Inflation scenario from canonical quantum cosmology

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Using the canonical quantization for the flat FRW cosmological model with scalar field ϕ and scalar potential $\mathrm{V}(\phi)$, we obtain a differential equation for this potential. Also we present the quantum solution for this model, and under the WKB approximation to get the semiclassical formulation, we give a family of potentials. Finally, the classical evolution of the system is given in the inflation scenary.

Keywords: Exact solutions; quantum cosmology.

Usando la cuantización canónica para el modelo cosmológico del FRW plano, con campo escalar ϕ y potencial escalar $\mathrm{V}(\phi)$, obtenemos una ecuación diferencial para éste potencial. También presentamos la solución cuántica para este modelo, y bajo la aproximación semi-clásica WKB, damos una familia de potenciales. Finalmente, la evolución clásica del sistema es dada en el escenario de inflación.

Descriptores: Soluciones exactas; cosmología cuántica.

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1. Introduction

It is a common issue in Cosmology nowadays to make use of scalar fields ϕ as the responsible agents of some of the most intriguing aspects of our universe. Just to mention a few, we find that scalar fields are used as the inflaton, which seeds the primordial perturbations for structure formation during an early inflationary epoch; as the cold dark matter candidate responsible for the formation of the actual cosmological structure, and as the dark energy component which seems to be driving the current accelerated expansion of the universe [1–16].

The key feature for such flexibility of the concept of scalar fields (spin-0 bosons) is the freedom one has to propose a *scalar potential* $V(\phi)$, which encodes in itself the (non gravitational) self-interactions among the scalar particles. The literature on scalar potentials is enourmosly vast, and most of the recent papers are aimed to explain the SnIa results that suggest the existence of dark energy [1–3, 5, 12–14, 17].

Recently, scalar fields coupled to gravity (in an FRW background) have also appeared in connection to the so called string theory landscape [18–20], where the scalar potential $\mathrm{V}(\phi)$ is usually thought as having many valleys, which represent the different vacua solutions. The hope is that the statistics of these vacua could explain, for example, the smallness the cosmological constant (the simplest candidate for dark energy).

Scalar fields also appear in the study of tachyon dynamics. For instance, in the unstable D-brane scenario, the scalar potential in the tachyon effective action around the minimum of the potential is of the form $\mathrm{V}(\phi) = \mathrm{e}^{-\alpha\phi/2}$ [21, 22]. Currently, there has been a lot of interest in the study of tachyon driven cosmology [23, 24].On the other hand, scalar fields have also been used within the so called (canonical) Quantum Cosmology (QC) formalism, which deals with a very early quantum epoch of the cosmos. Again, scalar fields act as matter sources, and then play an important role in determining the evolution of such an early universe.

Quantum cosmology means the quantization of minisuperspace models, in which the gravitational and matter variables have been reduced to a finite number of degrees of freedom. These models were extensively studied by means of Hamiltonian methods in the 1970s (for reviews see Refs. 25 and 26). It was first remarked by Kodama [27, 28], that solutions to the Wheeler-DeWitt equation (WDW) in the formulation of Arnowitt, Deser and Misner (ADM) and Ashtekar (in the connection representation) are related by $\Psi_{\mathrm{ADM}} = \Psi_{\mathrm{A}}\mathrm{e}^{\pm \mathrm{i}\Phi_{\mathrm{A}}}$, where $\Phi_A$is the homogeneous specialization of the generating functional [29] of the canonical transformation from the ADM variables to Ashtekar's. This function was calculated explicitly for the diagonal Bianchi type IX model by Kodama, who also found $\Psi_{\mathrm{A}} = \mathrm{constant}$ as solution. Since Φ_{A} is pure imaginary, for a certain factor ordering, one expects a solution of the form $\Psi = \mathrm{W}\mathrm{e}^{\pm\Phi}$.

Our aim in this paper is to determine which scalar potentials can arise as exact solutions to the WDW equation of QC, as well as which can be valid at the semiclassical level. For this we will use some of the ideas presented in the previous paragraphs to find the WDW equation and find exact solutions to the quantum cosmological model. Though there are many solutions in principle, we will focus only on the relevant solutions for the early universe.

2. Obtaining the potentials and the quantum model I

We begin by writing down the line element for a homogeneous and isotropic universe, the so called Friedmann-

Robertson-Walker (FRW) metric, for the flat case

$$ds^2 = -N^2(t)dt^2 + e^{2\alpha(t)}\left[dr^2 + r^2 d\Omega^2\right], \qquad (1)$$

where $a(t) = e^{\alpha}$ is the scale factor, $N(t)$ is the lapse function. The effective action we are going to work on[i]

$$S_{tot}=S_g+S_\phi=\int dx^4\sqrt{-g}\left[R-2\Lambda-\frac{\dot\phi^2}{2}+V(\phi)\right], \qquad (2)$$

where ϕ is a scalar field endowed with a scalar potential $V(\phi)$, and Λ is a cosmological constant[ii]. The Lagrangian become

$$\mathcal{L} = e^{3\alpha}\left[3\frac{\dot\alpha^2}{N} - \frac{\dot\phi^2}{2N} + N\left(V - 2\Lambda\right)\right], \qquad (3)$$

and then the canonical momenta are found to be

$$\Pi_\alpha = \frac{\partial\mathcal{L}}{\partial\dot\alpha} = 6e^{3\alpha}\frac{\dot\alpha}{N}, \qquad \dot\alpha = \frac{N\Pi_\alpha}{6}e^{-3\alpha},$$
$$\Pi_\phi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = -e^{3\alpha}\frac{\dot\phi}{N}, \qquad \dot\phi = -Ne^{-3\alpha}\Pi_\phi. \qquad (4)$$

We are now in position to write the corresponding Hamiltonian

$$\mathcal{H} = e^{-3\alpha}\left[\frac{1}{12}\Pi_\alpha^2 - \frac{1}{2}\Pi_\phi^2 - e^{6\alpha}V(\phi,\Lambda)\right], \qquad (5)$$

where we have written $V(\phi,\Lambda) = V(\phi) - 2\Lambda$.

The WDW equation for this model is achieved by replacing Π_{q^μ} by $-i\partial_{q^\mu}$ in Eq. (5); here $q^\mu = (a,\phi)$

$$\mathcal{H} = e^{-3\alpha}\left[-\frac{1}{12}\frac{\partial^2}{\partial\alpha^2} + \frac{1}{2}\frac{\partial^2}{\partial\phi^2} - e^{6\alpha}V(\phi,\Lambda)\right] = 0. \qquad (6)$$

Following the suggestion by Hartle and Hawking [30] we do a *semi-general factor ordering*, so that we can factor order $e^{-3\alpha}$ with Π_α, with this in mind we use

$$-e^{-(3-Q)\alpha}\,\partial_\alpha e^{-Q\alpha}\partial_\alpha = -e^{-3\alpha}\,\partial_\alpha^2 + Q\,e^{-3\alpha}\partial_\alpha, \qquad (7)$$

where Q is any real constant. With this factor ordering the WDW reads

$$\Box\Psi + Q\frac{\partial\Psi}{\partial\alpha} - e^{6\alpha}V(\phi,\Lambda)\Psi = 0, \qquad (8)$$

Ψ is called the wave function of the universe,

$$\Box \equiv -\frac{1}{12}\frac{\partial^2}{\partial\alpha^2} + \frac{1}{2}\frac{\partial^2}{\partial\phi^2}$$

is a (modified) two dimensional d'Alambertian operator in the q^μ coordinates. Taking the following ansatz for the wave function [31, 32]

$$\Psi(q^\mu) = W(q^\mu)e^{-S(q^\mu)}, \qquad (9)$$

where $S(q^\mu)$ is known as the *superpotential function*, Eq. (8) is transformed into

$$\Box W - W\Box S \;-\; 2\nabla W\cdot\nabla S + Q\frac{\partial W}{\partial\alpha} - QW\frac{\partial S}{\partial\alpha}$$
$$+\; W[(\nabla S)^2 - U] = 0, \qquad (10)$$

with

$$\nabla W\cdot\nabla S \equiv -\frac{1}{12}\left(\partial_\alpha W\right)\left(\partial_\alpha S\right) + \frac{1}{2}\left(\partial_\phi W\right)\left(\partial_\phi S\right),$$
$$(\nabla)^2 \equiv -\frac{1}{12}\left(\partial_\alpha\right)^2 + \frac{1}{2}\left(\partial_\phi\right)^2,$$

and $U(\phi,\Lambda) = e^{6\alpha}V(\phi,\Lambda)$.

Equation (10) can be easily solved if splitted in the following equations,

$$(\nabla S)^2 - U = 0, \qquad (11)$$

$$W\left(\Box S + \frac{Q}{12}\frac{\partial S}{\partial\alpha}\right) + 2\nabla W\cdot\nabla S = 0, \qquad (12)$$

$$\Box W + \frac{Q}{12}\frac{\partial W}{\partial\alpha} = 0. \qquad (13)$$

We will choose to solve Eqs. (11) and (12), and these solutions have to comply with Eq. (13), which will be our *constraint* equation.

Our next step is to find exact solutions of the WDW equation and determine the corresponding potential term $V(\phi,\Lambda)$. For this, let us start with Eq. (11), which is an equation for the superpotential function only. If $S(q^\mu) = f(\alpha)g(\phi)$, then

$$-\frac{1}{12f^2}\left(\frac{df}{d\alpha}\right)^2 + \frac{1}{2g^2}\left(\frac{dg}{d\phi}\right)^2 = e^{6\alpha}\frac{V(\phi,\Lambda)}{f^2g^2}. \qquad (14)$$

For simplicity, we assume that $f(\alpha) = e^{3\alpha}/\mu$, with μ an arbitrary constant. Hence, Eq. (14) becomes a differential equation for $g(\phi)$ in terms of the scalar potential as

$$\frac{1}{2}\left(\frac{dg}{d\phi}\right)^2 - \frac{3}{4}g^2 = \mu^2 V(\phi,\Lambda). \qquad (15)$$

This last equation has several exact solutions, which can be generated in the following way. Let us consider that $V(\phi,\Lambda) = g^2F(g)$, where $F(g)$ is an arbitrary function of its argument. Thus, eq. (15) can be in quadratures as

$$\Delta\phi = \pm\frac{1}{\sqrt{2}}\int\frac{d\ln g}{\sqrt{\frac{3}{4} + \mu^2F(g)}}. \qquad (16)$$

This last equation can be solved for g as a function of ϕ, and from it we can find the corresponding scalar potential of the model. Some solutions for the scalar potential are shown in Table I.

It turns out that, for this particular solution, Eqs. (12) and (13) become linear differential equations for W, whose solution can be given in the form

$$W(\alpha,\phi) = e^{u(\alpha)+v(\phi)} \qquad (17)$$

TABLE I. Some exact solutions of Eq. (16). Here, n is any real number. There is indeed a solution of Eq. (16) for $\mathrm{F(g)} = (\ln \mathrm{g})^{\mathrm{n}}$ in terms of the hypergeometric functions. However, the form of the scalar potential in this case cannot be given in closed form.

$\mathrm{F(g)}$	$\mathrm{V}(\phi,\Lambda)$
V_0	$\mathrm{V}_0 \exp\left(-\lambda\Delta\phi\right); \quad 2\sqrt{2}\lambda = \sqrt{\frac{3}{4}+\mu^2\mathrm{V}_0}$
$\mathrm{g}^{-\mathrm{n}}, \mathrm{n}\neq 2$	$\left\{(4\mu^2/3)\left[\cosh^2\left(\frac{\sqrt{6}\mathrm{n}}{4}\Delta\phi\right)-1\right]\right\}^{(2-\mathrm{n})/\mathrm{n}}$
$\ln \mathrm{g}$	$\mathrm{u}(\phi)\mathrm{e}^{2\mathrm{u}(\phi)}, \ \mathrm{u} = (\mu\Delta\phi/2)^2 - \frac{3}{4\mu^2}$
$(\ln \mathrm{g})^2$	$\mathrm{u}^2\mathrm{e}^{2\mathrm{u}}, \ \mathrm{u} = \sqrt{\frac{3}{4\mu^2}}\sinh\left(\mu\sqrt{2}\Delta\phi\right)$

after a bit of algebra, we obtain

$$\mathrm{W}(\alpha,\phi) = \exp\left[\mathrm{k}\left(2\alpha + \int\frac{\mathrm{d}\phi}{\partial_\phi(\ln \mathrm{g})}\right)\right] \times \exp\left[\frac{\mathrm{Q}}{2}\alpha - \frac{\mu^2}{2}\int\frac{\mathrm{d}[\mathrm{V}(\phi,\Lambda)]}{(\partial_\phi \mathrm{g})^2}\right], \qquad (18)$$

where k is an arbitrary constant. One only needs to verify under which conditions solutions in Eqs. (15) and (18) comply with the constraint equation (13), which takes the following form

$$\partial^2_\phi \mathrm{v} + (\partial_\phi \mathrm{v})^2 - \frac{16\mathrm{k}^2 - \mathrm{Q}^2}{24} = 0$$
$$\partial_\phi \mathrm{v} - \frac{\mathrm{k}}{\partial_\phi(\ln \mathrm{g})} + \frac{\mu^2}{2}\frac{\partial_\phi[\mathrm{V}(\phi,\Lambda)]}{(\partial_\phi \mathrm{g})^2} = 0, \qquad (19)$$

It is interesting to note that the functions S and W can be easily generated once a particular $\mathrm{g}(\phi)$ is found from Eq. (19).

3. The quantum model II

Using the potential $\mathrm{V}(\phi) = \mathrm{V}_0\mathrm{e}^{-\lambda\phi}$ and Making the following transformation between the coordinates

$$\mathrm{x} = -6\alpha + \lambda\phi, \qquad \mathrm{y} = -\alpha + \frac{1}{\lambda}\phi, \qquad (20)$$

the resulting WDW equation (8) is

$$\frac{\partial^2\Psi}{\partial \mathrm{x}^2} - \frac{1}{12\lambda^2}\frac{\partial^2\Psi}{\partial \mathrm{y}^2} - \frac{2\mathrm{V}_0}{\lambda^2-6}\mathrm{e}^{-\mathrm{x}}\Psi = 0 \qquad (21)$$

and by separation variables, and $\Psi = \mathrm{X(x)Y}(\tilde{\mathrm{y}})$ with $\tilde{\mathrm{y}} = 12\lambda\mathrm{y}$, we obtain the set of differential equation for the functions X and Y

$$\frac{\mathrm{d}^2\mathrm{X}}{\mathrm{dx}^2} + \left(\left(\frac{\eta}{2}\right)^2 - \beta\mathrm{e}^{-\mathrm{x}}\right)\mathrm{X} = 0,$$
$$\frac{\mathrm{d}^2\mathrm{Y}}{\mathrm{d}\tilde{\mathrm{y}}^2} + \left(\frac{\eta}{2}\right)^2\mathrm{Y} = 0, \qquad (22)$$

where $\beta = 2\mathrm{V}_0/(\lambda^2-6)$ and η is a separation constant. The solutions for these equations are given in term of complex order Bessel functions

$$\mathrm{X(x)} = \mathrm{I}_{\pm\mathrm{i}\eta}\left(\pm 2\sqrt{\beta}\mathrm{e}^{-\mathrm{x}/2}\right) + \mathrm{K}_{\pm\mathrm{i}\eta}\left(\pm 2\sqrt{\beta}\mathrm{e}^{-\mathrm{x}/2}\right)$$
$$\mathrm{Y(y)} = \mathrm{A}_0\mathrm{e}^{\mathrm{i}6\eta\lambda\mathrm{y}} + \mathrm{A}_1\mathrm{e}^{-\mathrm{i}6\eta\lambda\mathrm{y}}, \qquad (23)$$

for $|\lambda| < \sqrt{6}$, and for other values of λ,

$$\mathrm{X(x)} = \mathrm{J}_{\mathrm{i}\eta}\left(\pm 2\sqrt{\beta}\mathrm{e}^{-\mathrm{x}/2}\right) + \mathrm{J}_{-\mathrm{i}\eta}\left(\pm 2\sqrt{\beta}\mathrm{e}^{-\mathrm{x}/2}\right)$$
$$\mathrm{Y(y)} = \mathrm{A}_0\mathrm{e}^{\mathrm{i}6\eta\lambda\mathrm{y}} + \mathrm{A}_1\mathrm{e}^{-\mathrm{i}6\eta\lambda\mathrm{y}}, \qquad (24)$$

with $\mathrm{I}_{\pm\mathrm{i}\eta}$ and $\mathrm{K}_{\pm\mathrm{i}\eta}$ are modified Bessel Functions. In this way, the wave function is written as

$$\Psi(\mathrm{x},\mathrm{y}) = \left[\mathrm{I}_{\pm\mathrm{i}\eta}\left(\pm 2\sqrt{\beta}\mathrm{e}^{-\mathrm{x}/2}\right) + \mathrm{K}_{\pm\mathrm{i}\eta}\left(\pm 2\sqrt{\beta}\mathrm{e}^{-\mathrm{x}/2}\right)\right] \times \left[\mathrm{A}_0\mathrm{e}^{\mathrm{i}6\eta\lambda\mathrm{y}} + \mathrm{A}_1\mathrm{e}^{-\mathrm{i}6\eta\lambda\mathrm{y}}\right] \quad \text{for } |\lambda| < \sqrt{6}$$
$$\Psi(\mathrm{x},\mathrm{y}) = \left[\mathrm{J}_{\mathrm{i}\eta}\left(\pm 2\sqrt{\beta}\mathrm{e}^{-\mathrm{x}/2}\right) + \mathrm{J}_{-\mathrm{i}\eta}\left(\pm 2\sqrt{\beta}\mathrm{e}^{-\mathrm{x}/2}\right)\right] \times \left[\mathrm{A}_0\mathrm{e}^{\mathrm{i}6\eta\lambda\mathrm{y}} + \mathrm{A}_1\mathrm{e}^{-\mathrm{i}6\eta\lambda\mathrm{y}}\right] \text{ for } |\lambda| > \sqrt{6}. \qquad (25)$$

To extract a normalazible wave function we need to construct wave packets (see for example Ref. 34 and 35) to form a Gaussian state.

4. Semiclassical approximation

We shall make use of a semiclassical approximation to extract the dynamics of the WDW equation. Such approximation hides the problem of time, and thus the dynamical evolution of the minisuperspace variable can be obtained, and checked against the solutions of classical General Relativity. The semiclassical limit of the WDW equation is achieved by taking

$$\Psi(\alpha,\phi) = \mathrm{e}^{-\mathrm{S}} \qquad (26)$$

and imposing the usual conditions on the superpotential function S, namely

$$\left(\frac{\partial \mathrm{S}}{\partial \mathrm{a}}\right)^2 \gg \left|\frac{\partial^2\mathrm{S}}{\partial \mathrm{a}^2}\right|, \qquad \left(\frac{\partial \mathrm{S}}{\partial \phi}\right)^2 \gg \left|\frac{\partial^2\mathrm{S}}{\partial \phi^2}\right|. \qquad (27)$$

Hence, the WDW equation, under a particular factor ordering ($\mathrm{Q}=0$), becomes exactly the aforementioned EHJ equation (11) (this approximation is equivalent to a zero quantum potential in the Bohmian interpretation of quantum cosmology) [33]. This same equation is recovered directly, when we introduce the transformation on the canonical momentas

$$\Pi_{\mathrm{q}^\mu} \to \frac{\partial \mathrm{S}}{\partial \mathrm{q}^\mu}, \qquad (28)$$

in Eq. (6), with particular factor ordering and $\mathcal{H} = 0$. Using the same procedure used in the quantum case, we get again Eq. (15).

In this way, we get the classical behavior solving the relations between (28) and eqs. (4,4).

For example, an exponential scalar field potential $\mathrm{V}(\phi) = \mathrm{e}^{-\lambda\phi}$, the exact solution of the WDW equation reads

$$\Psi = \exp\left[2\mathrm{k}\left(\alpha - \frac{\phi}{\lambda}\right)\right] \times \exp\left[\frac{\mathrm{Q}}{2}\alpha + \frac{2\mu^2\mathrm{V}_0}{\lambda}\phi - \frac{1}{\mu}\mathrm{e}^{3\alpha-\frac{\lambda}{2}\phi}\right], \qquad (29)$$

$$\mathrm{k} = -\frac{3}{4}\left[1 \pm \frac{\lambda}{2\sqrt{2}}\sqrt{\frac{4}{3} + \frac{Q^2}{9\mu^2 V_0}}\right], \qquad (30)$$

where the last equation is the solution to the constraint equation (13) and the classical trayectories are given by

$$\alpha - \frac{\phi}{\lambda} = \mathrm{const.}, \qquad (31)$$

$$\phi(\tau) = \frac{2}{\lambda}\ln\left(\frac{\lambda^2}{4\mu}\Delta\tau\right), \qquad (32)$$

$$\mathrm{a}(\tau) = \left(\frac{\lambda^2}{4\mu}\Delta\tau\right)^{2/\lambda^2}, \qquad (33)$$

using (33), for obtain an increasing behaviour in the scale factor, *i.e.*, an inflationary solution is achieved if $\lambda < \sqrt{2}$ (power law), which is the well known inflationary attractor solution of an exponential potential [1, 2]. However, there exist the following constraint for the existence of the classical solution (31), $\lambda \leq \sqrt{6}$, see Ref. 2. Such a restriction does not appear explicitly in the QC formalism, but we can to show that for $\lambda = \sqrt{6}$, we recover exactly the free case, when $\mathrm{V}(\phi, \Lambda) = 0$ and $\mathrm{Q} = 0$. Recall that the latter is exactly the extreme case of stiff matter, because the scale factor of the universe evolves as $\mathrm{a}(\tau) \sim \tau^{1/3}$.

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i This action appear sin connection to the string theory landscape, see for example arXiv:hep-th/0311111 and arXiv:hep-th/0410213.

ii We are taking units such that c=G=1.

1. B. Ratra and P.J.E. Peebles, *Phys. Rev. D* **37** (1988) 3406.
2. L.P. Chimento and A.S. Jakubi, *Int. J. Mod. Phys. D* **5** (1996) 71, gr-qc/9506015.
3. E.J. Copeland, A.R. Liddle, and D. Wands, *Phys. Rev. D* **57** (1998) 4686, gr-qc/9711068.
4. D. Lyth and A. Liddle, *Cosmological Inflation and Large Scale Structure* (Cambridge University Press, 2000).
5. V. Sahni and L. Wang, *Phys. Rev. D* **62** (2000) 103517.
6. T. Matos and L.A. Ureña-López, *Class. Quantum Grav.* **17** (2000) L75, astro-ph/0004332.
7. T. Matos and L.A. Ureña-López, *Phys. Rev. D* **63** (2001) 063506, astro-ph/0006024.
8. A. Arbey, J. Lesgourgues, and P. Salati, *Phys. Rev. D* **64** (2001) 123528, astro-ph/0105564.
9. J. Lidsey, T. Matos, and L.A. Ureña-López, *Phys. Rev. D* **66** (2002) 023514, astro-ph/0111292.
10. L.G. Cabral-Rosetti, T. Matos, D. Núñez, and R. Sussman, *Class. Quantum Grav.* **19** (2002) 3603, gr-qc/0112044.
11. M. Alcubierre, F.S. Guzmán, T. Matos, D. Núñez, L.A. Ureña-López, and P. Wiederhold, *Class. Quantum Grav.* **19** (2002) 5017, gr-qc/0110102.
12. P.J.E. Peebles and B. Ratra, *Rev. Mod. Phys.* **75** (2003) 559, astro-ph/0207347.
13. U. Alam, V. Sahni, and A.A. Starobinsky, *JCAP* **0406** (2004) 008, astro-ph/0403687.
14. V. Sahni (2004), astro-ph/0403324.
15. R. Maartens (2002), prepared for 10th Greek Relativity Meeting on New Developments in Gravity (NEBX), Chalkidiki, Greece, 30 May-2 Jun 2002.
16. K. Dimopoulos, G. Lazarides, D. Lyth, and R. Ruiz de Austri, *Phys. Rev. D* **68** (2003) 123515, hep-ph/0308015.
17. L.A. Urena-Lopez and T. Matos, *Phys. Rev. D* **62** (2000) 081302, astro-ph/0003364.
18. A. Kobakhidze and L. Mersini-Houghton (2004), hep-th/0410213.
19. M.R. Douglas, *JHEP* **0305** (2003) 046, hep-th/0303194.
20. L. Susskind (2003), hep-th/0302219.
21. A. Sen, *Mod. Phys. Lett. A* **17** (2002) 1797 [arXiv:hep-th/0204143].
22. A. Sen, *Int. J. Mod. Phys. A* **18** (2003) 4869[arXiv:hep-th/0209122].
23. V. Gorini, A.Y. Kamenshchik, U. Moschella, and V. Pasquier, *Phys. Rev. D* **69** (2004) 123512 [arXiv:hep-th/0311111].
24. H. Garcia-Compean, G. Garcia-Jimenez, O. Obregon, and C.Ramirez, *Phys. Rev. D* **71** (2005) 063517.
25. M. Ryan, *Hamiltonian Cosmology* (Springer, Berlin, 1972).
26. M. MacCallum, *in General Relativity: An Einstein Centenary Survey*, edited by S. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
27. H. Kodama, *Prog. Theo. Phys.* **80** (1988) 1024.
28. H. Kodama, *Phys. Rev. D* **42** (1990) 2548.

29. A. Ashtekar, *Phys. Rev. Lett.* **57** (1986) 2244.

30. J. Hartle and S. Hawking, *Phys. Rev. D* **28** (1983) 2960.

31. V. Moncrief and M. Ryan, *PRD* **44** (1991) 2375.

32. O. Obregón and J. Socorro, *Int. J. Theor. Phys.* **35** (1996) 1381.

33. G.D. Barbosa and N. Pinto Neto, *Phys. Rev. D* **70** (2004) 103512.

34. O. Obregón and M.P. Ryan, *Mod. Phys. Lett. A* **40** (1998) 3251.

35. I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, 1980) p. 773.