Spin-weighted spherical harmonics and their applications

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A brief introduction to the spin-weighted spherical harmonics is given, and some applications of these functions in the solution by separation of variables of various systems of partial differential equations are presented. The examples considered here are the source-free Maxwell equations in flat space-time and in the Schwarzschild space-time, the Einstein vacuum field equations linearized about the flat space-time and the Dirac equation.

Keywords: Spherical harmonics; Maxwell's equations; Schwarzschild solution; linearized Einstein equations; Dirac equation.

Se da una breve introducción a los armónicos esféricos con peso de espín, y se presentan algunas aplicaciones de estas funciones en la solución de varios sistemas de ecuaciones diferenciales parciales. Los ejemplos considerados aquí son las ecuaciones de Maxwell sin fuentes en espacio-tiempo plano y en el espacio-tiempo de Schwarzschild, las ecuaciones de Einstein para el vacío linealizadas alrededor del espacio-tiempo plano y la ecuación de Dirac.

Descriptores: Armónicos esféricos; ecuaciones de Maxwell; solución de Schwarzschild; ecuaciones de Einstein linealizadas; ecuación de Dirac.

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1. Introduction

The spin-weighted spherical harmonics and the spin weight raising and lowering operators were introduced by Newman and Penrose [1] in the study of the asymptotic behavior of the gravitational field (see also Refs. 2 and 3). These functions are essentially the "monopole harmonics" that arise in the solution of the Schrödinger equation for a charged particle in the field of a magnetic monopole [4,5] and can also be expressed in terms of the Wigner D-functions [6,7], the Jacobi polynomials, the generalized associated Legendre functions and the hypergeometric functions [7]. However, the fact that the spin weight raising and lowering operators, $\eth$ ("eth") and $\overline{\eth}$ ("eth bar"), appear in a natural way when the equations for nonzero spin fields are written in spherical coordinates in terms of certain combinations of the field components (those with a definite spin weight), makes the spin-weighted spherical harmonics particularly useful (see also Refs. 7 to 15).

In the standard treatment of nonscalar fields in spherical coordinates, a variety of vector, tensor, or spinor fields is employed with widely variable notations and conventions; in some cases these fields are constructed by coupling the ordinary spherical harmonics with eigenfunctions of the corresponding spin operators. By contrast, the spin-weighted spherical harmonics provide a straightforward and uniform formalism applicable to fields of any spin.

In this paper, the spin-weighted spherical harmonics are defined following Refs. 16 and 17, making use of the representation of vectors by means of two-component spinors, and some illustrative examples of their application in the solution by separation of variables of equations for fields of spin 1/2, 1, and 2 are given. In Sec. 2, the spin-weighted spherical harmonics are defined. In Sec. 3, the source-free Maxwell equations in flat space-time are solved and in Sec. 4, a similar integration is presented assuming that the background space-time is that represented by the Schwarzschild metric. In Sec. 5, the Einstein vacuum field equations linearized about the Minkowski metric are solved and in Sec. 6, the Dirac equation in spherical coordinates is solved.

The results given in Secs. 3 and 5, below, coincide with those obtained in Refs. 7, 8, and 12 by means of a different approach, while the derivation of the expression for the electromagnetic field in the Schwarzschild space-time is presented here for the first time.

2. Spin-weighted spherical harmonics

2.1. Spherical harmonics

The Laplacian operator in the three-dimensional Euclidean space expressed in terms of the spherical coordinates is given by

$$\nabla^2 f = \frac{1}{r}\partial_r^2(rf) - \frac{1}{r^2}L^2 f, \tag{1}$$

where $L^2 = (-\mathrm{i}\mathbf{r}\times\nabla)^2$ is the square angular momentum operator in units such that $\hbar = 1$. Hence, a function of the form $f(r,\theta,\phi) = r^l g(\theta,\phi)$ $(l = 0,1,\ldots)$ satisfies the Laplace equation, $\nabla^2 f = 0$, provided that

$$L^2 g = l(l+1)g, \tag{2}$$

i.e. $g(\theta,\phi)$ is a spherical harmonic of order l. A simple way of finding the spherical harmonics then follows from the fact that any polynomial in the *Cartesian coordinates* $(x,y,z) = (x^1,x^2,x^3)$ of the form

$$f(x,y,z) = d_{ij\ldots k}x^i x^j \cdots x^k, \tag{3}$$

$(i, j, \ldots = 1, 2, 3)$, where the $d_{ij\ldots k}$ are constant (real or complex) coefficients totally symmetric, satisfies the Laplace equation if and only if the trace of $d_{ij\ldots k}$ vanishes,

$$d_{iik\ldots m} = 0. \tag{4}$$

(Throughout this paper there is summation over repeated indices.) In effect, the Laplacian operator is given by $\nabla^2 f = \partial_i \partial_i f$, where $\partial_i = \partial/\partial x^i$; hence, if $d_{ij\ldots k}$ has l indices, making use of the symmetry of $d_{ij\ldots k}$,

$$\begin{aligned}\nabla^2(d_{ij\ldots k}x^i x^j \cdots x^k) &= \partial_m \partial_m (d_{ij\ldots k}x^i x^j \cdots x^k) \\ &= \partial_m (l d_{mjs\ldots k} x^j x^s \cdots x^k) \\ &= l(l-1) d_{mms\ldots k} x^s \cdots x^k.\end{aligned}$$

Thus, assuming that Eq. (4) holds,

$$f(x, y, z) = d_{ij\ldots k}x^i x^j \cdots x^k = r^l d_{ij\ldots k} N^i N^j \cdots N^k,$$

where $N^i \equiv x^i/r$, is a solution of the Laplace equation and

$$d_{ij\ldots k} N^i N^j \cdots N^k \tag{5}$$

is a spherical harmonic of order l (see also Ref. 18). Among other things, Eq. (5) shows that, under the inversion $(x, y, z) \mapsto (-x, -y, -z)$, a spherical harmonic of order l is multiplied by a factor $(-1)^l$, which means that the parity of a spherical harmonic of order l is equal to $(-1)^l$.

Despite the simplicity of the condition (4), expression (5) is not particularly useful for finding the explicit form of the spherical harmonics. However, by expressing the components N^i in terms of spinors, one obtains a useful representation for the standard spherical harmonics Y_{lm}.

2.2. Spinors and spin weight

Let V be a three-dimensional real vector space with a positive definite interior product, and let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis of V. As we shall show, it is convenient to introduce another set of three vectors labelled by two indices that take two values only, *e.g.* 1 and 2,

$$\begin{aligned}\mathbf{e}_{11} &\equiv \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathrm{i}\mathbf{e}_2), \qquad \mathbf{e}_{12} = \mathbf{e}_{21} \equiv -\frac{1}{\sqrt{2}}\mathbf{e}_3, \\ \mathbf{e}_{22} &\equiv \frac{1}{\sqrt{2}}(-\mathbf{e}_1 + \mathrm{i}\mathbf{e}_2).\end{aligned} \tag{6}$$

These vectors form a basis of (the complexification of) V; in fact, defining

$$\begin{aligned}v_{11} &\equiv \frac{1}{\sqrt{2}}(v_1 + \mathrm{i}v_2), \qquad v_{12} = v_{21} \equiv -\frac{1}{\sqrt{2}}v_3, \\ v_{22} &\equiv \frac{1}{\sqrt{2}}(-v_1 + \mathrm{i}v_2)\end{aligned} \tag{7}$$

[*cf.* Eqs. (6)], where the v_i are the components with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of an arbitrary vector, one finds that

$$v^i \mathbf{e}_i = -v_{11}\mathbf{e}_{22} + 2v_{12}\mathbf{e}_{12} - v_{22}\mathbf{e}_{11} = -v^{AB}\mathbf{e}_{AB}, \tag{8}$$

where, as in what follows, the capital Latin indices $A, B, \ldots$, take on the values 1 and 2, and these indices (which will be called spinor indices) are lowered or raised following the rules

$$\psi_A = \varepsilon_{AB}\psi^B, \qquad \psi^A = -\varepsilon^{AB}\psi_B, \tag{9}$$

where ε_{AB} is the Levi-Civita symbol: ε_{12}=1, $\varepsilon_{21}= -1$, ε_{11}=0=ε_{22}. Then we have, for instance, ψ^2=ψ_1, ψ^1=$-\psi_2$, v^{11}=v_{22}, v^{12}=$-v_{21}$. (On the other hand, the tensor indices $i, j, \ldots$ are raised or lowered using the Kronecker delta and therefore, *e.g.*, $v^i = v_i$.) In this manner, instead of representing a vector by means of an array of the form (v^1, v^2, v^3), we will have a symmetric matrix $\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22}\end{pmatrix}$.

If $v^i\mathbf{e}_i$ and $w^i\mathbf{e}_i$ are two arbitrary vectors, then their scalar product is given by

$$v^i w_i = -v^{AB}w_{AB}. \tag{10}$$

Owing to the rules (9), we have

$$\begin{aligned}\psi_A\phi^A &= \psi_1\phi^1 + \psi_2\phi^2 = -\psi_1\phi_2 + \psi_2\phi_1 \\ &= -\psi^2\phi_2 - \psi^1\phi_1 = -\psi^A\phi_A;\end{aligned} \tag{11}$$

therefore $v^{AB}w_{AB} = v_{AB}w^{AB}$ and $\phi^A\phi_A = 0$.

Let λ^1, λ^2 be two auxiliary complex variables; the double sum $v_{AB}\lambda^A\lambda^B$ is explicitly given by

$$\begin{aligned}v_{AB}\lambda^A\lambda^B &= v_{11}(\lambda^1)^2 + 2v_{12}\lambda^1\lambda^2 + v_{22}(\lambda^2)^2 \\ &= (\lambda^2)^2\left[v_{11}(\lambda^1/\lambda^2)^2 + 2v_{12}(\lambda^1/\lambda^2) + v_{22}\right] \\ &= (\lambda^2)^2 v_{11}\Big((\lambda^1/\lambda^2) - r_a\Big)\Big((\lambda^1/\lambda^2) - r_b\Big),\end{aligned}$$

where r_a and r_b are the roots of the polynomial $v_{11}z^2$+$2v_{12}z$+v_{22}. Hence,

$$v_{AB}\lambda^A\lambda^B = v_{11}(\lambda^1 - r_a\lambda^2)(\lambda^1 - r_b\lambda^2). \tag{12}$$

The values of r_a and r_b are [see Eq. (7)]

$$\begin{aligned}r_{a,b} &= \frac{-2v_{12} \pm \sqrt{(2v_{12})^2 - 4v_{11}v_{22}}}{2v_{11}} \\ &= \frac{v_3 \pm \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}}{v_1 + \mathrm{i}v_2} \\ &= \frac{v\cos\theta \pm v}{v\sin\theta\,\mathrm{e}^{\mathrm{i}\phi}},\end{aligned}$$

where v, θ, and ϕ are the norm and the polar and azimuth angles of $v^i\mathbf{e}_i$. Thus

$$r_{a,b} = \mathrm{e}^{-\mathrm{i}\phi}\frac{\cos\theta \pm 1}{\sin\theta} = \mathrm{e}^{-\mathrm{i}\phi}\frac{\cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta \pm 1}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta},$$

i.e.

$$r_a = \mathrm{e}^{-\mathrm{i}\phi}\cot\frac{1}{2}\theta, \qquad r_b = -\mathrm{e}^{-\mathrm{i}\phi}\tan\frac{1}{2}\theta,$$

and therefore, from (12)

$$\begin{aligned} v_{AB}\lambda^A\lambda^B &= \frac{1}{\sqrt{2}} v\, \mathrm{e}^{\mathrm{i}\phi} \sin\theta(\lambda^1 - \mathrm{e}^{-\mathrm{i}\phi}\cot\frac{1}{2}\theta\,\lambda^2) \\ &\quad \times (\lambda^1 + \mathrm{e}^{-\mathrm{i}\phi}\tan\frac{1}{2}\theta\,\lambda^2) \\ &= \sqrt{2}\, v\, \mathrm{e}^{\mathrm{i}\phi}\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta(\lambda^1 - \mathrm{e}^{-\mathrm{i}\phi}\cot\frac{1}{2}\theta\,\lambda^2) \\ &\quad \times (\lambda^1 + \mathrm{e}^{-\mathrm{i}\phi}\tan\frac{1}{2}\theta\,\lambda^2) \\ &= \sqrt{2}\, v(\mathrm{e}^{\mathrm{i}\phi/2}\sin\frac{1}{2}\theta\,\lambda^1 - \mathrm{e}^{-\mathrm{i}\phi/2}\cos\frac{1}{2}\theta\,\lambda^2) \\ &\quad \times (\mathrm{e}^{\mathrm{i}\phi/2}\cos\frac{1}{2}\theta\,\lambda^1 + \mathrm{e}^{-\mathrm{i}\phi/2}\sin\frac{1}{2}\theta\,\lambda^2). \end{aligned}$$

Now, letting

$$\begin{pmatrix} o^1 \\ o^2 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2}\cos\frac{1}{2}\theta \\ \mathrm{e}^{\mathrm{i}\phi/2}\sin\frac{1}{2}\theta \end{pmatrix}, \tag{13}$$

we can write $\mathrm{e}^{\mathrm{i}\phi/2}\sin\frac{1}{2}\theta\,\lambda^1 - \mathrm{e}^{-\mathrm{i}\phi/2}\cos\frac{1}{2}\theta\,\lambda^2 = o_A\lambda^A$. Defining the conjugate or mate of o^A by

$$\widehat{o}_A \equiv \overline{o^A}, \tag{14}$$

we have

$$\begin{pmatrix} \widehat{o}^1 \\ \widehat{o}^2 \end{pmatrix} = \begin{pmatrix} -\mathrm{e}^{-\mathrm{i}\phi/2}\sin\frac{1}{2}\theta \\ \mathrm{e}^{\mathrm{i}\phi/2}\cos\frac{1}{2}\theta \end{pmatrix}, \tag{15}$$

and, therefore, $\mathrm{e}^{\mathrm{i}\phi/2}\cos\frac{1}{2}\theta\,\lambda^1 + \mathrm{e}^{-\mathrm{i}\phi/2}\sin\frac{1}{2}\theta\,\lambda^2 = \widehat{o}_B\lambda^B$. Hence,

$$v_{AB}\lambda^A\lambda^B = \sqrt{2}\, v\, o_A\lambda^A\widehat{o}_B\lambda^B,$$

which implies that

$$v_{AB} = \sqrt{2}\, v\, o_{(A}\widehat{o}_{B)}, \tag{16}$$

where the parentheses denote symmetrization on the indices enclosed [*e.g.*, $o_{(A}\widehat{o}_{B)} = \frac{1}{2}(o_A\widehat{o}_B + o_B\widehat{o}_A)$]. In the case of a complex vector $\mathbf{w}$, we can write $w_{AB} = \alpha_{(A}\beta_{B)}$, where β_A is not proportional to the mate of α_A.

If $(U^A{}_B)$ is a 2×2 matrix belonging to SU(2), and $o'^A \equiv U^A{}_B o^B$, then the mate of o^A transforms in the same manner, $\widehat{o}'^A = U^A{}_B\widehat{o}^B$, inducing the transformation $v'^{AB} = U^A{}_C U^B{}_D v^{CD}$ on the spinor components of a vector [see Eq. (16)], which corresponds to a rotation about the origin. This conclusion follows from the fact that the transformation $v^{AB} \mapsto v'^{AB}$ is linear and the norm of a vector is preserved under this transformation:

$$v'^{AB}v'_{AB} = U^A{}_C U^B{}_D v^{CD} U_{AR} U_{BS} v^{RS}. \tag{17}$$

But, according to the rules (9),

$$\begin{aligned} U^A{}_C U_{AR} &= U^1{}_C U_{1R} + U^2{}_C U_{2R} \\ &= U^1{}_C U^2{}_R - U^2{}_C U^1{}_R \\ &= U^1{}_C U^2{}_R - U^1{}_R U^2{}_C \\ &= (U^1{}_1 U^2{}_2 - U^1{}_2 U^2{}_1)\,\varepsilon_{CR}. \end{aligned}$$

One can convince oneself of the validity of the last equality noting that the two expressions coincide for each combination of values of the spinor indices C and R, taking into account the definition of the Levi-Civita symbol ε_{CR}. Hence, for any matrix with unit determinant, $(U^A{}_B)$,

$$U^A{}_C U_{AR} = \varepsilon_{CR},$$

and Eqs. (17) and (9) give

$$v'^{AB}v'_{AB} = \varepsilon_{CR}\varepsilon_{DS}v^{CD}v^{RS} = v^{CD}v_{CD}$$

(which amounts to $v'^i v'_i = v^i v_i$ [see Eq. (10)]). It can be shown that the determinant of the induced transformation $v^i \mapsto v'^i$ is positive.

Equations (6) and (7) can be written in the form

$$\mathbf{e}_{AB} = \frac{1}{\sqrt{2}}\sigma^i{}_{AB}\,\mathbf{e}_i \qquad \text{and} \qquad v_{AB} = \frac{1}{\sqrt{2}}\sigma^i{}_{AB}v_i \tag{18}$$

with

$$(\sigma_{1AB}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (\sigma_{2AB}) = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix},$$

$$(\sigma_{3AB}) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{19}$$

Then, from Eq. (10) we can see that

$$\sigma_i{}^{AB}\sigma^j{}_{AB} = -2\delta_i^j \tag{20}$$

and therefore

$$\mathbf{e}_i = -\frac{1}{\sqrt{2}}\sigma_i{}^{AB}\,\mathbf{e}_{AB} \quad \text{and} \quad v_i = -\frac{1}{\sqrt{2}}\sigma_i{}^{AB}v_{AB}. \tag{21}$$

In the case of a unit vector, N^i, from Eqs. (16) and (21) it follows that $N^i = -\sigma^i{}_{AB}o^A\widehat{o}^B$. Hence, going back to expression (5) for spherical harmonics, we find that any spherical harmonic of order l can also be expressed in the form

$$\begin{aligned} & d_{ij\ldots k}\underbrace{N^iN^j\cdots N^k}_{l} \\ &= (-1)^l d_{ij\ldots k}\,\sigma^i{}_{AB}o^A\widehat{o}^B\,\sigma^j{}_{CD}o^C\widehat{o}^D\cdots\sigma^k{}_{EF}o^E\widehat{o}^F \\ &= (-\sqrt{2})^l d_{ABCD\ldots EF}o^A\widehat{o}^B o^C\widehat{o}^D\cdots o^E\widehat{o}^F \end{aligned}$$

with

$$d_{ABCD\ldots EF} \equiv \left(\frac{1}{\sqrt{2}}\right)^l \sigma^i{}_{AB}\,\sigma^j{}_{CD}\cdots\sigma^k{}_{EF}\,d_{ij\ldots k} \tag{22}$$

[*cf.* Eqs. (18)]. Since $\sigma_{iAB} = \sigma_{iBA}$ [see Eqs. (19)], the coefficients $d_{ABCD\ldots}$ satisfy

$$d_{ABCD\ldots EF} = d_{BACD\ldots EF} = d_{ABDC\ldots EF} = d_{ABCD\ldots FE}$$

and, as a consequence of the symmetry of $d_{ij\ldots k}$, $d_{ABCD\ldots EF} = d_{CDAB\ldots EF}$. Furthermore, a difference of the form $d_{ABCD\ldots EF} - d_{ACBD\ldots EF}$ is given by

$$\begin{aligned} d_{ABCD\ldots EF} - d_{ACBD\ldots EF} &= (d_{A12D\ldots EF} - d_{A21D\ldots EF})\,\varepsilon_{BC} \\ &= d_A{}^R{}_{RD\ldots EF}\,\varepsilon_{BC} \end{aligned}$$

and, using Eqs. (22), (11), (20), and (4),

$$\begin{aligned} 2^{l/2} d_A{}^R{}_{RD\ldots EF} &= \sigma^i{}_A{}^R \sigma^j{}_{RD} \cdots \sigma^k{}_{EF}\, d_{ij\ldots k} \\ &= \frac{1}{2}(\sigma^i{}_A{}^R\sigma^j{}_{RD} + \sigma^j{}_A{}^R\sigma^i{}_{RD}) \cdots \sigma^k{}_{EF}\, d_{ij\ldots k} \\ &= \frac{1}{2}(\sigma^i{}_A{}^R\sigma^j{}_{RD} - \sigma^i{}_D{}^R\sigma^j{}_{RA}) \cdots \sigma^k{}_{EF}\, d_{ij\ldots k} \\ &= \frac{1}{2}\sigma^{iSR}\sigma^j{}_{RS}\varepsilon_{AD} \cdots \sigma^k{}_{EF}\, d_{ij\ldots k} \\ &= -\delta^{ij}\varepsilon_{AD} \cdots \sigma^k{}_{EF}\, d_{ij\ldots k} \\ &= -\varepsilon_{AD} \cdots \sigma^k{}_{EF}\, d_{ii\ldots k} = 0, \end{aligned}$$

which means that $d_{ABCD\ldots EF}$ is totally symmetric under all transpositions of its indices. Hence, any spherical harmonic of order l has the form

$$d_{AB\ldots CDE\ldots F}\, \underbrace{o^A o^B \cdots o^C}_{l}\, \underbrace{\widehat{o}^D \widehat{o}^E \cdots \widehat{o}^F}_{l}, \qquad (23)$$

with $d_{ABCD\ldots EF}$ being totally symmetric in its $2l$ indices. For example, any spherical harmonic of order 2 has the form

$$\begin{aligned} d_{ABCD} o^A o^B \widehat{o}^C \widehat{o}^D &= d_{1111} o^1 o^1 \widehat{o}^1 \widehat{o}^1 + d_{1112}(2o^1o^1\widehat{o}^1\widehat{o}^2 + 2o^1o^2\widehat{o}^1\widehat{o}^1) \\ &\quad + d_{1122}(o^1o^1\widehat{o}^2\widehat{o}^2 + 4o^1o^2\widehat{o}^1\widehat{o}^2 + o^2o^2\widehat{o}^1\widehat{o}^1) \\ &\quad + d_{1222}(2o^1o^2\widehat{o}^2\widehat{o}^2 + 2o^2o^2\widehat{o}^1\widehat{o}^2) + d_{2222} o^2o^2\widehat{o}^2\widehat{o}^2 \\ &= d_{1111}\,\tfrac{1}{4}\mathrm{e}^{-2\mathrm{i}\phi}\sin^2\theta - d_{1112}\,\mathrm{e}^{-\mathrm{i}\phi}\sin\theta\cos\theta \\ &\quad + d_{1122}\,\frac{1}{2}(3\cos^2 - 1) + d_{1222}\,\mathrm{e}^{\mathrm{i}\phi}\sin\theta\cos\theta \\ &\quad + d_{2222}\,\tfrac{1}{4}\mathrm{e}^{2\mathrm{i}\phi}\sin^2\theta \\ &= d_{1111}\sqrt{\frac{2\pi}{15}}\,Y_{2,-2} - d_{1112}\sqrt{\frac{8\pi}{15}}\,Y_{2,-1} \\ &\quad + d_{1122}\sqrt{\frac{4\pi}{5}}\,Y_{2,0} - d_{1222}\sqrt{\frac{8\pi}{15}}\,Y_{2,1} + d_{2222}\sqrt{\frac{2\pi}{15}}\,Y_{2,2}, \end{aligned}$$

where d_{1111}, d_{1112}, d_{1122}, d_{1222}, and d_{2222}, are arbitrary complex numbers and we have made use of the standard definition of the spherical harmonics Y_{lm}.

A quantity η is said to have spin weight s if, under the transformation $o^A \mapsto \mathrm{e}^{\mathrm{i}\chi/2} o^A$, it transforms into $\mathrm{e}^{\mathrm{i}s\chi}\eta$ (hence, the components o^A have spin weight 1/2). Then, from Eq. (14) it follows that $\widehat{o}^A$ has spin weight $-1/2$ and Eq. (23) shows that an ordinary spherical harmonic has spin weight equal to zero. By definition, a spin-weighted spherical harmonic of order j and spin weight s will be an expression of the form

$${}_sP_j = d_{AB\ldots CDE\ldots F}\, \underbrace{o^A o^B \cdots o^C}_{j+s}\, \underbrace{\widehat{o}^D \widehat{o}^E \cdots \widehat{o}^F}_{j-s}, \qquad (24)$$

where the coefficients $d_{AB\ldots F}$ are totally symmetric in their $2j$ indices ($j = 0, \frac{1}{2}, 1, \ldots$). According to the definition given above, the function (24) has spin weight s. Since $j + s$ and $j - s$ must be integral numbers, j and s must both be integers or half-integers and

$$|s| \leqslant j. \qquad (25)$$

An alternative characterization of the spin-weighted spherical harmonics, analogous to Eq. (2), can be given making use of the operators ð and $\overline{\text{ð}}$ defined by [1]

$$\begin{aligned} \eth\eta &= -\left(\partial_\theta + \frac{\mathrm{i}}{\sin\theta}\partial_\phi - s\cot\theta\right)\eta \\ &= -\sin^s\theta\left(\partial_\theta + \frac{\mathrm{i}}{\sin\theta}\partial_\phi\right)(\eta\sin^{-s}\theta), \\ \overline{\eth}\eta &= -\left(\partial_\theta - \frac{\mathrm{i}}{\sin\theta}\partial_\phi + s\cot\theta\right)\eta \\ &= -\sin^{-s}\theta\left(\partial_\theta - \frac{\mathrm{i}}{\sin\theta}\partial_\phi\right)(\eta\sin^s\theta), \end{aligned} \qquad (26)$$

where s is the spin weight of η. Then one finds that, for $A = 1, 2$,

$$\eth o^A = 0, \quad \eth\widehat{o}^A = o^A, \quad \overline{\eth} o^A = -\widehat{o}^A, \quad \overline{\eth}\widehat{o}^A = 0, \qquad (27)$$

and that ð and $\overline{\text{ð}}$ are linear and satisfy the Leibniz rule. A straightforward computation shows that

$$\begin{aligned} \overline{\eth}\eth\,{}_sP_j &= -[j(j+1) - s(s+1)]\,{}_sP_j, \\ \eth\overline{\eth}\,{}_sP_j &= -[j(j+1) - s(s-1)]\,{}_sP_j. \end{aligned} \qquad (28)$$

Furthermore, $\overline{\eth}\eth f = -L^2 f = \eth\overline{\eth} f$, if f is a function with spin weight equal to zero and applying ð or $\overline{\text{ð}}$ to a spin-weighted spherical harmonic of the form (24), one obtains another spin-weighted spherical harmonic with spin weight $s+1$ or $s-1$, respectively. That is, ð and $\overline{\text{ð}}$ raise and lower, respectively, the spin weight in one unit.

The symbol ${}_sY_{jm}$ will denote a spin-weighted spherical harmonic with spin weight s, order j, with a dependence on ϕ of the form $\mathrm{e}^{\mathrm{i}m\phi}$ such that

$$\int_0^{2\pi}\int_0^{\pi} |{}_sY_{jm}|^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = 1.$$

The phase of these functions can be chosen in such a way that [1,7,17]

$$\eth\, {}_sY_{jm} = [j(j+1) - s(s+1)]^{1/2}\, {}_{s+1}Y_{jm},$$
$$\overline{\eth}\, {}_sY_{jm} = -[j(j+1) - s(s-1)]^{1/2}\, {}_{s-1}Y_{jm}, \qquad (29)$$

with ${}_0Y_{jm}$=Y_{jm}. From Eqs. (13), (15), and (24) one finds that, as in the case of the ordinary spherical harmonics, $m = -j, -j+1, \ldots, j$.

An important fact is that for a fixed value s of the spin weight, the set of the spin-weighted spherical harmonics $\{{}_sY_{jm}\}$ is complete (as well as orthonormal) in the sense that any function with spin weight s can be written as a series in ${}_sY_{jm}$ [1,7].

3. Solution of the source-free Maxwell equations

By construction, $N_i = -\sigma_i{}^{AB} o_A \widehat{o}_B$ are the Cartesian components of a unit radial vector, which is one of the vectors of the orthonormal basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$, induced by the spherical coordinates (r, θ, ϕ). It turns out that $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are the real and imaginary parts of the vector with Cartesian components $\sigma_i{}^{AB} o_A o_B$, *i.e.*

$$(\mathbf{e}_r)_i = -\sigma_i{}^{AB} o_A \widehat{o}_B, \quad (\mathbf{e}_\theta + \mathrm{i}\mathbf{e}_\phi)_i = \sigma_i{}^{AB} o_A o_B. \qquad (30)$$

An arbitrary vector field $\mathbf{F}$ in the three-dimensional Euclidean space can be written in the form

$$\mathbf{F} = F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi$$

or, equivalently,

$$\mathbf{F} = -\sqrt{2}\, F_0\, \mathbf{e}_r - \frac{1}{\sqrt{2}} F_{-1} (\mathbf{e}_\theta + \mathrm{i}\mathbf{e}_\phi) + \frac{1}{\sqrt{2}} F_{+1} (\mathbf{e}_\theta - \mathrm{i}\mathbf{e}_\phi), \qquad (31)$$

with

$$F_0 \equiv -\frac{1}{\sqrt{2}}\, F_r, \qquad F_{\pm 1} \equiv \pm \frac{1}{\sqrt{2}} (F_\theta \pm \mathrm{i} F_\phi). \qquad (32)$$

According to Eqs. (18) and (30) we also have

$$F_0 = F_{AB} o^A \widehat{o}^B,$$
$$F_{+1} = F_{AB} o^A o^B, \qquad (33)$$
$$F_{-1} = F_{AB} \widehat{o}^A \widehat{o}^B.$$

These last expressions show that F_s has spin weight equal to s ($s = 0, 1, -1$). The functions F_s will be referred to as the spin-weighted components of $\mathbf{F}$.

Making use of the standard expression for the divergence of a vector field,

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2} \partial_r (r^2 F_r) + \frac{1}{r \sin\theta} \partial_\theta (F_\theta \sin\theta) + \frac{1}{r \sin\theta} \partial_\phi F_\phi,$$

one obtains [see Eqs. (26)]

$$\nabla \cdot \mathbf{F} = -\frac{\sqrt{2}}{r^2} \partial_r (r^2 F_0) + \frac{1}{\sqrt{2}\, r} (\eth F_{-1} - \overline{\eth} F_{+1}). \qquad (34)$$

Similarly, one finds that

$$\nabla \times \mathbf{F} = \frac{\mathrm{i}}{\sqrt{2}\, r} (\eth F_{-1} + \overline{\eth} F_{+1})\, \mathbf{e}_r + \frac{\mathrm{i}}{\sqrt{2}\, r} \left[\partial_r (r F_{-1}) + \overline{\eth} F_0\right] (\mathbf{e}_\theta + \mathrm{i}\mathbf{e}_\phi) + \frac{\mathrm{i}}{\sqrt{2}\, r} \left[\partial_r (r F_{+1}) - \eth F_0\right] (\mathbf{e}_\theta - \mathrm{i}\mathbf{e}_\phi). \qquad (35)$$

The source-free Maxwell equations in a vacuum (in cgs units) can be written as

$$\nabla \cdot \mathbf{F} = 0, \qquad \nabla \times \mathbf{F} = \frac{\mathrm{i}}{c} \partial_t \mathbf{F}, \qquad (36)$$

where $\mathbf{F} \equiv \mathbf{E} + \mathrm{i}\mathbf{B}$. Expressed in terms of the spin-weighted components of $\mathbf{F}$, these equations read [see Eqs. (34) and (35)]

$$-\frac{2}{r} \partial_r (r^2 F_0) + \eth F_{-1} - \overline{\eth} F_{+1} = 0,$$
$$-\frac{1}{r} (\eth F_{-1} + \overline{\eth} F_{+1}) = \frac{2}{c} \partial_t F_0,$$
$$-\frac{1}{r} [\partial_r (r F_{-1}) + \overline{\eth} F_0] = \frac{1}{c} \partial_t F_{-1},$$
$$\frac{1}{r} [\partial_r (r F_{+1}) - \eth F_0] = \frac{1}{c} \partial_t F_{+1}. \qquad (37)$$

Looking for separable solutions of this set of equations of the form

$$F_s = g_s(t, r)\, {}_sY_{jm}(\theta, \phi), \qquad s = 0, \pm 1, \qquad (38)$$

making use of Eqs. (29,) one obtains

$$-\frac{2}{r} \partial_r (r^2 g_0) + \sqrt{j(j+1)} (g_{-1} + g_1) = 0,$$
$$\frac{\sqrt{j(j+1)}}{r} (g_1 - g_{-1}) = \frac{2}{c} \partial_t g_0,$$
$$-\frac{1}{r} [\partial_r (r g_{-1}) - \sqrt{j(j+1)} g_0] = \frac{1}{c} \partial_t g_{-1},$$
$$\frac{1}{r} [\partial_r (r g_1) - \sqrt{j(j+1)} g_0] = \frac{1}{c} \partial_t g_1. \qquad (39)$$

These equations can be combined to obtain a second-order equation for g_0, g_1, or g_{-1}. In order to find expressions equivalent to those given in Refs. 7, 8, 19, and 20 we apply the operator ∂_t to both sides of the second equation (39),

$$-\frac{2}{c^2} \partial_t^2 (r g_0) = \sqrt{j(j+1)}\, \frac{1}{c} \partial_t (g_{-1} - g_1)$$

and then, making use of the last two equations (39),

$$\begin{aligned}
-\frac{2}{c^2}\partial_t^2(rg_0)&\\
&= \frac{\sqrt{j(j+1)}}{r}[-\partial_r(rg_{-1}) + 2\sqrt{j(j+1)}\,g_0 - \partial_r(rg_1)]\\
&= 2j(j+1)\frac{g_0}{r} - \frac{1}{r}\partial_r[\sqrt{j(j+1)}\,r(g_{-1}+g_1)]\\
&= 2j(j+1)\frac{g_0}{r} - \frac{2}{r}\partial_r^2(r^2 g_0).
\end{aligned}$$

Letting

$$\chi \equiv -\frac{\sqrt{2}\,rg_0\,Y_{jm}}{j(j+1)},$$

one finds that χ satisfies the scalar wave equation,

$$\nabla^2\chi - (1/c^2)\partial_t^2\chi = 0$$

[see Eqs. (1) and (2)], and from Eqs. (29), (38), and (39),

$$\begin{aligned}
F_{+1} &= -\frac{1}{\sqrt{2}\,r}\left(\frac{1}{c}\partial_t + \partial_r\right) r\eth\chi,\\
F_0 &= \frac{1}{\sqrt{2}\,r}\bar{\eth}\eth\chi, \qquad (40)\\
F_{-1} &= -\frac{1}{\sqrt{2}\,r}\left(\frac{1}{c}\partial_t - \partial_r\right) r\bar{\eth}\chi.
\end{aligned}$$

One can verify that these are the spin-weighted components of the vector field

$$\mathbf{F} = -\frac{\mathrm{i}}{c}\partial_t(\mathbf{r}\times\nabla\chi) - \nabla\times(\mathbf{r}\times\nabla\chi). \qquad (41)$$

The labels j and m contained in the separable solution obtained above have a direct physical meaning; they determine the eigenvalues of the operators representing the square total angular momentum and the z-component of the total angular momentum of the field, respectively [8,7]. By virtue of the completeness of the spin-weighted spherical harmonics and of the linearity of the Maxwell equations and of the expressions (40) or (41), any solution of the Maxwell equations can be represented in the form (41), where χ is a (possibly complex) solution of the scalar wave equation.

4. Electromagnetic perturbations of the Schwarzschild solution

In this section, we directly integrate the source-free Maxwell equations assuming that the background space-time is the Schwarzschild solution, neglecting the effect of the energy-momentum of the electromagnetic field on the space-time curvature. The result obtained here by elementary means is equivalent to that obtained using other approaches (see, *e.g.*, Refs. 21 to 25) which are applicable to more general backgrounds (*e.g.* the algebraically special vacuum space-times).

As is well-known, the Maxwell equations for the source-free electromagnetic field on a possibly curved space-time can be written as

$$\begin{aligned}
\partial_\alpha f_{\beta\gamma} + \partial_\gamma f_{\alpha\beta} + \partial_\beta f_{\gamma\alpha} &= 0,\\
\frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}f^{\alpha\beta}) &= 0, \qquad (42)
\end{aligned}$$

where $f^{\alpha\beta}$ denotes the electromagnetic field tensor, $\partial_\alpha{\equiv}\partial/\partial x^\alpha$, the x^α are space-time coordinates, $g{=}\det(g_{\alpha\beta})$, with $g_{\alpha\beta}$ being the components of the metric tensor in the coordinate system x^α, $f_{\alpha\beta}{=}g_{\alpha\gamma}g_{\beta\delta}f^{\gamma\delta}$ and the Greek lower case indices run from 0 to 3 (see for example Ref. 26). The Schwarzschild metric, which corresponds to the exterior gravitational field of a spherically symmetric matter distribution, is usually written in the form

$$\mathrm{d}s^2 = -\left(1-\frac{2GM}{c^2 r}\right)c^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\dfrac{2GM}{c^2 r}} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2),$$

in terms of coordinates $(x^0,x^1,x^2,x^3)=(ct,r,\theta,\phi)$, where M is a constant; hence

$$\begin{aligned}
(g_{\mu\nu}) &= \mathrm{diag}(-h(r), h(r)^{-1}, r^2, r^2\sin^2\theta),\\
h(r) &\equiv 1-\frac{2GM}{c^2 r} \qquad (43)
\end{aligned}$$

and $\sqrt{|g|} = r^2\sin\theta$.

Making use of Eq. (43) and the definitions

$$\begin{aligned}
F_0 &= -\frac{1}{\sqrt{2}}(f^{01} + \mathrm{i}r^2\sin\theta\, f^{23}),\\
F_{\pm1} &= \pm\frac{r}{\sqrt{2}}\left[f^{02} + \mathrm{i}\frac{\sin\theta}{h}f^{31} \pm \mathrm{i}\left(\sin\theta\, f^{03} + \mathrm{i}\frac{f^{12}}{h}\right)\right], \qquad (44)
\end{aligned}$$

a straightforward computation shows that the Maxwell equations (42) are equivalent to

$$\begin{aligned}
-\frac{2}{r}\partial_r(r^2F_0) - \bar{\eth}F_{+1} + \eth F_{-1} &= 0,\\
\frac{2r}{c}\partial_t F_0 + h(\bar{\eth}F_{+1} + \eth F_{-1}) &= 0,\\
\frac{r}{c}\partial_t F_{+1} - \partial_r(rhF_{+1}) + \eth F_0 &= 0,\\
\frac{r}{c}\partial_t F_{-1} + \partial_r(rhF_{-1}) + \bar{\eth}F_0 &= 0. \qquad (45)
\end{aligned}$$

Then, looking for separable solutions of Eqs. (45) of the form

$$F_s = g_s(t,r)\,{}_sY_{jm}(\theta,\phi), \qquad s = 0,\pm1, \qquad (46)$$

applying Eqs. (29) again one obtains

$$\begin{aligned} -\frac{2}{r}\partial_r(r^2 g_0) + \sqrt{j(j+1)}\,(g_1 + g_{-1}) &= 0, \\ \frac{2r}{c}\partial_t g_0 - \sqrt{j(j+1)}\,h(g_1 - g_{-1}) &= 0, \\ \frac{r}{c}\partial_t g_1 - \partial_r(rhg_1) + \sqrt{j(j+1)}\,g_0 &= 0, \\ \frac{r}{c}\partial_t g_{-1} + \partial_r(rhg_{-1}) - \sqrt{j(j+1)}\,g_0 &= 0. \end{aligned} \tag{47}$$

(Note that these equations reduce to Eqs. (39) when $h = 1$.) These equations can be combined to obtain a *second order* partial differential equation for g_0, g_1, or g_{-1}. For instance, starting from the second equation of Eqs. (47), making use then of the third, fourth and first equation in (47), one finds that

$$\begin{aligned} &\frac{1}{c^2}\partial_t^2(rg_0) \\ &= \frac{\sqrt{j(j+1)}\,h}{2}\frac{1}{c}\partial_t(g_1 - g_{-1}) \\ &= \frac{\sqrt{j(j+1)}\,h}{2r}\left[\partial_r(rhg_1 + rhg_{-1}) - 2\sqrt{j(j+1)}\,g_0\right] \\ &= h\left[\frac{1}{r}\partial_r(h\partial_r(r^2 g_0)) - j(j+1)\frac{g_0}{r}\right]. \end{aligned} \tag{48}$$

Clearly this last equation can be reduced to an ordinary differential equation assuming, for example, that $rg_0 = f(r)\,e^{-i\omega t}$,

$$-\frac{\omega^2}{c^2}f = h\left[\frac{1}{r}\frac{d}{dr}\left(h\frac{d}{dr}(rf)\right) - j(j+1)\frac{f}{r^2}\right].$$

A simple solution of Eq. (48) is given by $g_0 = (\text{const.})r^{-2}$, with $j=0$ and $F_{\pm 1}=0$ [see Eq. (17)]. When the constant is real, the resulting field is the one present in the Reissner–Nordström solution.

Now letting

$$\chi \equiv -\frac{\sqrt{2}\,rg_0\,Y_{jm}}{j(j+1)},$$

for $j \neq 0$, one finds that Eq. (48) is equivalent to

$$\frac{1}{c^2}\partial_t^2\chi = h\left[\frac{1}{r}\partial_r\left(h\partial_r(r\chi)\right) + \frac{1}{r^2}\bar{\eth}\eth\chi\right], \tag{49}$$

which reduces to the scalar wave equation when $h = 1$. Combining the first two equations in Eq. (47), one obtains

$$g_{\pm 1} = \frac{1}{\sqrt{j(j+1)}}\left[\frac{1}{r}\partial_r(r^2 g_0) \pm \frac{1}{hc}\partial_t(rg_0)\right];$$

hence

$$\begin{aligned} F_{+1} &= -\frac{1}{\sqrt{2}\,r}\left(\frac{1}{hc}\partial_t + \partial_r\right) r\eth\chi, \\ F_0 &= \frac{1}{\sqrt{2}\,r}\bar{\eth}\eth\chi, \\ F_{-1} &= -\frac{1}{\sqrt{2}\,r}\left(\frac{1}{hc}\partial_t - \partial_r\right) r\bar{\eth}\chi. \end{aligned} \tag{50}$$

As in the previous example, the linearity of Eqs. (42), (49), and (50) and the completeness of the spin-weighted spherical harmonics imply that the general solution to the Maxwell equations on the Schwarzschild space-time is given by Eqs. (50), with χ being a solution of Eq. (49).

5. Solution of the linearized Einstein vacuum field equations

The Einstein vacuum field equations linearized about the Minkowski metric can be obtained assuming that the space-time metric, $g_{\alpha\beta}$, differs slightly from the flat Minkowski metric $(\eta_{\alpha\beta}) = \mathrm{diag}(-1, 1, 1, 1)$,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}. \tag{51}$$

To the first order in $h_{\alpha\beta}$ and its derivatives, the curvature is given by

$$\begin{aligned} K_{\alpha\beta\gamma\delta} = \frac{1}{2}(&\partial_\alpha\partial_\delta h_{\beta\gamma} - \partial_\alpha\partial_\gamma h_{\beta\delta} \\ &+\partial_\beta\partial_\gamma h_{\alpha\delta} - \partial_\beta\partial_\delta h_{\alpha\gamma}), \end{aligned} \tag{52}$$

which possesses the symmetries of the full curvature tensor,

$$\begin{aligned} K_{\alpha\beta\gamma\delta} = -K_{\beta\alpha\gamma\delta} = -K_{\alpha\beta\delta\gamma} = K_{\gamma\delta\alpha\beta}, \\ K_{\alpha\beta\gamma\delta} + K_{\alpha\delta\beta\gamma} + K_{\alpha\gamma\delta\beta} = 0 \end{aligned} \tag{53}$$

and

$$\partial_\alpha K_{\beta\gamma\delta\epsilon} + \partial_\beta K_{\gamma\alpha\delta\epsilon} + \partial_\gamma K_{\alpha\beta\delta\epsilon} = 0. \tag{54}$$

The linearized Einstein equations are then given by

$$K_{\alpha\beta} = 0, \tag{55}$$

where $K_{\alpha\beta} \equiv K^\gamma{}_{\alpha\gamma\beta}$, with the indices being lowered or raised by means of $\eta_{\alpha\beta}$ and its inverse $\eta^{\alpha\beta}$. From Eq. (54) (contracting with $\eta^{\gamma\delta}$), one obtains

$$\partial^\gamma K_{\alpha\beta\gamma\epsilon} = 0. \tag{56}$$

The symmetries (53) and the field equations (55) reduce to ten the number of algebraically independent components of the curvature $K_{\alpha\beta\gamma\delta}$, which can be represented by the tensor fields (for example Ref. 27)

$$E_{ij} \equiv K_{0i0j}, \qquad B_{ij} \equiv -\frac{1}{2}\varepsilon_{ilm}K_{lm0j}. \tag{57}$$

Owing to Eqs. (53) and (55), E_{ij} and B_{ij} are symmetric and their traces are equal to zero. The differential conditions (54) and (56) are equivalent to

$$\partial_i F_{ij} = 0, \qquad \varepsilon_{ijk}\partial_j F_{km} = \frac{i}{c}\partial_t F_{im}, \tag{58}$$

with $F_{ij} \equiv E_{ij} + iB_{ij}$ [*cf.* Eq. (36)].

As shown in Sec. 2.2, the spinor equivalent of a symmetric tensor with vanishing trace is totally symmetric. Thus, the spinor equivalent of F_{ij}, F_{ABCD}, is totally symmetric and has five algebraically (complex) components, which can be represented by the five spin-weighted components [*cf.* Eqs. (33)]

$$\begin{aligned}
F_{+2} &= F_{ABCD}o^A o^B o^C o^D = \frac{1}{2}(F_{\theta\theta} - F_{\phi\phi} + 2\mathrm{i}F_{\theta\phi}), \\
F_{+1} &= F_{ABCD}o^A o^B o^C \widehat{o}^D = -\frac{1}{2}(F_{r\theta} + \mathrm{i}F_{r\phi}), \\
F_{0} &= F_{ABCD}o^A o^B \widehat{o}^C \widehat{o}^D = \frac{1}{2}F_{rr}, \\
F_{-1} &= F_{ABCD}o^A \widehat{o}^B \widehat{o}^C \widehat{o}^D = \frac{1}{2}(F_{r\theta} - \mathrm{i}F_{r\phi}), \\
F_{-2} &= F_{ABCD}\widehat{o}^A \widehat{o}^B \widehat{o}^C \widehat{o}^D = \frac{1}{2}(F_{\theta\theta} - F_{\phi\phi} - 2\mathrm{i}F_{\theta\phi}),
\end{aligned} \tag{59}$$

where $F_{\theta\theta}$, $F_{\phi\phi}$, ... denote the components of F_{ij} with respect to the orthonormal basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$. In order to make use of the spin-weighted spherical harmonics to solve the set of equations (58), we must express these equations in spherical coordinates (replacing the partial derivatives appearing in Eqs. (58) by covariant derivatives) and then combine them in such a way that only quantities with a well-defined spin weight appear [see Eq. (44)]. Alternatively, we can express Eqs. (58) in terms of three-dimensional spinors, which directly yields the desired equations [7]. The result is that the first equations in (58) are equivalent to

$$\begin{aligned}
\frac{1}{r}\eth F_{-2} - \frac{2}{r^3}\partial_r(r^3 F_{-1}) - \frac{1}{r}\bar{\eth} F_0 &= 0, \\
\frac{1}{r}\eth F_{-1} - \frac{2}{r^3}\partial_r(r^3 F_{0}) - \frac{1}{r}\bar{\eth} F_{+1} &= 0, \\
\frac{1}{r}\eth F_{0} - \frac{2}{r^3}\partial_r(r^3 F_{+1}) - \frac{1}{r}\bar{\eth} F_{+2} &= 0,
\end{aligned} \tag{60}$$

while the second set of equations in (58) is equivalent to

$$\begin{aligned}
-\frac{1}{r}[\bar{\eth}F_{-1} + \partial_r(rF_{-2})] &= \frac{1}{c}\partial_t F_{-2}, \\
-\frac{1}{r}[\eth F_{-2} + 2\partial_r(rF_{-1}) + 3\bar{\eth}F_0] &= \frac{4}{c}\partial_t F_{-1}, \\
-\frac{1}{r}(\eth F_{-1} + \bar{\eth}F_{+1}) &= \frac{2}{c}\partial_t F_{0}, \\
-\frac{1}{r}[\bar{\eth}F_{+2} - 2\partial_r(rF_{+1}) + 3\eth F_0] &= \frac{4}{c}\partial_t F_{+1}, \\
-\frac{1}{r}[\eth F_{+1} - \partial_r(rF_{+2})] &= \frac{1}{c}\partial_t F_{+2}.
\end{aligned} \tag{61}$$

Then, for a separable solution of the form

$$F_s = g_s(t,r)\,{}_sY_{jm}(\theta,\phi), \qquad s = 0, \pm 1, \pm 2, \tag{62}$$

making use of Eqs. (29), one finds that Eqs. (60) and (61) reduce to a set of differential equations for the g_s

$$\begin{gathered}
\sqrt{j(j+1)-2}g_{-2} - \frac{2}{r^2}\partial_r(r^3 g_{-1}) + \sqrt{j(j+1)}\,g_0 = 0, \qquad \sqrt{j(j+1)}(g_{-1}+g_1) - \frac{2}{r^2}\partial_r(r^3 g_0) = 0, \\
\sqrt{j(j+1)}g_0 - \frac{2}{r^2}\partial_r(r^3 g_1) + \sqrt{j(j+1)-2}\,g_2 = 0, \qquad -\frac{1}{r}[\partial_r(rg_{-2}) - \sqrt{j(j+1)-2}\,g_{-1}] - \frac{1}{c}\partial_t g_{-2} = 0, \\
-\frac{1}{r}[\sqrt{j(j+1)-2}\,g_{-2} + 2\partial_r(rg_{-1}) - 3\sqrt{j(j+1)}g_0] - \frac{4}{c}\partial_t g_{-1} = 0, \\
-\frac{\sqrt{j(j+1)}}{r}(g_{-1}-g_1) - \frac{2}{c}\partial_t g_0 = 0, \qquad -\frac{1}{r}[-\sqrt{j(j+1)-2}\,g_2 - 2\partial_r(rg_1) + 3\sqrt{j(j+1)}g_0] - \frac{4}{c}\partial_t g_1 = 0, \\
-\frac{1}{r}[\sqrt{j(j+1)-2}g_1 - \partial_r(rg_2)] - \frac{1}{c}\partial_t g_2 = 0.
\end{gathered} \tag{63}$$

These equations can now be combined to obtain a second-order decoupled equation for one of the functions g_s. For instance, one finds that

$$\frac{1}{c^2}\partial_t^2(r^2 g_0) = \frac{1}{r}\partial_r^2(r^3 g_0) - j(j+1)g_0, \tag{64}$$

which is equivalent to the condition that

$$\chi \equiv -\frac{2r^2 g_0\, Y_{jm}}{[j(j+1)][j(j+1)-2]}$$

be a solution of the scalar wave equation. Making use again of Eqs. (63), (64), and (29), one can show that all the spin-weighted components of F_{ij} are given by

$$\begin{aligned}
F_{+2} &= -\frac{1}{2r^2}\left(\frac{1}{c}\partial_t + \partial_r\right)^2 r^2\eth\eth\chi, \\
F_{+1} &= \frac{1}{2r^2}\left(\frac{1}{c}\partial_t + \partial_r\right) r\bar{\eth}\eth\eth\chi, \\
F_{0} &= -\frac{1}{2r^2}\bar{\eth}\bar{\eth}\eth\eth\chi, \\
F_{-1} &= \frac{1}{2r^2}\left(\frac{1}{c}\partial_t - \partial_r\right) r\eth\bar{\eth}\bar{\eth}\chi, \\
F_{-2} &= -\frac{1}{2r^2}\left(\frac{1}{c}\partial_t - \partial_r\right)^2 r^2\bar{\eth}\bar{\eth}\chi.
\end{aligned} \tag{65}$$

As in the case of the electromagnetic field treated in Secs. 3 and 4, the labels j and m determine the eigenvalues of the square total angular momentum and of the z-component of the total angular momentum of the field (62). Expressions (65) are equivalent to those obtained in Ref. 20.

6. The Dirac equation

The Dirac equation is given by

$$\mathrm{i}\hbar\partial_t\psi = -\mathrm{i}\hbar c\boldsymbol{\alpha}\cdot\nabla\psi + Mc^2\beta\psi, \tag{66}$$

where ψ is a four-component column and the 4×4 matrices α_i and β satisfy the conditions $\alpha_i\alpha_j + \alpha_j\alpha_i = \delta_{ij}I$, $\alpha_i\beta + \beta\alpha_i = 0$, $\beta^2 = I$ (see for example Refs. 28 to 30). These conditions are satisfied with

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where the σ_i are the usual Pauli matrices. Expressing ψ in the form $\psi = \begin{pmatrix} u \\ v \end{pmatrix}$, where u and v are two-component columns, Eq. (66) amounts to

$$\begin{aligned} \mathrm{i}\hbar\partial_t u &= -\mathrm{i}\hbar c\boldsymbol{\sigma}\cdot\nabla v + Mc^2 u, \\ \mathrm{i}\hbar\partial_t v &= -\mathrm{i}\hbar c\boldsymbol{\sigma}\cdot\nabla u - Mc^2 v. \end{aligned} \tag{67}$$

These equations are invariant under spatial rotations if one assumes that u and v are two-component spinors that transform by means of SU(2) matrices under rotations.

By analogy with Eqs. (33) and (59), a two-component spinor,

$$u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix},$$

has the spin-weighted components

$$u_+ \equiv u_A o^A, \qquad u_- \equiv u_A \widehat{o}^A, \tag{68}$$

which have spin weight 1/2 and $-1/2$, respectively. As can be readily verified, the analog of Eq. (31) is

$$u = u_+\widehat{o} - u_- o, \tag{69}$$

where o and $\widehat{o}$ are the two-component spinors defined by Eqs. (13) and (15). A straightforward computation yields [7,9]

$$\boldsymbol{\sigma}\cdot\nabla u = -\frac{1}{r}[\partial_r(ru_-) + \overline{\eth}u_+]\, o - \frac{1}{r}[\partial_r(ru_+) - \eth u_-]\, \widehat{o};$$

therefore, Eqs. (67) are equivalent to

$$\begin{aligned} \frac{1}{c}\partial_t u_- &= -\frac{1}{r}\partial_r(rv_-) - \frac{1}{r}\overline{\eth}v_+ - \frac{\mathrm{i}Mc}{\hbar}u_-, \\ \frac{1}{c}\partial_t u_+ &= \frac{1}{r}\partial_r(rv_+) - \frac{1}{r}\eth v_- - \frac{\mathrm{i}Mc}{\hbar}u_+, \\ \frac{1}{c}\partial_t v_- &= -\frac{1}{r}\partial_r(ru_-) - \frac{1}{r}\overline{\eth}u_+ + \frac{\mathrm{i}Mc}{\hbar}v_-, \\ \frac{1}{c}\partial_t v_+ &= \frac{1}{r}\partial_r(ru_+) - \frac{1}{r}\eth u_- + \frac{\mathrm{i}Mc}{\hbar}v_+. \end{aligned} \tag{70}$$

The system of equations (70) allows separable solutions of the form

$$\begin{aligned} u_\pm &= f_\pm(r)\,{}_{\pm\frac{1}{2}}Y_{jm}(\theta,\phi)\,\mathrm{e}^{-\mathrm{i}Et/\hbar}, \\ v_\pm &= g_\pm(r)\,{}_{\pm\frac{1}{2}}Y_{jm}(\theta,\phi)\,\mathrm{e}^{-\mathrm{i}Et/\hbar} \end{aligned} \tag{71}$$

($j = 1/2, 3/2, \ldots$), where E is a constant. Substituting Eqs. (71) into Eqs. (70), making use of Eqs. (29), one obtains the system of *ordinary* differential equations

$$\begin{aligned} \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rg_-) - (j+\frac{1}{2})\frac{g_+}{r} + \frac{\mathrm{i}Mc}{\hbar}f_- &= \frac{\mathrm{i}E}{\hbar c}f_-, \\ -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rg_+) + (j+\frac{1}{2})\frac{g_-}{r} + \frac{\mathrm{i}Mc}{\hbar}f_+ &= \frac{\mathrm{i}E}{\hbar c}f_+, \\ \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rf_-) - (j+\frac{1}{2})\frac{f_+}{r} - \frac{\mathrm{i}Mc}{\hbar}g_- &= \frac{\mathrm{i}E}{\hbar c}g_-, \\ -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rf_+) + (j+\frac{1}{2})\frac{f_-}{r} - \frac{\mathrm{i}Mc}{\hbar}g_+ &= \frac{\mathrm{i}E}{\hbar c}g_+. \end{aligned} \tag{72}$$

If we now let

$$\begin{aligned} A &\equiv f_+ + f_-, \qquad B \equiv f_+ - f_-, \\ C &\equiv g_+ + g_-, \qquad D \equiv g_+ - g_-, \end{aligned}$$

Eqs. (72) take the form

$$\begin{aligned} \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rA) - (j+\frac{1}{2})\frac{A}{r} &= -\frac{\mathrm{i}}{\hbar c}(E+Mc^2)D, \\ \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rD) + (j+\frac{1}{2})\frac{D}{r} &= -\frac{\mathrm{i}}{\hbar c}(E-Mc^2)A, \end{aligned} \tag{73}$$

together with

$$\begin{aligned} \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rC) - (j+\frac{1}{2})\frac{C}{r} &= -\frac{\mathrm{i}}{\hbar c}(E-Mc^2)B, \\ \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rB) + (j+\frac{1}{2})\frac{B}{r} &= -\frac{\mathrm{i}}{\hbar c}(E+Mc^2)C. \end{aligned} \tag{74}$$

Equations (73) imply that A satisfies the differential equation

$$\frac{\mathrm{d}^2A}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}A}{\mathrm{d}r} + \left[k^2 - \frac{(j-\frac{1}{2})(j+\frac{1}{2})}{r}\right]A = 0,$$

where $k \equiv \sqrt{E^2 - M^2c^4}/\hbar c$. Hence, A must be proportional to a spherical Bessel function,

$$A(r) = a j_{j-\frac{1}{2}}(kr), \tag{75}$$

where a is a constant; and, making use of Eq. (73) and the recurrence relations for the spherical Bessel functions, one obtains

$$D(r) = -\mathrm{i}a\sqrt{\frac{E-Mc^2}{E+Mc^2}}\, j_{j+\frac{1}{2}}(kr). \tag{76}$$

In a similar way, one finds that

$$B(r) = b j_{j+\frac{1}{2}}(kr), \qquad C(r) = \mathrm{i}b\sqrt{\frac{E-Mc^2}{E+Mc^2}}\, j_{j-\frac{1}{2}}(kr),$$

where b is a constant. Further details can be found in Refs. 7 and 9.

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