Exact analytic description of nuclear shape phase transitions

G. Lévai

Institute of Nuclear Research of the Hungarian Academy of Sciences (ATOMKI),
P.O. Box 51, H-4001 Debrecen, Hungary
e-mail: levai@atomki.hu

Recibido el 30 de enero de 2005; aceptado el 15 de marzo de 2005

We apply the sextic oscillator as a γ-independent potential in the Bohr Hamiltonian and present exact analytic results for the energy eigenvalues and wavefunctions of the lowest few levels. Further properties, such as the flexible shape of the potential are also discussed. As illustration a potential reproducing the spectroscopic properties of the ^{134}Ba nucleus, the first candidate for the E(5) symmetry is constructed. Fits to the energy spectrum of the ^{102}Ru, ^{104}Ru and ^{106}Ru isotopes are also presented, and it is shown that in this region the potential undergoes changes characteristic for a transition from the spherical vibrator to the deformed γ-unstable domain. Possible generalizations of the model are also pointed out.

Keywords: Nuclear shape; Quadrupole collectivity; Quasi-exactly solvable potentials.

En este trabajo aplicamos el oscilador sextico como un potencial independiente de la variable γ en el Hamiltoniano de Bohr y presentamos resultados analíticos para los valores propios y las funciones propias de los niveles de menor energía. Se discuten igualmente otras propiedades como la forma flexible del potencial. A manera de ilustración, se construye un potencial que reproduce las propiedades espectroscópicas del núcleo ^{134}Ba, siendo esta el primer candidato de simetría E(5). Se presentan ajustes al espectro energético de los isótopos ^{102}Ru, ^{104}Ru y ^{106}Ru y se muestra que en esta región el potencial sufre cambios característicos de la transición de vibrador esférico al dominio de γ inestable. Se mencionan las posibles generalizaciones del modelo.

Descriptores: Forma del nucleo; colectividad cuadrupolar; potenciales con solucion cuasi-exacta.

PACS: 21.10.Re; 03.65.Ge

1. The Bohr Hamiltonian and various nuclear shape phases

In one of the fundamental models in nuclear physics the nucleus is pictured as a liquid drop that undergoes collective oscillations. The Hamiltonian describing this phenomenon contains a kinetic term and a potential that keeps the nucleus in the vicinity of stable configurations that correspond to various nuclear shapes. When only the quadrupole mode is considered, the Hamiltonian reduces to an operator containing vibrational and rotational kinetic terms depending on the β and γ shape variables, and a potential term $V(\beta,\gamma)$ that is the function of the same variables in general [1]

$$H = -\frac{\hbar^2}{2B}\left(\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_k \frac{Q_k^2}{\sin^2(\gamma-\frac{2}{3}\pi k)}\right) + V(\beta,\gamma)\ . \quad (1)$$

The minima of the $V(\beta,\gamma)$ potential correspond to equilibrium nuclear shapes. In spherical nuclei the minimum is at $\beta=0$, while in the case of deformed nuclei β (and γ) have finite equilibrium values. An interesting case is that of the γ-unstable nuclei, for which the potential is assumed to depend only on β. The most characteristic shapes also correspond to various dynamical symmetries associated with the Interacting Boson Model [2], which focuses on the quadrupole excitation of nuclei: the U(5), SU(3) and O(6) symmetries correspond to vibrational, axially deformed rotor and γ-unstable deformed nuclei, respectively.

More recently further symmetries have also been associated with certain types of $V(\beta,\gamma)$ potentials. First the E(5) symmetry was proposed [3], which is expected to occur as the nuclear shape evolves from the spherical to the γ-unstable domain, as one moves along an isotope chains for example. Starting from the spherical side the potential, which in this case is thought to depend only on the β variable has a minimum at $\beta=0$, while in the γ-unstable domain the minimum should appear at $\beta>0$. In the transition between these two domains one expects that there is a potential shape in which the two minima are nearly degenerate and are separated by a small barrier only. In Ref. 3 this is defined as a critical point, and the corresponding potential is approximated with a square well, which is flat in the allowed region and then abruptly reaches infinity. This potential is solvable exactly, and it yields characteristic ratios of the excitation energies of various states, and of the strength of electromagnetic transitions between them. These numbers corresponding to the E(5) symmetry can then be compared with the experimentally observed data in order to locate nuclei associated with this symmetry.

In order to describe this situation with the Bohr Hamiltonian, the potential is assumed to be γ-independent $V(\beta,\gamma)=U(\beta)$, which allows the separation of the variables as

$$\Psi(\beta,\gamma,\theta_i) = \beta^{-2}\phi(\beta)\Phi(\gamma,\theta_i)\ . \quad (2)$$

Eventually this leads to a one-dimensional Schrödinger-like equation

$$-\frac{\mathrm{d}^2\phi}{\mathrm{d}\beta^2}+\left(\frac{(\tau+1)(\tau+2)}{\beta^2}+u(\beta)\right)\phi=\epsilon\phi\ , \qquad (3)$$

where $\epsilon = (2B/\hbar^2)E$, $u(\beta) = (2B/\hbar^2)U(\beta)$, and τ is a quantum number originating from the γ-dependent part of the wavefunction, and plays a role in this five-dimensional setting as the l orbital angular momentum does in the case of three-dimensional problems. Its allowed values have been determined in Ref. 4.

In the present paper we analyze the exact solutions of Eq. (3), and in particular, we propose the application of the sextic oscillator in the Bohr Hamiltonian. We do not deal with potentials that depend on the γ variable also: in this case further assumptions have to be made in order to obtain approximate solutions. For a review on the subject see, *e.g.* Ref. 5.

2. The sextic oscillator as a γ-independent potential

There are only a handful of radial potentials that possess analytic solution in the presence of centrifugal type term, which originates from the kinetic part of the Hamiltonian. Such a term, $(\tau+1)(\tau+2)\beta^{-2}$ also appears in the radial equation (3), so it is straightforward to apply those potentials in it, which are solvable for arbitrary angular momentum in the three-dimensional radial Schrödinger equation. The most trivial examples are the harmonic oscillator [6] and square well [7] potentials, which are solvable in terms of generalized Laguerre polynomials $L_n^{(\alpha)}(z)$ and Bessel functions $J_\nu(z)$, respectively. One further trivial example is the Coulomb potential (also solvable in terms of $L_n^{(\alpha)}(z)$), the application of which appears less useful in the Bohr Hamiltonian due to its asymptotic behaviour. The Davidson and the Kratzer potentials are straightforward generalizations of the harmonic oscillator and Coulomb potentials in such a way that a β^{-2}-type term appears in them, which can be treated together with the centrifugal term. In technical terms this means that equation (3) is solved formally with non-integer values of τ (or l in three dimensions). These latter two potentials have been applied in the Bohr Hamiltonian in Refs. 8 and 9.

This is the complete list of potentials for which the solution can be given for any state with arbitrary τ and node number. There are, however, some potentials for which exact solutions can be given for a limited number of states. These potentials are called quasi-exactly solvable (QES) [10], and they can be solved exactly up to a finite value of the principal quantum number (node number). The general solutions of these potentials are written in terms of power series, which, however, can be reduced to polynomials for the first few states with special choices of the potential parameters.

The first application of QES potentials in the Bohr Hamiltonian was proposed in Ref. 11, where the sextic oscillator was considered. This is probably the best known example for the QES potentials, and its conventional form in a radial equation is [10]

$$H=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{(2s-1/2)(2s-3/2)}{x^2} + \left(b^2-4a(s+\frac{1}{2}+M)\right)x^2+2abx^4+a^2x^6, \quad (4)$$

where $x\in[0,\infty)$ and M is a non-negative integer. For any value of M, $M+1$ solutions of (4) can be obtained in an algebraic way. The (unnormalized) solutions are written in the form

$$\phi_n(x)=P_n(x^2)(x^2)^{s-\frac{1}{4}}\exp\left(-\frac{a}{4}x^4-\frac{b}{2}x^2\right), \qquad n=0,1,2,\dots \quad (5)$$

where P_n is a polynomial of order n, and $a\geq 0$ is required for the proper normalization of the solutions. For $a=0$ (4) reduces to the harmonic oscillator, and P_n in (5) turns into $L_n^{(\alpha)}$.

The simplest solutions are obtained for $M = 0$ and $M = 1$ [10]. For $M = 0$ only one nodeless (*i.e.* ground-state) solution appears at $E_0^{(M=0)} = 4bs$, with the corresponding wavefunction being

$$\phi_0^{(M=0)}(x)\sim(x^2)^{s-\frac{1}{4}}\exp\left(-\frac{a}{4}x^4-\frac{b}{2}x^2\right). \qquad (6)$$

For $M = 1$ two solutions appear, one nodeless, and another with one node for $x > 0$. These correspond to the ground-state and the first excited state, respectively, at energies $E_0^{(M=1)} = 4bs+\lambda_-(s)$ and $E_1^{(M=1)} = 4bs+\lambda_+(s)$, where

$$\lambda_\pm(s)=2b\pm2(b^2+8as)^{1/2} \qquad (7)$$

are the roots of the equation $\lambda^2-4b\lambda-32as=0$. The corresponding wavefunctions are

$$\phi_n^{(M=1)}(x)\sim\left(1-\frac{\lambda}{8s}x^2\right)(x^2)^{s-\frac{1}{4}} \times\exp\left(-\frac{a}{4}x^4-\frac{b}{2}x^2\right), \quad (8)$$

and the $\lambda=\lambda_-(s)$ and $\lambda=\lambda_+(s)$ choice has to be made for $n=0$ and $n=1$, respectively [10]. (Note that $a\geq 0$ and $s\geq 0$ imply $\lambda_-(s)\leq 0$, so the polynomial part of (8) is nodeless.) It has to be mentioned that the solutions for $M=0$ and $M=1$ belong to *different* sextic potentials if s is the same, as the coefficient of the quadratic term is different then. We shall see, however, that with appropriate combinations of s and M it is possible to solve sextic potentials that differ only in the strength of the centrifugal term.

The normalization of the wavefunctions can also be given in closed form. For this one has to evaluate integrals of the type

$$I^{(A)} \equiv \int_0^\infty x^A \exp\left(-\frac{a}{2}x^4 - bx^2\right)$$
$$= \frac{1}{2}\Gamma\left(\frac{A+1}{2}\right) a^{-\frac{A+1}{4}} \exp\left(\frac{b^2}{4a}\right) D_{-\frac{A+1}{2}}\left(\frac{b}{a^{1/2}}\right) \quad (9)$$
$$= \frac{1}{2}\Gamma\left(\frac{A+1}{2}\right)(2a)^{-\frac{A+1}{4}} U\left(\frac{A+1}{4}, \frac{1}{2}; \frac{b^2}{2a}\right), \quad (10)$$

where $D_p(z)$ is the parabolic cylinder function and $U(\alpha, \beta; z)$ is one of the forms of the confluent hypergeometric function [12].

Larger values of M can also be considered (*e.g.* for $M = 2$ three different solutions are obtained for the three roots of a cubic algebraic equation for λ), but $M = 0$ and $M = 1$ are sufficient to describe the most important collective states.

In order to apply the sextic oscillator in Eq. (3) it is necessary to identify s with $\tau/2 + 5/4$ (to account for the centrifugal term) and to set the coefficient of the quadratic term to a constant value for each state. These requirements together set a condition for the combination of τ and M in the following way:

$$s + M + \frac{1}{2} = \frac{1}{2}\left(\tau + 2M + \frac{7}{2}\right) \equiv c = const. \quad (11)$$

In practical terms this means that M, which runs from 0 to a finite positive integer value determining the number of solutions is uniquely related to the τ quantum number. In particular, increasing M with one unit corresponds to decreasing τ with 2. This also means that the constant c in (11) must be different for even and odd values of τ, since $\tau + 2M$ is even and odd in the two cases, respectively. This also implies that the coefficient of the quadratic term also depends on the parity of τ, however, the magnitude of this difference can be minimized with respect to that of the quartic and sextic terms with appropriate choice of a and b. For $b^2 > 10a$, for example, this deviation becomes marginal.

With all these considerations the sextic oscillator Hamiltonian can be cast in the following form of (3) with $u(\beta)$ being

$$u^\pi(\beta) = (b^2 - 4ac^\pi)\beta^2 + 2ab\beta^4 + a^2\beta^6 + u_0^\pi\ , \quad (12)$$

where the index $\pi = +/-$ is included to distinguish the potential for even/odd τ's. We have also introduced a constant u_0^π in order to control the relative position of the τ-even and τ-odd part of the spectrum.

For illustration let us consider the case with $M = 0$ and 1, which allows solutions with node number 0 and 1 ($\xi = 1$ and 2 in the conventional notation) and $\tau = 0$, 1, 2 and 3. Table I contains the explict form of the energy and radial wave function of the first few levels, while Fig. 1 displays the schematic structure of the spectrum.

TABLE I. Explicit form of the lowest few energy eigenvalues and wavefunctions for M=0 and 1 with $c = 11/4$ for τ even and $c = 13/4$ for τ odd. Note that $\lambda_\pm = 2b \pm 2(b^2 + 10a)^{1/2}$ and $\tilde{\lambda}_\pm = 2b \pm 2(b^2 + 14a)^{1/2}$, while u_0^- is defined in Eq. (14).

ξ	τ	M	$E_{\xi,\tau}$	$\phi_{\xi,\tau}/\exp(-\frac{a\beta^4}{4} - \frac{b\beta^2}{2})$
1	0	1	$5b + \lambda_-$	$N_{10}\beta^2(1 - \lambda_-/10\beta^2)$
1	1	1	$7b + \tilde{\lambda}_- + u_0^-$	$N_{11}\beta^3(1 - \tilde{\lambda}_-/14\beta^2)$
1	2	0	$9b$	$N_{12}\beta^4$
1	3	0	$11b + u_0^-$	$N_{13}\beta^5$
2	0	1	$5b + \lambda_+$	$N_{20}\beta^2(1 - \lambda_+/10\beta^2)$
2	1	1	$7b + \tilde{\lambda}_+ + u_0^-$	$N_{21}\beta^3(1 - \tilde{\lambda}_+/14\beta^2)$

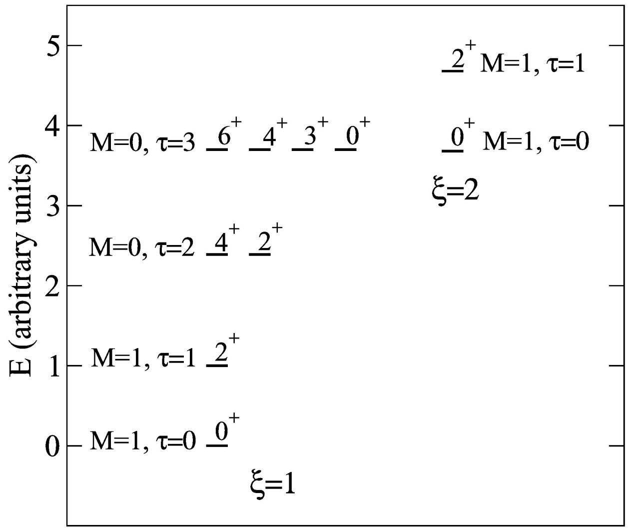

FIGURE 1. Schematic typical spectrum for the sextic oscillator with indication of the relevant quantum numbers.

Let us now analyze the different potential shapes that can be produced by different choices of the parameters in Eq. (12). From (12) we find that the shape of the potential $u^\pi(\beta)$ depends on the sign of $b^2 - 4ac^\pi$ and b, which set the coefficients of the quadratic and quartic terms. (The coefficient of the leading sextic term is always positive.) When $b^2 > 4ac^\pi$ and $b > 0$ hold (*i.e.* for $b > 2(ac^\pi)^{1/2}$), the potential has a minimum at $\beta = 0$ and it increases monotonously with β. When $b^2 < 4ac^\pi$, irrespective of the sign of b (*i.e.* for $-2(ac^\pi)^{1/2} < b < 2(ac^\pi)^{1/2}$), a minimum appears for $\beta > 0$, while for $b^2 > 4ac^\pi$ and $b < 0$ (*i.e.* for $b < -2(ac^\pi)^{1/2}$), first a maximum appears and then a minimum as β increases. In all three cases the exact location of the extremal point(s) can be obtained from the real and positive solutions of

$$(\beta_0^\pi)^2 = \frac{1}{3a}[-2b \pm (b^2 + 12ac^\pi)^{1/2}]\ . \quad (13)$$

Due to the relatively small difference in c^+ and c^-, the τ-even and τ-odd potentials have the same types of extrema at about the same β, except for some peculiar combinations of a and b. Assuming that there are no complications of this

TABLE II. Ratios of some energy eigenvalues and electric quadrupole transition strengths from the sextic oscillator with $a = 40000$, $b = 200$, the infinite square well [3] and the β^4 potential [13], together with the experimentally observed quantities for ^{134}Ba.

	$\frac{E(4^+_{1,2})}{E(2^+_{1,1})}$	$\frac{E(0^+_{2,0})}{E(2^+_{1,1})}$	$\frac{E(6^+_{1,3})}{E(2^+_{1,1})}$	$\frac{B(E2;4^+_{1,2}\to 2^+_{1,1})}{B(E2;2^+_{1,1}\to 0^+_{1,0})}$	$\frac{B(E2;2^+_{2,0}\to 2^+_{1,1})}{B(E2;2^+_{1,1}\to 0^+_{1,0})}$	$\frac{B(E2;0^+_{1,3}\to 2^+_{1,2})}{B(E2;2^+_{1,1}\to 0^+_{1,0})}$
sextic osc.	2.39	3.68	3.70	1.70	1.03	2.12
E(5)	2.20	3.03	3.59	1.68	0.86	2.21
β^4	2.09	2.39	3.27	1.82	1.41	2.52
^{134}Ba (exp.)	2.31	3.57	3.65	1.56(18)	0.42(12)	

TABLE III. Excitation energies (in keV) for the lowest few states of the ^{102}Ru, ^{104}Ru and ^{106}Ru isotopes, and the parameters a and b obtained from a fit to the spectrum. Energies in parenthesis account for levels with ambiguous J^π assignment.

J^π	ξ	τ	^{102}Ru	^{104}Ru	^{106}Ru
2^+_1	1	1	475	358	270
2^+_2	1	2	1103	893	792
4^+_1			1106	888	(715)
0^+_2	1	3	1837		
3^+_1			1522	1242	(1092)
4^+_2			(1799)		
6^+_1			1863	1556	(1296)
0^+_3	2	0	944	988	991
a			[0]	1496	4190
b			283	216	143

kind, we can now return to the question of renormalizing the minima of the τ-even and τ-odd potentials. For $b > 2(ac^\pi)^{1/2}$, $\pi = +, -$ the minima of the two potentials will be u_0^+ and u_0^- at $\beta = 0$, so they coincide if $u_0^+ = u_0^-$ holds. For $b < 2(ac^\pi)^{1/2}$ we can equate the minima of $u^+(\beta)$ and $u^-(\beta)$ if we set $u_0^+ = 0$ and

$$u_0^- = (b^2 - 11a)(\beta_0^+)^2 - (b^2 - 13a)(\beta_0^-)^2 + 2ab[(\beta_0^+)^4 - (\beta_0^-)^4] + a^2[(\beta_0^+)^6 - (\beta_0^-)^6], \quad (14)$$

where the β_0^π are obtained from (13) with the choice of the "+" sign. With this the two potentials have their minima at the same energy, but they take on different values at the origin. Illustrations for possible potential shapes and for the dependence of the energy levels on a and b the reader should consult Ref. 11.

The electric quadrupole transition rates can also be determined analytically by calculating the matrix elements of the transition operator [3, 7]

$$T^{(\mathrm{E2})} = t\alpha_{2\mu} = t\beta \left[D^{(2)}_{\mu,0}\cos\gamma + 2^{-1/2}(D^{(2)}_{\mu,2} + D^{(2)}_{\mu,-2})\sin\gamma \right]. \quad (15)$$

The radial integrals that appear in the β variable in the matrix elements of $T^{(\mathrm{E2})}$ can again be determined using (9). In order to obtain the *total* matrix elements, one has to calculate also the components depending on γ and the Euler angles θ_i. This can be done following the techniques described in Ref. 4. These parts introduce certain selection rules not only for the angular momenta, but also for τ, *i.e.* $\Delta\tau = \pm 1$.

3. Illustration for selected nuclei

As an illustrative first application of the sextic oscillator as a γ-independent potential the low-lying spectrum and the $B(\mathrm{E2})$ rates of the ^{134}Ba nucleus, the first candidate for the E(5) symmetry were approximated with (12), taking $a = 40000$ and $b = 200$ [11]. These parameters result in a potential that has a shallow local minimum at $\beta > 0$ and a relatively flat bottom, so it has features that are expected from a nucleus with E(5) symmetry.

In Table II we summarize the ratio of the most important energy eigenvalues and those of the most characteristic $B(\mathrm{E2})$ transition rates obtained from the sextic oscillator with parameters $a = 40000$, $b = 200$, the infinite square well potential [3] and the numerically solved β^4 potential [13] together with the corresponding experimental values for ^{134}Ba, whenever available. It is seen that the energy ratios corresponding to the E(5) symmetry systematically fall between the values of the β^4 potential and the sextic oscillator. The situation is less obvious for the ratio of the $B(\mathrm{E2})$ values: here the sextic oscillator and the infinite square well seem to yield similar ratios, while the numbers obtained from the β^4 potential are systematically higher. This might be due to the fact that the sextic oscillator potential goes to infinity steeper than the β^4 potential, so the asymptotic behaviour of its wavefunctions can be closer to that of the wavefunctions of the infinite square well.

For further examples we consider some even Ru isotopes near $A = 104$, which is also thought to be located at a phase transition from the spherical to the γ-unstable domain [14]. In particular, we analyze the $^{102-106}$Ru isotopes by fitting the a and b parameters to the excitation energies of their low-lying collective states. Table III contains the energy of these states (in keV) and the fitted parameters a and b. All the indicated states were considered with equal weight in the fits, except for those with ambiguous J^π assignments, which were taken with half the weight of the others. Let us now comment on the results for each isotope separately.

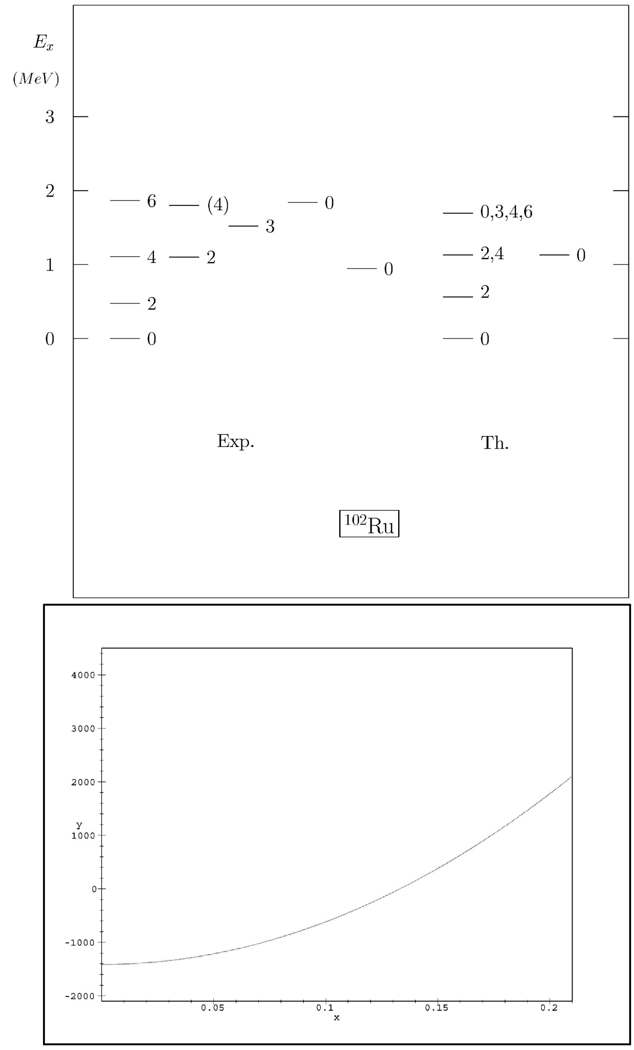

FIGURE 2. The low-lying experimental and calculated energy spectrum of the ^{102}Ru nucleus (upper panel) and the corresponding potential (lower panel). The scales in the lower panel are $x = \beta$ (arbitrary unit) and $y = E_x$ (keV).

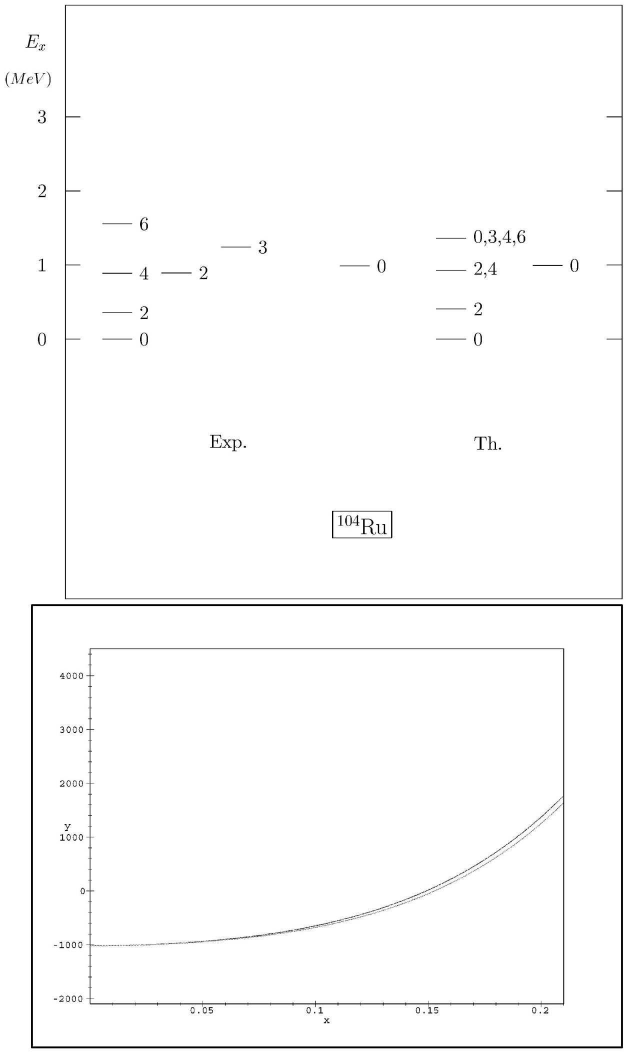

FIGURE 3. The same as Fig. 2 for ^{104}Ru. The solid and the dashed lines correspond to $u^+(\beta)$ and $u^-(\beta)$, respectively.

For ^{102}Ru the naive fit has resulted in a small *negative* value for a, which is clearly incompatible with a normalizable solution (5). The reason why a negative a was obtained lies in the relative position of the levels $E_{2,0}$ and $E_{1,2}$: for ^{102}Ru $E_{2,0} - E_{1,2} = 2(b^2 + 10a)^{1/2} - 2b$ (see Table I) should be negative based on the experimental data, which would require a negative a. Therefore we considered $a = 0$, which corresponds to the harmonic limit. This nucleus, in fact, is close to the harmonic vibrator, as can be seen from Fig. 2. It has to be noted that the $E_{2,0} - E_{1,2}$ energy difference is positive for potentials with a dominant term x^N, $N > 2$ [15,16], and this is fully in line with our analytical results.

In ^{104}Ru $E_{2,0} > E_{1,2}$, and we obtain a positive a. The shape of the potential in Fig. 3 is still close to the harmonic limit, but it is flatter. This is due to the smaller (but still positive) coefficient of the quadratic term of the potential ($b^2 - 4ac$). As discussed in the previous Section, such a combination of a and b results in a potential with a minimum at $\beta = 0$. The ^{104}Ru nucleus has been suggested as an example for a phase transition form the spherical to the γ-unstable domain [14]. As opposed to the case of the harmonic potential for ^{102}Ru, here the two potential curves $u^+(\beta)$ and $u^-(\beta)$ slightly differ.

The trend of the a and b parameter continues for the ^{106}Ru nucleus, as can be seen from Table III, and this is also reflected in Fig. 4. Now $b^2 - 4ac$ is negative, and this corresponds to a potential with a local minimum at $\beta > 0$. (This is natural, as now the coefficient of the quadratic term is negative, so the potential curve has negative derivative close to the origin.) The $u(\beta)$ resembles even more to a flat-bottomed potential expected at a phase transition. Based on this finding alone, ^{106}Ru could also be associated with an E(5) symmetry. It has to be noted, however, that this nucleus is less well-known experimentally (*e.g.* no $B(\mathrm{E2})$ values are known) than its neighbors, so there is less ground to compare its spectroscopic properties with the key numbers associated with the E(5) symmetry.

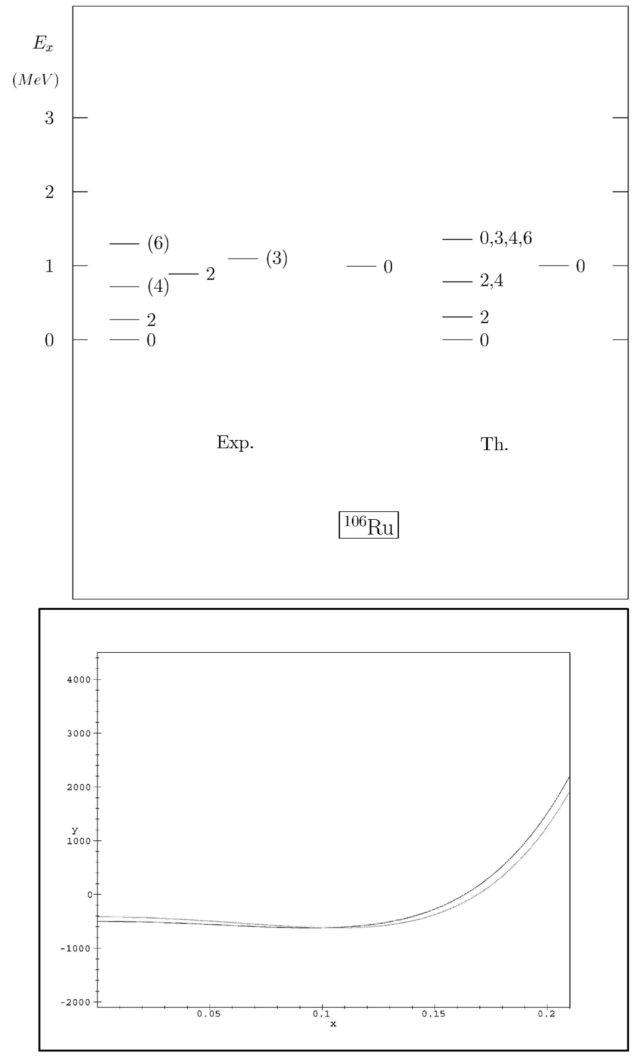

FIGURE 4. The same as Fig. 3 for ^{106}Ru.

Based on the results for the three Ru isotopes discussed here we can establish that in accordance with the expectations the changes in the spectrum and in the corresponding potential are in line with a transition from the spherical vibrator to the deformed γ-unstable phase. The results also demonstrate the flexible nature of the sextic oscillator and confirm its usefulness in the analysis of realistic nuclei. Work is in progress to calculate further spectroscopic data, such as electromagnetic transition rates.

4. Summary and outlook

We have shown that the sextic oscillator proposed previously for application in the Bohr Hamiltonian as a γ-independent potential is indeed capable of describing realistic nuclear spectra and can also account for the fine effects associated with them, such as phase transition through critical points. This is due to its flexible shape that can reproduce potentials with minimum at $\beta = 0$ and $\beta > 0$ alike, furthermore, in the latter case a local maximum can also be obtained at $0 < \beta_{\rm max} < \beta_{\rm min}$. In addition to the energy eigenvalues and the wavefunctions, the $B(\mathrm{E2})$ rates can also be calculated analytically in this model, and this makes the sextic oscillator potential a valuable tool in the analysis of collective nuclear phenomena.

The model can be extended further along various lines. Allowing larger values for M in (4) higher states can also be included in the spectrum. (For $M = 2$ these are $\xi,\ \tau =$ 1,0; 1,1; 1,2; 1,3; 2,0; 2,1.)

The flexible shape might be used to describe nuclei corresponding to other symmetries (X(5)), where local minima and maxima are expected to occur in $u(\beta)$. For this a γ-dependent potential term also has to be included in the Bohr Hamiltonian, so the exact analytic results have to be combined with approximations usually applied in this situation [5, 17].

Finally, there are further quasi-exactly solvable potentials with an x^{-2}-type term, and these can also be applied in the Bohr Hamiltonian.

Acknowledgments

This work was supported by the OTKA grant No. T37502 (Hungary). The author thanks J.M. Arias for the valuable discussions on the topic.

1. A. Bohr and B. Mottelson, *Nuclear Structure* (Benjamin, Reading, MA 1975) Vol. II.
2. F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, Cambridge 1987).
3. F. Iachello, *Phys. Rev. Lett.* **85** (2000) 3580.
4. D. R. Bès, *Nucl. Phys.* **10** (1959) 379.
5. L. Fortunato, nucl-th/0411087.
6. A. Bohr, *Dan. Mat. Fys. Medd.* **26** (1952).
7. L. Wilets and M. Jean, *Phys. Rev.* **102** (1956) 788.
8. J.P. Elliott, J.A. Evans, and P. Park, *Phys. Lett.* **169B** (1986) 309.
9. L. Fortunato and A. Vitturi, *J. Phys. G* **29** (2003) 1341.
10. A. G. Ushveridze, *Quasi-exactly solvable models in quantum mechanics* (IOP Publishing, Bristol 1994).
11. G. Lévai and J.M. Arias, *Phys. Rev. C* **69** (2004) 014304.
12. M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover, New York 1970).
13. J.M. Arias *et al.*, *Phys. Rev. C* **68** (2003) 041302(R).
14. A. Frank, C.E. Alonso, and J.M. Arias, *Phys. Rev. C* **65** (2001) 014301.

15. J.M. Arias, J. Dukelsky, and J.E. García-Ramos, *Phys. Rev. Lett.* **91** (2003) 162502.
16. D. Bonatsos, D. Lenis, N. Minkov, P.P. Raychev, and P.A. Terziev, *Phys. Rev. C* **69** (2004) 044316.
17. F. Iachello, *Phys. Rev. Lett.* **87** (2001) 052502.