduction of an $U(1)$ factor as done in Ref. 8.

Noncommutativity in field theories has been the subject of an important number of works in the last few years. In particular, the Seiberg-Witten construction [10] and its generalization for any gauge group [11] have been studied. This construction allows to express the noncommutative gauge fields in terms of the usual ones and their derivatives, maintaining the same degrees of freedom. It has been extended for noncommutative matter fields, which also can be generated in terms of the commutative matter fields and gauge fields of interest [11]. By this procedure, noncommutative versions of the standard model and consequently the electroweak interaction sector have been give in Ref. 12 (see also Ref. 13). As a consequence, new interactions among the fields of the theory are predicted. In this work, we will investigate the noncommutative generalization of the bosonic sector of Gauge Higgs unification models in 6D based on the $SU(3)$ gauge group compactified on T^2/Z_2 [9]. The noncommutative extension is obtained by means of the Seiberg-Witten map. We calculate, for the bosonic sector, the resulting first order corrections and compare them with the results obtained in other works.

2. The 6-Dimensional Model

Let us consider a Yang-Mills theory in 6-dimensional space-time with a $SU(3)$ gauge group, the Lagrangian of the theory is

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}F_{mn}F^{mn},$$

the field strength tensor is defined by

$$F_{mn} = \partial_m A_n - \partial_n A_m - ig_6[A_m, A_n],$$

and g_6 is the coupling constant in 6D. This action is interpreted by a dimensional reduction on an orbifold T^2/Z_N for $N = 3, 4, 6$ [9]. The result of this reduction is given by

$$\mathcal{L}=-\frac{1}{2}\text{Tr}F_{\mu\nu}F^{\mu\nu}+2\text{Tr}D_\mu A_{\overline{z}}D^\mu A_z-g^2\text{Tr}\left[A_z, A_{\overline{z}}\right]^2, \quad (1)$$

where $g = g_6\sqrt{V}$ is the gauge coupling of the 4-dimensional effective theory, V is the volume of the two extra dimensions and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu],$$
$$D_\mu A_{z,\overline{z}} = \partial_\mu A_{z,\overline{z}} - ig[A_\mu, A_{z,\overline{z}}] = F_{\mu z,\overline{z}}. \quad (2)$$

The orbifold reduction [9] for the gauge fields A_m leads to the 4-dimensional A_μ and the two complex components of the scalar boson doublet (Higgs), which are contained in the A_z and $A_{\overline{z}}$ gauge fields,

$$A_\mu = \begin{pmatrix} W_\mu & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2\sqrt{3}} \begin{pmatrix} B_\mu I & 0 \\ 0 & -2B_\mu \end{pmatrix},$$
$$A_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \phi \\ 0 & 0 \end{pmatrix},$$
$$A_{\overline{z}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ \phi^\dagger & 0 \end{pmatrix}.$$

Substituting these expressions in the Lagrangian (1) we find

$$\mathcal{L} = -\frac{1}{2}\text{Tr}F_{\mu\nu}(W)F^{\mu\nu}(W) - \frac{1}{4}F_{\mu\nu}(B)F^{\mu\nu}(B) + \left(D_\mu\phi\right)^\dagger\left(D^\mu\phi\right) - V\left(\phi\right), \quad (3)$$

where

$$D_\mu\phi = \left(\partial_\mu - \frac{1}{2}igW_\mu^a\tau_a - \frac{1}{2}ig\tan\theta_W B_\mu\right)\phi,$$
$$\tan\theta_W = \sqrt{3} \quad \text{and} \quad V\left(\phi\right) = \frac{g^2}{2}\left|\phi\right|^4.$$

Thus this Lagrangian has a $SU(2) \times U(1)$ invariance with a scalar massless doublet with a quartic potential. However, as show in Ref. 14, quantum fluctuations induce corrections to the potential $V(\phi)$ which can trigger radiative symmetry breaking. The leading terms in the one-loop effective potential for the Higgs are,

$$V_{\text{eff}}\left(\phi\right) = -\mu^2\left|\phi\right|^2 + \lambda\left|\phi\right|^4.$$

Assuming $\mu^2 > 0$, so that electroweak symmetry breaking can occur, we have that $\langle|\phi|\rangle = \nu/\sqrt{2}$ with $\nu = \mu/\sqrt{\lambda}$.

3. Noncommutative Gauge Theories

3.1. Noncommutative space-time

Noncommutative space-time incorporates coordinates $\widehat{x}^\mu$, given by operators that satisfy the following relations,

$$[\widehat{x}^\mu, \widehat{x}^\nu] = i\theta^{\mu\nu}, \quad (4)$$

where $\theta^{\mu\nu} = -\theta^{\nu\mu}$ are real numbers. The Weyl-Wigner-Moyal correspondence establishes an equivalence between the Heisenberg algebra of the operators $\widehat{x}^\mu$ and the function algebra in $\mathbb{R}^m$. It has an associative and noncommutative star product, the Moyal $\star$-product, given by,

$$f(x)\star g(x) \equiv \left[\exp\left(\frac{i}{2}\frac{\partial}{\partial x^\alpha}\,\theta^{\alpha\beta}\frac{\partial}{\partial y^\beta}\right)f(x)g(y)\right]_{y\to x}$$
$$= f\left(x\right)g\left(x\right) + \frac{i}{2}\,\theta^{\alpha\beta}\partial_\alpha f\left(x\right)\partial_\beta g\left(x\right) + \mathcal{O}(\theta^2). \quad (5)$$

Since we will work with a non-Abelian gauge group, our functions are matrix valued, and the corresponding matrix Moyal product is denoted by an $*$. Therefore, a theory on the noncommutative space of the $\widehat{x}$ is equivalent to a theory of usual fields, where the function product is substituted by the Moyal $*$ product. This suggests that any theory can be converted into a noncommutative one by replacing the ordinary function product with the $*$ product.

3.2. The Seiberg-Witten Map

For an ordinary Yang-Mills theory, the gauge field and the strength field tensor transformations can be written as:

$$\delta_\lambda A_\mu = \partial_\mu\lambda + i\lambda A_\mu - iA_\mu\lambda$$
$$\delta_\lambda F_{\mu\nu} = i\lambda F_{\mu\nu} - iF_{\mu\nu}\lambda. \quad (6)$$

For the noncommutative gauge theory, we use the same equations (6) except that the matrix multiplications are replaced by the $*$ product. Then the gauge field and the strength field tensor transformations are [10]:

$$\widehat{\delta}_{\widehat{\lambda}}\widehat{A}_\mu = \partial_\mu\widehat{\lambda} + i\widehat{\lambda} * \widehat{A}_\mu - i\widehat{A}_\mu * \widehat{\lambda},$$
$$\widehat{\delta}_{\widehat{\lambda}}\widehat{F}_{\mu\nu} = i\widehat{\lambda} * \widehat{F}_{\mu\nu} - i\widehat{F}_{\mu\nu} * \widehat{\lambda}, \quad (7)$$

from which the original Yang-Mills theory (6) results in the limit $\theta \to 0$. As shown by Kontsevich [15], at the level of the physical degrees of freedom there is a one to one relation between the commutative and the noncommutative theories. Nevertheless both theories are quite different, as noncommutativity generates new couplings. An infinitesimal commutative gauge transformation $\delta_\lambda A_\mu = \partial_\mu\lambda + i\lambda A_\mu - iA_\mu\lambda$, will induce the noncommutative one,

$$\widehat{A}_\mu(A + \delta_\lambda A) = \widehat{A}_\mu(A) + \widehat{\delta}_{\widehat{\lambda}}\widehat{A}_\mu(A). \quad (8)$$

This is the so called Seiberg-Witten map.

The solution to (8) can be obtained by setting $\widehat{A}_\mu = A_\mu + A'_\mu(A)$ and $\widehat{\lambda} = \lambda + \lambda'(\lambda, A)$, where A'_μ and λ' are local functions of λ and A_μ of first order in θ. Then substituting in (8) and expanding to first order,

$$A'_\mu(A+\delta_\lambda A) - A'_\mu(A) - \partial_\mu\lambda' - i[\lambda', A_\mu] - i[\lambda, A'_\mu]$$
$$= -\frac{1}{2}\,\theta^{\alpha\beta}(\partial_\alpha\lambda\partial_\beta A_\mu + \partial_\beta A_\mu\partial_\alpha\lambda). \quad (9)$$

One solution of this equation is given by [10],

$$\widehat{A}_\mu(A) = A_\mu + A'_\mu(A)$$
$$= A_\mu - \frac{1}{4}\,\theta^{\alpha\beta}\left\{A_\alpha, \partial_\beta A_\mu + F_{\beta\mu}\right\} + \mathcal{O}(\theta^2), \quad (10)$$
$$\widehat{\lambda}(\lambda, A) = \lambda + \lambda'(\lambda, A)$$
$$= \lambda + \frac{1}{4}\,\theta^{\alpha\beta}\left\{\partial_\alpha\lambda, A_\beta\right\} + \mathcal{O}(\theta^2), \quad (11)$$

from which it turns out that,

$$\widehat{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{4}\theta^{\alpha\beta}\left(2\left\{F_{\mu\alpha}, F_{\nu\beta}\right\}\right.$$
$$\left. - \left\{A_\alpha, \left(D_\beta + \partial_\beta\right) F_{\mu\nu}\right\}\right) + \mathcal{O}(\theta^2). \quad (12)$$

These Eqs. (10), (11), (12) are the explicit form of the Seiberg-Witten map, which in this way can be constructed for any Lie algebra of transformations [11].

4. The noncommutative model

As previously mentioned, our purpose is the construction of a noncommutative version of the 6-dimensional $SU(3)$ gauge theory presented in Sec. 2.

The noncommutative action is given by:

$$\widehat{\mathcal{S}}_{NC} = -\frac{1}{2}\mathrm{Tr}\int d^6x \widehat{F}_{mn}\widehat{F}^{mn}, \quad (13)$$

where

$$\widehat{F}_{mn} = F_{mn} + \frac{1}{4}\theta^{kl}\left(2\left\{F_{mk}, F_{nl}\right\}\right.$$
$$\left. - \left\{A_k, \left(D_l + \partial_l\right) F_{mn}\right\}\right) + \mathcal{O}(\theta^2). \quad (14)$$

Here the indexes m, n, k and l take the values $0, \ldots, 3, z$ and $\overline{z}$. Thus the noncommutative parameter θ^{kl} can be: $\theta^{\mu\nu}$ (noncommutativity among the 4-dimensional space-time coordinates), $\theta^{\mu z}$, $\theta^{\mu\overline{z}}$ (noncommutativity among the 4-dimensional space-time coordinates and the extra dimensions coordinates) and $\theta^{z\overline{z}}$ (noncommutativity between the extra dimensions). After somewhat cumbersome computations, we obtain the following expression for these corrections in terms of the $SU(2)$ and $U(1)$ field strengths $W^{\mu\nu}$ and $B^{\mu\nu}$ respectively, the corresponding gauge fields W^μ and B^μ and the Higgs field ϕ,

$$\begin{aligned}
\widehat{\mathcal{L}}_{NC} = &-\frac{1}{2}\mathrm{Tr}\, W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + (D_\mu\phi)^\dagger\,(D^\mu\phi) - \frac{g^2}{2}\,|\phi|^4 \\
&- \frac{1}{4}\theta^{\alpha\beta}\bigg\{\frac{1}{2\sqrt{3}}\mathrm{Tr}\Big[4\{W_{\mu\alpha}, B_{\nu\beta}\}W^{\mu\nu} + 2\{W_{\mu\alpha}, W_{\nu\beta}\}B^{\mu\nu}I - \{W_\alpha, D_\beta W_{\mu\nu}\}B^{\mu\nu}I - \{B_\alpha, D_\beta W_{\mu\nu}\}W^{\mu\nu}\Big] \\
&+ \frac{1}{2\sqrt{3}}B_\alpha\partial_\beta B_{\mu\nu}B^{\mu\nu} - \frac{1}{2\sqrt{3}}B_{\mu\alpha}B_{\nu\beta}B^{\mu\nu} + 2(D^\mu\phi)^\dagger\Big(W_{\mu\alpha} - \frac{1}{2\sqrt{3}}B_{\mu\alpha}I\Big)(D_\beta\phi) + H.c. \\
&+ (D^\mu\phi)^\dagger\Big(W_\alpha - \frac{1}{2\sqrt{3}}B_\alpha I\Big)\Big(\overrightarrow{\partial}_\beta + \overrightarrow{D}_\beta\Big)(D_\mu\phi) + (D^\mu\phi)^\dagger\Big(\overleftarrow{\partial}_\beta + \overleftarrow{D}_\beta\Big)\Big(W_\alpha - \frac{1}{2\sqrt{3}}B_\alpha I\Big)(D_\mu\phi) \\
&+ ig\Big[\phi^\dagger(D_\alpha\phi)(D_\beta\phi)^\dagger\phi - (D_\beta\phi)^\dagger(D_\alpha\phi)\phi^\dagger\phi\Big] - ig^3\phi^\dagger\phi\phi^\dagger W_\beta W_\alpha\phi - g^2\Big[\phi^\dagger\Big(W_\alpha + \frac{1}{2\sqrt{3}}B_\alpha I\Big)\partial_\beta(\phi\phi^\dagger)\phi \\
&- \frac{2}{\sqrt{3}}B_\alpha\partial_\beta(\phi\phi^\dagger)\phi^\dagger\phi + \phi^\dagger\partial_\beta(\phi\phi^\dagger)\Big(W_\alpha + \frac{1}{2\sqrt{3}}B_\alpha I\Big)\phi\Big]\bigg\} \\
&+ \frac{i}{2}\theta^{z\overline{z}}\bigg\{-2i(D_\mu\phi)^\dagger\Big(W^{\mu\nu} + \frac{1}{\sqrt{3}}B^{\mu\nu}I\Big)(D_\nu\phi) + \frac{g}{2}\Big[\phi^\dagger\phi(D_\mu\phi)^\dagger(D^\mu\phi) - (D_\mu\phi)^\dagger\phi\phi^\dagger(D^\mu\phi)\Big] \\
&- g\,\phi^\dagger\Big(W_{\mu\nu}W^{\mu\nu} + \frac{1}{\sqrt{3}}W_{\mu\nu}B^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}\Big)\phi\bigg\}. \quad (15)
\end{aligned}$$

In this equation there are new interactions with respect to the ones found in the 4D noncommutative formulations of the SM [12, 16], for instance the interactions between the weak gauge fields and the electromagnetic field which appear in the first terms that multiply the four-dimensional noncommutativity parameter $\theta^{\alpha\beta}$. Of particular interest are the corrections corresponding to noncommutativity between the extra dimensions, *i.e.* the terms multiplied by $\theta^{z\overline{z}}$, given by interactions among the Higgs and the gauge bosons, and also higher order Higgs self-interactions. Considering only these sort of corrections, we have,

$$\begin{aligned}\widehat{\mathcal{L}}_{NC} = &-\frac{1}{2}\mathrm{Tr}\, W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + (D_\mu\phi)^\dagger\,(D^\mu\phi)\\ &- \frac{g^2}{2}\,|\phi|^4 + \frac{i}{2}\theta^{z\bar{z}}\Big\{ -2i\,(D_\mu\phi)^\dagger\left(W^{\mu\nu} + \frac{1}{\sqrt{3}}B^{\mu\nu}I\right)\\ &\times (D_\nu\phi) + \frac{g}{2}\big[\phi^\dagger\phi\,(D_\mu\phi)^\dagger\,(D^\mu\phi)\\ &- (D_\mu\phi)^\dagger\,\phi\phi^\dagger(D^\mu\phi)\big]\\ &-g\phi^\dagger\left(W_{\mu\nu}W^{\mu\nu}+\frac{1}{\sqrt{3}}W_{\mu\nu}B^{\mu\nu}-\frac{1}{4}B_{\mu\nu}B^{\mu\nu}\right)\phi\Big\}. \qquad (16)\end{aligned}$$

The noncommutative corrections in this Lagrangian are dimension-six operators, well known from the electroweak effective Lagrangian technique [17], a scheme in which the effects of these terms can be studied in a model-independent manner.

5. Conclusions

In this work we explore the consequences of noncommutativity in a 6-dimensional model, by means of the Seiberg-Witten map. We consider the $SU(3)$ gauge Higgs unification model of the electroweak interactions of Refs. 14 and 9, compactified to 4D on an orbifold T^2/Z_N for $N = 3, 4, 6$. We analyze noncommutativity among all the 6-dimensional coordinates. As a consequence of the orbifold symmetries, it turns out that there are no corrections to the model due to noncommutativity among the 4D coordinates and the two-extra dimensions. We find that the corrections we obtain, in particular those corresponding to noncommutativity among the 4D coordinates, differ from the ones of noncommutative models calculated directly in 4D, also by means of the Seiberg-Witten map [12], for instance interactions between the weak gauge fields and the electromagnetic field which appear in the first terms that multiply the four-dimensional noncommutativity parameter $\theta^{\alpha\beta}$ in Eq. (15).

As well as in the commutative model, the spontaneous symmetry breaking should arise dynamically, from first order quantum corrections. Thus it would be interesting to include matter and to study the corresponding noncommutative corrections, which could be done following [11], progress in this direction will be reported elsewhere.

As mentioned in the introduction, the model we are considering here has a too high value for the weak angle. However, accordingly with Ref. 9, it can be extended in such a way that it correctly reproduces the standard model data at low energies. Thus we can expect that in a noncommutative version of this extended model, the kind of corrections presented here will still be present, in particular those corresponding to noncommutativity between extra dimensions. Finally, from the results of the particular noncommutative model we started with, which could be interesting on its own, we can conclude that noncommutativity in higher dimensional models can have interesting consequences and phenomenological effects beyond those of four dimensional noncommutative theories. The study of the phenomenological consequences and more realistic models, including matter fields, is in progress.

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