REVISTA MEXICANA DE FÍSICA S **53** (2) 50–53 FEBRERO 2007

Noncommutative field theory approach to the fractional quantum Hall effect with the filling factor one-half

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Recibido el 18 de julio de 2005; aceptado el 14 de marzo de 2005

The fractional quantum Hall effect is studied in the context of the noncommutative quantum field theory in (2+1) dimensions. For the filling factor $\nu = 1/2$, the noncommutative effective field theory incorporates a Chern-Simons gauge field (in the temporal gauge) coupled to the matter in the presence of a suitable quenched external magnetic field. After providing the Feynman rules for this system, the noncommutative corrections to the self-energy of quasiparticles are computed, showing that it is zero at Hartree-Fock approximation. Finally, in this approach it is proved that the density ρ satisfies a noncommutative deformation of the w_∞-algebra.

Keywords: Noncommutative field theory; fractional quantum Hall effect; Chern-Simons theory.

El efecto Hall cuántico fraccionario se estudia en el contexto de la teoría cuántica de campos no conmutativa en (2+1)-dimensiones. Para el factor de llenado $\nu = 1/2$, la teoría de campos efectiva incorpora un campo de norma de Chern-Simons (en la norma temporal) acoplado a la materia en la presencia de un campo magnético externo apropiadamente cancelado. Después de dar las reglas de Feynman para este sistema, las correcciones no conmutativas de la autoenegía de las cuasipartículas son calculadas y se muestra que son cero en la aproximación de Hartree-Fock. Finalmente, en este enfoque se prueba que la densidad ρ satisface una deformación no conmutativa del álgebra w_∞.

Descriptores: Teoría de campos no conmutativa; efecto Hall cuántico fraccionario; teoría de Chern-Simons.

PACS: 11.10.Nx; 73.43.Lp; 11.10.Gh

1. Introduction

Recently noncommutative field theory has attracted a great deal of interest. This renewed attention was motivated mainly by the developments of D-branes in the presence of a constant Neveu-Schwarz background B-field and in M-theory (for some in recent reviews see, for instance, [1, 2].)

However, in the context of the effective low energy field theory description of condensed matter phenomena, there are also a number of works [3–6]. For instance, an electric charge moving in a plane with a strong perpendicular external magnetic field at the lowest Landau level, can be regarded as living in a noncommutative space [7, 8]. Consequently, there is a strong relation between the quantum Hall systems and the systems of noncommutative field theory.

In condensed matter physics, particles are usually regarded as effective particles (or quasiparticles); that is, the particles can be provided with some interactions that, for the moment, we are not interested in describing explicitly. This is very similar to effective field theories in quantum field theory [9]. Thus, these quasiparticles are assumed to have a kind of non-local interaction of the original particles due to their dressing, which is also a characteristic of the interacting noncommutative field theory. In addition, when we have a system in the presence of a strong external magnetic field, the system behaves as a noncommutative system [7, 10]. Then it is natural to describe some condensed matter systems in the context of the noncommutativity of the space.

In this study, we shall take a particular case, which is the fractional quantum Hall effect (FQHE) with filling factor $\nu = 1/2$. This is because, for this particular value of ν, the presence of a Fermi surface was noticed and the external magnetic field can be suppressed by attaching a magnetic flux to each particle. This also allows us to compute the effective mass of the quasiparticles. Following these ideas, a number of works have been worked out in different gauges [11–13]. The aim of the present study is to look for a noncommutative correction to this effective mass and then to establish an upper limit for the noncommutative parameter Θ.

A fruitful approach to the description of the quantum Hall systems is the introduction of a Chern-Simons gauge field that interacts with the electrons. This interaction causes to attach to each electron a magnetic flux tube [14]. For the case of a filling factor $\nu = 1/2$, there will be two fluxes attached to each electron. If there is an external magnetic field B, then the fictitious magnetic field that arises from the flux tubes exactly cancels this external magnetic field. Ignoring the fluctuation in the gauge field, we now have a residual model which consists of spinless fermions in a zero magnetic field.

The organization of our paper is as follows: In Sec. 2 we construct our model. In Sec. 3 we explicitly calculate the correction to the first order for the mass of the quasiparticles. Finally, Sec. 4 discusses our results and gives our conclusions.

2. FQHE Through Noncommutative Chern-Simons Theory

The starting point is to consider a two-dimensional system placed in a uniform magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$, which is perpendicular to the plane. We assume that all electrons are polarized, so we can ignore the spin state of the electrons.

The FQHE can be studied from the point of view of field theory in which a number $\widetilde{\phi}$ of fluxes of the magnetic field coming from a Chern-Simons gauge field are included. These new particles are known as composite fermions if $\widetilde{\phi}$ is even. In order to construct our model, we will consider only the bulk states; that is, we must have a gauge invariant theory. In this context, we will assume the temporal gauge for the Chern-Simons gauge field, *i.e.* $a_0 = 0$. The explicit transformation from Coulomb gauge to this gauge has been worked out in Ref. 15.

We are considering the fluctuation of the Chern-Simons gauge field around the mean field $\mathbf{a}_{mf} = \mathbf{A}$, where $\mathbf{A}$ is defined in the symmetric gauge: $\mathbf{A} = (B/2)\mathbf{z} \times \mathbf{x}$. This fluctuation $\delta\mathbf{a} = \mathbf{A} - \mathbf{a}$ we will denote hereafter as $\mathbf{a}$.

In this work, we use the path integral formulation of field theory. We are considering a fermion field coupled to a Chern-Simons gauge field theory, and an external electromagnetic field, which is quenched exactly at filling factor $\nu = 1/2$. We also consider an interaction between fermions. The total action describing the system is given by

$$S = S_{CS} + S_{mat} + S_{int}. \tag{1}$$

In order to construct the Feynman rules for the noncommutative model, we need to write each term of the action and make some observations:

$$S_{CS} = -\frac{1}{2\pi\widetilde{\phi}} \int dx a_i \varepsilon^{ij} \partial_0 a_j, \tag{2}$$

$$S_{mat} = \int dx \Big[\psi^\dagger(x) \star D_0 \star \psi(x) - \frac{1}{2m} (D_i \psi^\dagger(x)) \star (D_i \psi(x)) \Big] \tag{3}$$

and

$$S_{int} = \int dx dy \delta\rho(x) \star V(x,y) \star \delta\rho(y), \tag{4}$$

where the Chern-Simons term was written in the temporal gauge. In the above equations $\star$ is the Moyal product defined by

$$(f \star g)(x) = e^{\frac{i}{2}\Theta^{\mu\nu}\partial^y_\mu \partial^z_\nu f(y) g(z)|_{y=z=x}}, \tag{5}$$

where $\Theta^{\mu\nu}$ is an anti-symmetric matrix representing the noncommutativity parameter defined by $[x^\mu, x^\nu] = i\Theta^{\mu\nu}$, with $\mu, \nu = 0, 1, 2$. For applications to our system, we are considering only spatial noncommutativity, which means: $\Theta^{01} = \Theta^{02} = 0$, where the temporal coordinate is in the usual commutative form.

This total action is invariant under the following gauge transformations:

$$\begin{aligned} \psi &\rightarrow \psi + i\lambda \star \psi, \\ \psi^\dagger &\rightarrow \psi^\dagger - i\psi^\dagger \star \lambda, \\ a_\mu &\rightarrow a_\mu - \partial_\mu \lambda - i[a_\mu, \lambda]_\star. \end{aligned} \tag{6}$$

Notice that the ordering in the Moyal product is important. With this we can now define a *covariant* derivative as

$$\begin{aligned} D_\mu &= \partial_\mu \psi + i a_\mu \star \psi, \\ D^\dagger_\mu &= \partial_\mu \psi^\dagger - i\psi^\dagger \star a_\mu. \end{aligned} \tag{7}$$

Finally, note that in the interaction term of the action, $\delta\rho = \rho - \rho_0$, but in the noncommutative case it is necessary to redefine the density now with the star product:

$$\rho = \psi^\dagger \star \psi. \tag{8}$$

Following Shankar and Murthy is approach [15], we shall use the physical constraint

$$\delta\rho = \frac{1}{2\pi\widetilde{\phi}} \nabla \times \mathbf{a}. \tag{9}$$

In noncommutative field theory, it is always more convenient to work in momentum space, so that this action becomes

$$\begin{aligned} S = &\int_k \psi^\dagger \left(i\omega - \frac{1}{2m}k^2 + \mu \right) \psi - \frac{1}{2\pi\widetilde{\phi}} \int_k \epsilon^{ij} a_i(k) \omega a_j(k) \\ &- \frac{1}{2m} \Big\{ \int_k \psi^\dagger(k_1) \mathbf{k}_1 \cdot \mathbf{a}(k_2) \psi(k_3) e^{\frac{i}{2}k_3 \wedge k_2} \delta(k_3 + k_2 - k_1) \\ &+ \int_k \psi^\dagger(k_1) \mathbf{a}(k_2) \cdot \mathbf{k}_3 \psi(k_3) e^{\frac{i}{2}k_3 \wedge k_2} \delta(k_3 + k_2 - k_1) \\ &+ \int_k \psi^\dagger(k_1) \mathbf{a}(k_2) \cdot \mathbf{a}(k_3) \psi(k_4) e^{\frac{i}{2}k_4 \wedge k_3} e^{\frac{i}{2}(k_4 + k_3) \wedge k_2} \\ &\times \delta(k_4 + k_3 + k_2 - k_1) \Big\} + \int_q \delta\rho(q) V(q) \delta\rho(-q), \end{aligned} \tag{10}$$

where in the last row, we used the following definition for the Fourier transform of the density:

$$\begin{aligned} \rho(q) &= \int_x e^{iqx} \star \psi^\dagger(x) \star \psi(x) \\ &= \int_k e^{\frac{i}{2} q \wedge k \psi^\dagger(k+q) \psi(k)}, \end{aligned} \tag{11}$$

where $k \wedge q = k_\mu \Theta^{\mu\nu} q_\nu$. With this redefinition we note that the deformation of the space is absorbed into the definition of density, so that we can replace $\psi^\dagger\psi$ with $(1/2\pi\widetilde{\phi})\nabla \times \mathbf{a} + \rho_0$.

Let us reorder the action terms as

$$S_{mat} = \int_k \psi^\dagger \left(i\omega - \frac{k^2}{2m} + \mu \right) \psi,$$

$$S_{CS} = -\frac{1}{2\pi\widetilde{\phi}} \int \epsilon^{ij} a_i a_j \omega + \frac{1}{(2\pi\widetilde{\phi})^2} \int_q (\mathbf{q} \times \mathbf{a})^2 V(q)$$

$$- \frac{1}{2m}\rho_0 \int_q \mathbf{a} \cdot \mathbf{a},$$

$$S_i = \frac{1}{2m} \int \psi^\dagger(k_1)(\mathbf{k}_1 + \mathbf{k}_3) \cdot \mathbf{a}(k_2)\psi(k_3) e^{\frac{i}{2}k_3 \wedge k_2}$$

$$\times \delta(k_3 - k_2 - k_1),$$

$$S_{ia} = \frac{1}{4\pi m\widetilde{\phi}} \int ((\mathbf{k}_1 \times \mathbf{a}(k_1))\mathbf{a}(k_2) \cdot \mathbf{a}(k_3) e^{\frac{i}{2}k_3 \wedge k_2}$$

$$\times \delta(k_3 + k_2 - k_1). \quad (12)$$

The parts of the action that have modifications due to noncommutativity are the interaction part and the self-interaction of the gauge field.

The next task is to find the Feynman rules. As the propagators comes from quadratic terms of the action, then they do not undergo any modification from the noncommutativity of the space. The free propagator for the composite fermions is given by

$$G_0(k, \omega) = \frac{1}{i\omega - (\varphi(k) - \mu)}, \quad (13)$$

where $\varepsilon(k) - \mu$ is the energy of the particles around the Fermi surface. The gauge fluctuation propagates according to the equation

$$D_0(q, \omega) = U, \quad (14)$$

with

$$U^{-1} = \begin{pmatrix} -\frac{\rho_0}{2m} & -\frac{i\omega}{2\pi\widetilde{\phi}} \\ \frac{i\omega}{2\pi\widetilde{\phi}} & -\frac{\rho_0}{2m}\left[1 - \frac{2mq^2}{(2\pi\widetilde{\phi})^2\rho_0}V(q)\right] \end{pmatrix}. \quad (15)$$

As we have pointed out above, the only modification due to noncommutativity comes from the interaction vertices, so that the Feynman rule for the component l of the fermion-gauge field vertex is given by

$$\frac{1}{2m} e^{\frac{i}{2}k_3 \wedge k_2} (\mathbf{k}_1 + \mathbf{k}_3)_l. \quad (16)$$

The other vertex that corresponds to the self-interacting statistical gauge field is not written explicitly because we shall not compute corrections involving it.

3. Calculation of Self-Energy

We have now all the necessary for the computation of the first order correction of the self-energy. As mentioned above, in the one-loop correction we have two vertices with a noncommutative contribution in principle non-zero, so we need to check in this context what the self-energy is. The vertex that we are considering is such that this has two internal lines (one loop), one that corresponds to the fermion internal line, and the other one is a gauge field. The vertex to analyze takes on the following explicit form:

$$\int dk dq (\mathbf{k}_1 + \mathbf{q})_l e^{-k_1 \wedge k} G(q) D_{lm}(k)(\mathbf{k}_2 + \mathbf{q})_m$$

$$\times e^{q \wedge k} \delta(k_1 - k - q)\delta(q - k_2 + k), \quad (17)$$

where k_1 and k_2 are the external momenta, and k and q are the internal momenta for the gauge propagator and fermion propagator respectively.

When we evaluate the integration in k, this vertex becomes

$$\int dq (\mathbf{k}_1 + \mathbf{q})_l G(q) D_{lm}(k_1 - q)(\mathbf{k}_2 + \mathbf{q})_m \delta(k_1 + k_2). \quad (18)$$

From this equation, we see that the phase factor coming from the noncommutative vertex vanishes exactly; thus we recover the standard commutative result for the value of the self-energy obtained in the Hartree-Fock approximation for the self-energy of the fermion field. This result with an explicit Coulomb potential of interaction, $V(q) = \pi/\varepsilon q$, was reported before [13], and the Hartree-Fock self-energy that results is in agreement whit the results reported in Ref. 15.

4. Final Remarks

We have computed noncommutative corrections for the FQHE for the filling factor $\nu = 1/2$. In particular we calculated the correction for the self-energy diagram at the Hartree-Fock approximation. Because the procedure we have used to construct the theory, we absorbed the noncommutative deformation of the space in the density function ρ defined in Eq. (8). Consequently, the noncommutative correction for the self-energy of the quasiparticles vanishes, giving the usual correction [13, 15].

However this redefinition of the density function incorporating the noncommutativity parameter Θ leads to the definition of a new function such that it satisfies a generalization of the w_∞-algebra, given by

$$[\rho(q), \rho(q')] = 2i\rho(q + q') \sin\left(\frac{q \wedge q'}{2}\right). \quad (19)$$

This algebra was reported in the context of the quantum mechanics approach to the description of the FQHE [16,17] with a cross product rather the wedge that we use here. In the limit that $\Theta \to 1$, the wedge product leads to the cross product. This coincidence with the w_∞- algebra shows the intrinsic noncommutative nature of the FQHE.

Also it is worth noting that we only compute the self-energy to the leading order; however it is possible that the computation to a higher order in perturbation theory gives a correction for the self-energy depending on the noncommutative parameter Θ. Some results along these lines will be reported elsewhere.

Acknowledgments

This work was supported in part by a CONACyT México grant No. 33951E. The work of S.E.-J. is supported by a CONACyT graduate fellowship. S. Estrada-Jiménez whishes to thank Prof. Yong-Shi Wu for useful discussions and suggestions concerning the subject of this paper and for his kind hospitality at the University of Utah, where part of this work was outlined.

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