The Einstein-Hamilton-Jacobi equation: searching the classical solution for barotropic FRW

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The dynamical evolution of the scale factor of FRW cosmological model is presented, when the equation of state of the material content takes the form $\mathrm{p} = \gamma\rho$, $\gamma = \mathrm{constant}$, including the cosmological term. We use the WKB approximation and the relation with the Einstein-Hamilton-Jacobi equation to obtain the exact solutions.

Keywords: Einstein-Hamilton-Jacobi equation; classical cosmology; exact solutions

Se presenta la evolución dinámica del factor de escala de los modelos cosmológicos, cuando la ecuación de estado toma la forma $\mathrm{p} = \gamma\rho$, $\gamma =$ constante, incluyendo el termino cosmológico. Usamos la aproximación WKB y la relación con la ecuación de Einstein-Hamilton-Jacobi para obtener soluciones exactas

Descriptores: Ecuación de Einstein-Hamilton-Jacobi; cosmología clásica; soluciones exactas.

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1. Introduction

The behaviour of the cosmological scale factor $\mathrm{A(t)}$ in solutions of Einstein's field equations with the Friedmann-Robertson-Walker line element has been the subject of numerous studies, where the presentations tend to focus on models in which $\mathrm{p} = 0$ and there is no cosmological constant ($\Lambda = 0$). Some treatments include the cosmological constant [1–5] and the pressure p is given in terms of density ρ by an equation of state $\mathrm{p} = \mathrm{p}(\rho)$ and $\Lambda \neq 0$, for particular values in the γ parameter [6–9].

The standart model of cosmology is based on Einstein's General Relativity theory, which can be derived from the geometric Einstein-Hilbert Lagrangian

$$\mathcal{L}_{\mathrm{geo}} = \frac{1}{16\pi\mathrm{G}}\sqrt{-\mathrm{g}}\,\mathrm{R}, \tag{1}$$

where R is the Ricci scalar, G the Newton constant, and $\mathrm{g} = |\mathrm{g}_{\mu\nu}|$ the determinant of the metric tensor. By performing the metric variation of this equation, one obtains the well known Einstein's field equations

$$\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{R}\,\mathrm{g}_{\mu\nu} = -8\pi\mathrm{G}\mathrm{T}_{\mu\nu}, \tag{2}$$

where $\mathrm{T}_{\mu\nu}$ is the energy-momentum stress tensor, associated with a matter lagrangian, which is the source of gravitation, assigning the corresponding equation of state, which varies during different epochs of the history of the universe.

Introducing a symmetry through the metric tensor, in cosmology one assumes a simple one according to the cosmological principle that states that the universe is both homogeneous and isotropic. This homogeneous and isotropic space-time symmetry was originally studied by Friedmann, Robertson, and Walker (FRW). The symmmetry is encoded in the special form of the following line element:

$$\mathrm{ds}^2 = -\mathrm{N}^2(\mathrm{t})\mathrm{dt}^2 + \mathrm{A}^2(\mathrm{t})\left[\frac{\mathrm{dr}^2}{1-\kappa\mathrm{r}^2} + \mathrm{r}^2\mathrm{d}\Omega^2\right], \tag{3}$$

where $\mathrm{A(t)}$ is the scale factor, N(t) the lapse function, κ is the constant curvature, taking on the values $0, +1, -1$ (flat, closed and open space, respectively).

The FRW solutions to the Einstein field equation (2) represent a cornerstone in the development of modern cosmology, since with them it is posible to understand the expansion of the universe.

Recently, Faraoni [10] introduced a procedure based on the Riccati differential equation, obtained by the combination of the Einstein field equation, resulting in the same solutions obtained by the standard procedure [1–5,11], without the cosmological term. This alternative approach is more direct than the standard one, which was used in the factorization procedure in the supersymmetric level [12, 13] to obtain both the iso-spectral potential and function in particular one dimensional systems.

The set of differential equations for the FRW cosmological model, including the cosmological term, become

$$\frac{\ddot{\mathrm{A}}}{\mathrm{A}} = -\frac{4\pi\mathrm{G}}{3}(\rho + 3\mathrm{p}) - \frac{\Lambda}{3} \tag{4}$$

$$\left(\frac{\dot{\mathrm{A}}}{\mathrm{A}}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\Lambda}{3} - \frac{\kappa}{A^2}, \tag{5}$$

where the overdot means d/dt.

In the literature, one can find the well-known classical behaviour of the scale factor for $\kappa = 0$ [11]

$$A = \left[6\pi G M_\gamma (\gamma+1)^2\right]^{\frac{1}{3(\gamma+1)}} (t-t_0)^{\frac{2}{3(\gamma+1)}}. \qquad (6)$$

Taking different values for the constant, γ we have the following subcases:

$$A = \begin{cases} \left[\frac{32}{3}\pi G M_{\frac{1}{3}}\right]^{\frac{1}{4}} t^{\frac{1}{2}} & \text{for} \quad \gamma=\frac{1}{3} \quad \text{radiation} \\ \left[6\pi G M_0\right]^{\frac{1}{3}} t^{\frac{2}{3}} & \text{for} \quad \gamma=0 \quad \text{dust} \\ \left[24\pi G M_1\right]^{\frac{1}{6}} t^{\frac{1}{3}} & \text{for} \quad \gamma=1 \quad \text{stiff fluid} \end{cases} \qquad (7)$$

However, for the case $\gamma = -1$, the solution becomes exponential:

$$\mathrm{A} = \mathrm{A_0 e^{Ht}}, \qquad \text{with} \qquad \mathrm{H} = 2\sqrt{\frac{2}{3}\pi \mathrm{GM_{-1}}}, \qquad (8)$$

here, we consider the sign (+) in the exponential function, because we consider the inflationary behaviour. These solutions will be compared with the solutions found by our method.

The main purpose of this work is the introduction of the WKB-like procedure for calculating the function $\mathrm{A(t)}$, including the function $\Phi(A)$, which plays an important role in the supersymmetric fashion [14, 15], called the superpotential function, in the Hamiltonian formalism for solving the Einsten-Hamilton-Jacobi equation. Also, we include the cosmological term in the formalism.

The remainder of the paper is organized as follows. The procedure that includes the Einstein-Hamilton-Jacobi equation and the master equation is described in Sec. 2. In Sec. 3 we present the exact solutions for the master equation found for this model, including their corresponding analysis. Finally, Sec. 4 is devoted to comments.

2. Einstein-Hamilton-Jacobi equation: the WKB-like method

We will use the total Lagrangian for a homogeneous and isotropic universe (FRW cosmological model), and perfect-fluid like ordinary matter with pressure p and energy density ρ, and barotropic state equation $\mathrm{p} = \gamma\rho$, including the cosmological term Λ [16, 17]:

$$\mathrm{L} = \frac{6\mathrm{A}}{\mathrm{N}}\left(\frac{\mathrm{dA}}{\mathrm{dt}}\right)^2 - 6\kappa N A - 2N\Lambda A^3 + 16\pi G N M_\gamma A^{-3\gamma}. \qquad (9)$$

We define the canonical momentum conjugate of the generalized coordinate A (scale factor) as

$$\Pi_{\mathrm{A}} \equiv \frac{\partial \mathrm{L}}{\partial \dot{\mathrm{A}}} = \frac{12\mathrm{A}}{\mathrm{N}}\frac{\mathrm{dA}}{\mathrm{dt}}. \qquad (10)$$

The canonical hamiltonian function has the following form:

$$\mathrm{L} = \Pi_A \dot{A} - N\mathcal{H} = \Pi_A \dot{A} - N\left[\frac{\Pi_A^2}{24A} + 6\kappa A + 2\Lambda A^3 - 16\pi G M_\gamma A^{-3\gamma},\right] \qquad (11)$$

where

$$\mathcal{H} = \frac{1}{24A}\left[\Pi_A^2 + 144\kappa A^2 + 48\Lambda A^4 - 384\pi G M_\gamma A^{-3\gamma+1}\right]. \qquad (12)$$

Performing the variation of (11) with respect to N, $\partial\mathrm{L}/\partial\mathrm{N} = 0$ implies the well-known result $\mathcal{H} = 0$.

At this point we can do two things: i) the quantization procedure, imposing the quantization condition on $\mathcal{H} \to \hat{\mathcal{H}}$, where $\hat{\mathcal{H}}$ is an operator, and by applying this hamiltonian operator to the wave function Ψ, we obtain the Wheeler-DeWitt (WDW) equation in the minisuperspace

$$\hat{\mathcal{H}}\Psi = 0, \qquad (13)$$

and ii) the WKB-like method, if one performs the transformation

$$\Pi_{\mathrm{A}} = \frac{\mathrm{d}\Phi}{\mathrm{dA}} \qquad (14)$$

in (12), becomes the Einstein-Hamilton-Jacobi equation, when Φ is the superpotential function that is related to the physical potential under consideration.

We shall use part ii) as an alternative method for obtaining the classical solutions to the FRW cosmological model.

Introducing the ansatz (14) into Eq. (12) we get

$$\left[\left(\frac{\mathrm{d}\Phi}{\mathrm{dA}}\right)^2 + 144\kappa\mathrm{A}^2 + 48\Lambda\mathrm{A}^4 - 384\pi\mathrm{GM}_\gamma\mathrm{A}^{-3\gamma+1}\right] = 0,$$

$$\frac{\mathrm{d}\Phi}{\mathrm{dA}} = \pm 12\mathrm{A}\sqrt{\frac{8}{3}\pi\mathrm{GM}_\gamma\mathrm{A}^{-(3\gamma+1)} - \frac{\Lambda}{3}\mathrm{A}^2 - \kappa} \qquad (15)$$

Relating the Eqs. (10), (14) and (15), we obtain the classical evolution for the scale factor in term of the "cosmic time" τ defined by $\mathrm{d}\tau = \mathrm{N(t)dt}$, through the following master equation

$$\mathrm{d}\tau = \frac{\mathrm{dA}}{\sqrt{\frac{8}{3}\pi\mathrm{GM}_\gamma\mathrm{A}^{-(3\gamma+1)} - \frac{\Lambda}{3}\mathrm{A}^2 - \kappa}}, \qquad (16)$$

which corresponds to Eq. (5) in the gauge N=1.

This equation is not easy to solve in general way for all values in the γ parameter. However, we can solve this for particular values in two sectors in the γ parameter:

1. $\gamma < 0$, say $(-1/3, -2/3, -1)$, and $\Lambda \neq 0$. This is the phenomenon commonly known as inflation-like.

2. $\gamma = 1/3$, $\Lambda \neq 0$, any κ.

3. $\gamma > 0$, $\Lambda = 0$.

In the following section, we describe its behaviour for the scale factor.

3. Solution of the master equation

Here, we obtain analytic solutions for the scale factor, via the master equation rewritten in terms of a "conformal time" coordinate T. In some cases, it will be necessary to drop the cosmological term, in order to of obtain the corresponding exact solution.

3.1. $\gamma < 0$, inflation-like phenomenon

Considering some negative values for the γ parameter, namely, $\gamma = -1, -1/3, -2/3$, we have

1. $\gamma = -1$, the equation (16) is the following (for simplicity we choose the changes $\mathrm{A} \to \mathrm{x}$, $\mathrm{a}_\gamma = \frac{8}{3}\pi \mathrm{G}\mathrm{M}_\gamma$, $\mathrm{b} = -\Lambda/3$):

$$d\tau = \frac{\mathrm{dx}}{\sqrt{(\mathrm{a}_{-1} + \mathrm{b})\,\mathrm{x}^2 - \kappa}}. \tag{17}$$

Integrating (17) and inverting, we obtain

$$\mathrm{A}(\tau) = \sqrt{\frac{3\kappa}{\Lambda - 8\pi GM_{-1}}} \times \sinh\left[\sqrt{\frac{8\pi GM_{-1}}{3} - \frac{\Lambda}{3}}\tau\right]. \tag{18}$$

The character of this solution is related to the cosmological term Λ and the curvature parameter κ as follows:

(a) For $\kappa = 1$, the behaviour is inflationary.

(b) For $\kappa = -1$, the behaviour will be inflationary if $\mathrm{M}_{-1} > (\Lambda + 3)/8\pi \mathrm{G} > 0$.

(c) For $\kappa = 0$, we will solve the original equation (16), obtaining

$$\mathrm{A}(\tau) = \mathrm{m}_1 \exp\left[2\sqrt{\frac{2}{3}\pi \mathrm{GM}_{-1} - \frac{\Lambda}{3}}\tau\right] + \mathrm{m}_2 \exp\left[-2\sqrt{\frac{2}{3}\pi \mathrm{GM}_{-1} - \frac{\Lambda}{3}}\tau\right]. \tag{19}$$

Here m_1 and m_2 are integration constants. For inflation, the following conditions are necessary: $\mathrm{m}_1 > \mathrm{m}_2$ and $\mathrm{M}_{-1}, > (\Lambda + 3)/8\pi \mathrm{G} > 0$. This last result generalize that found in [11], and is the same if $m_2 = 0$ and $\Lambda = 0$ in the gauge $\mathrm{N} = 1$.

2. $\gamma = -1/3$, the equation (16) is written in the following form:

$$d\tau = \frac{\mathrm{dx}}{\sqrt{\mathrm{bx}^2 + \mathrm{a}_{-1/3} - \kappa}} \tag{20}$$

with solution

$$\mathrm{A}(\tau) = \sqrt{\frac{8\pi \mathrm{GM}_{-\frac{1}{3}} - 3\kappa}{|\Lambda|}} \quad \sinh\left[\sqrt{\frac{|\Lambda|}{3}}\tau\right], \tag{21}$$

with $\Lambda < 0$. For inflation, the following conditions are necessary: $\mathrm{M}_{-\frac{1}{3}} > 3\kappa/8\pi \mathrm{G} > 0$, implying $\kappa = 1$ and $|\Lambda| > 3M_{-\frac{1}{3}}$.

3. $\gamma = -2/3$. Eq. (16), read as

$$d\tau = \frac{\mathrm{dx}}{\sqrt{\mathrm{bx}^2 + \mathrm{a}_{-1/3}\mathrm{x} - \kappa}} \tag{22}$$

with the solution for $\kappa = -1$ and $\Lambda < 0$, is

$$\mathrm{A}(\tau) = \frac{3}{2|\Lambda|}\left\{\left(\sqrt{\frac{|\Lambda|}{3}} - \frac{4}{3}\pi \mathrm{GM}_{-\frac{2}{3}}\right)\mathrm{e}^{\sqrt{\frac{|\Lambda|}{3}}\tau} - \left(\sqrt{\frac{|\Lambda|}{3}} + \frac{4}{3}\pi GM_{-\frac{2}{3}}\right)e^{-\sqrt{\frac{|\Lambda|}{3}}\tau} - \frac{8}{3}\pi GM_{-\frac{2}{3}}\right\}, \tag{23}$$

having an inflationary behaviour.

3.2. $\gamma = 1/3$, $\Lambda \neq 0$, any κ

In this subcase, (16) is written as

$$d\tau = \frac{\mathrm{AdA}}{\sqrt{\frac{8}{3}\pi \mathrm{GM}_{\frac{1}{3}} - \frac{1}{3}\Lambda \mathrm{A}^4 - \kappa \mathrm{A}^2}}. \tag{24}$$

With the change of variables $\mathrm{u} = \mathrm{A}^2$, (24) is

$$\tau = \frac{1}{2}\int_0^{A^2} \frac{du}{\sqrt{\frac{8}{3}\pi GM_{\frac{1}{3}} - \frac{1}{3}\Lambda u^2 - \kappa u}}, \tag{25}$$

whose solutions are, depending on the sign of Λ,

1. $\Lambda > 0$

$$\sqrt{\frac{3}{4\Lambda}}\left\{\arcsin\left[\frac{\frac{2\Lambda A^2}{3} + \kappa}{\sqrt{\kappa^2 + \frac{32}{9}\pi G\Lambda M_{\frac{1}{3}}}}\right] - \arcsin\left[\frac{\kappa}{\sqrt{\kappa^2 + \frac{32}{9}\pi G\Lambda M_{\frac{1}{3}}}}\right]\right\} \tag{26}$$

2. $\Lambda < 0$

$$\sqrt{\frac{3}{4\Lambda}}\left\{\mathrm{Ln}\left[2\sqrt{-\frac{\Lambda}{3}\left(\frac{8}{3}\pi \mathrm{GM}_{\frac{1}{3}} - \frac{1}{3}\Lambda \mathrm{A}^4 - \kappa \mathrm{A}^2\right)} - \frac{2}{3}\Lambda A^2 - \kappa\right] - Ln\left[2\sqrt{-\frac{8}{9}\pi G\Lambda M_{\frac{1}{3}}} - \kappa\right]\right\}. \tag{27}$$

3.3. $\gamma > 0$, $\Lambda = 0$

For this subcase, (16) is given by

$$d\tau = \frac{dA}{\sqrt{\frac{8}{3}\pi G M_\gamma A^{-(3\gamma+1)} - \kappa}}, \quad (28)$$

and introducing the following conformal transformation $d\tau = x^{3\gamma+2} dT$, and the change of variable $u = a_\gamma x^{-(3\gamma+1)} - \kappa$, the scale factor becomes, in the "conformal time" T,

$$A(T) = \left[\frac{a_\gamma(3\gamma+1)^2}{4} T^2 - \sqrt{-\kappa}(3\gamma+1)T\right]^{-\frac{1}{3\gamma+1}}, \quad (29)$$

which is valid for $\kappa \leq 0$.

When we know the function $A(T)$, we can obtain the transformation rule between the times $d\tau$ and dT, for instance

$$d\tau = A^{3\gamma+2} dT,. \quad (30)$$

thus

$$d\tau = \left[\mu_\gamma T^2 - \nu_\gamma T\right]^{-\frac{3\gamma+2}{3\gamma+1}} dT, \quad (31)$$

where

$$\mu_\gamma = \frac{a_\gamma(3\gamma+1)^2}{4}$$

and $\nu_\gamma = \sqrt{-\kappa}(3\gamma+1)$.

Now, considering the flat universe ($\kappa = 0$), we can integrate (31), but for consistency between Eqs. (29) and (30), we introduce the parameter ϵ in the sense that when $\tau = 0$, $\epsilon = T$ and $\tau = \tau$, $\epsilon \to \infty$:

$$\int_0^\tau d\tau = \mu^{-\frac{3\gamma+2}{3\gamma+1}} \int_T^\epsilon (x)^{-\frac{2(3\gamma+1)}{3\gamma+1}} dx. \quad (32)$$

After a tedious calculation, we arrive at

$$T = \left[\mu^{\frac{3\gamma+2}{3\gamma+1}} \frac{3(\gamma+1)}{3\gamma+1}\tau + \epsilon^{-\frac{3(\gamma+1)}{3\gamma+1}}\right]^{-\frac{3\gamma+1}{3(\gamma+1)}}. \quad (33)$$

Introducing (33) into (29), we found the scale factor in a general way:

$$A(\tau) = \left[\sqrt{\mu_\gamma}\frac{3(\gamma+1)}{3\gamma+1}\tau + \mu^{-\frac{3(\gamma+1)}{2(3\gamma+1)}} \epsilon^{-\frac{3(\gamma+1)}{3\gamma+1}}\right]^{\frac{2}{3(\gamma+1)}}. \quad (34)$$

At this point, we can calculate the behaviour of the scale factor for some positive values to the parameter γ, and compare them with those found in the standard literature:

1. Dust era, $\gamma = 0$, the scale factor becomes, in the gauge $N = 1$

$$A(t) = \left\{[(6\pi G M_0]^{\frac{1}{2}} t + \mu_0^{-\frac{3}{2}} \epsilon^{-3}\right\}^{\frac{2}{3}}. \quad (35)$$

2. Radiation era, $\gamma = \frac{1}{3}$, in the gauge $N = 1$

$$A(t) = \left\{\left[\frac{32}{3}\pi G M_{\frac{1}{3}}\right]^{\frac{1}{2}} t + \mu_{\frac{1}{3}}^{-1} \epsilon^{-2}\right\}^{\frac{1}{2}}. \quad (36)$$

3. Stiff matter, $\gamma = 1$, in the gauge $N = 1$

$$A(t) = \left\{[24\pi G M_1]^{\frac{1}{2}} t + \mu_1^{-\frac{3}{4}} \epsilon^{-\frac{3}{2}}\right\}^{\frac{1}{3}}. \quad (37)$$

Choosing appropriately the parameter $\epsilon \to \infty$, we obtain the usual results for the scale factor for the FRW, with $\kappa = 0$ and $\Lambda = 0$ [11].

4. Comments

The inflationary scenarios for $\gamma < 0$ and matter epochs were considered in the FRW cosmological model. Also, our method is more general than that employed by Faraoni, because when the cosmological constant is included in the formalism, Eq.(3.1) in Ref. 10 does not reduce to the Riccati equation and the procedure fails.

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