

Transition from quantum to classical behavior for some simple model systems

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There is an increasing interest in the question of why typical quantum mechanical properties, such as those connected with the superposition of states or diffraction patterns for material systems, are not observed on the classical macroscopic level. By discussing two simple model problems connected via the free-particle propagator, we show under what circumstances typical quantum effects that show up in these systems can attain significant magnitudes so as to have a chance to be observable. The influence of the interaction with a dissipative environment will also be considered, and the time scale where the effects reach their maximum and how they decay afterwards will be discussed. Furthermore, a comparison with recent scattering experiments will be given.

Keywords: Transition from quantum to classical behavior.

Hay actualmente interés en el problema de por qué propiedades típicas de mecánica cuántica, tales como las conectadas con la superposición de estados o esquemas de difracción para sistemas de materiales, no se observan en el nivel macroscópico clásico. Discutiendo dos modelos simples de problemas conectados por el propagador de partícula libre, mostramos bajo qué circunstancias efectos cuánticos típicos que surgen en esos sistemas pueden alcanzar magnitudes significativas que dan la posibilidad de ser observables. La influencia de la interacción con un entorno disipativo será también considerada y la escala de tiempo donde los efectos toman su valor máximo y como decaen posteriormente también será discutida. Además la comparación con experimentos recientes de dispersión será analizada.

Descriptores: Transición del comportamiento clásico al cuántico.

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1. Introduction

Recently, increasing interest has been shown in the transition from the quantum mechanical to the classical world, not least of all because the region on the borderline of these two domains is not only of interest to the developing field of nanotechnology, but also in the process of becoming more and more experimentally accessible. New fields of physics and engineering - such as quantum computation, quantum cryptography and teleportation - gain experimental, as well as theoretical, interest. In particular, one wants to understand why typical quantum mechanical properties, like coherent superpositions of states, quantum entanglement or diffraction patterns of material systems, do not show up at the classical level. Most experiments investigating these problems usually consider that quantum systems, which are carefully isolated from their environment, *e.g.*, ultracold ions caught in magneto-optical traps, keep the quantum properties intact for a long enough time to allow for measurements to be taken. This is necessary because of the so-called decoherence (see, *e.g.*, [1,2]), a process that is assumed to destroy the quantum effects due to disturbances of the quantum states by interaction with an environment which can simply be represented by the measuring apparatus [3–6]. Therefore, it is widely believed that some quantum effects like coherent superpositions or entanglement are unimportant and/or not accessible to experiment at ambient conditions in condensed systems because of their extremely fast decoherence, caused by environmental disturbance [7]. Despite considerable effort, a well-established theory of decoherence in condensed matter does not yet exist either [7,8]. Hence, more surprising are the

recent spectacular results of Dreismann *et al.* [7–10] which show that quantum entanglement for example, is experimentally accessible in condensed matter at ambient conditions by means of scattering techniques. However, the effects observed are on a very short time-scale of about a hundred attoseconds.

In this paper, we consider simple examples with analytical solutions - namely, free motion and, with modified initial conditions, the problem of diffraction-in-time [11] - to show under what circumstances and on which time-scale typical quantum mechanical properties of these systems occur or disappear, respectively, and what the interrelations between these effects are.

First, the isolated systems will be investigated and, in a second step, the interaction with the environment will be included. The time-scales when the effects reach their maximum will be determined and compared with the above-mentioned experiments by Dreismann et al; the decay of the effects will also be discussed for the isolated and interacting systems.

In Sec. 2, the wave packet (WP) solution of the time-dependent Schrödinger equation (SE) and its Green's function will be given in a form that applies not only for free motion, but also allows for comparison with other problems, *e.g.*, diffraction in time. The occurrence of a term in the corresponding density equation that does not occur in the equation of motion for the classical density of this system will be discussed as an example of a typical quantum mechanical property.

In Sec. 3, the diffraction-in-time problem will be reformulated using Green's function as defined in Section 2. For

comparison with the classical situation, the solution will then be given in the form of the corresponding Wigner function.

In Sec. 4, the interaction of the quantum system with a dissipative environment will be included in an effective way that was introduced in earlier works. The resulting modifications of the effects described in the preceding two sections will be shown. A similarity with the quantum Zeno effect will be mentioned.

Finally, in Sec. 5, the conditions for the maximum effect and the disappearance of the quantum mechanical properties will be discussed, both without and with environmental influence, and the time-scales will be compared with the experimental work of Dreismann et al.

2. Green's function and wave packet properties for free motion

For given initial conditions $\Psi(x', 0)$ (in one dimension, as will be used throughout this paper), the solutions of the time-dependent SE can be obtained from

$$\Psi(x, t) = \int_{-\infty}^{+\infty} dx' G(x, x', t, t' = 0) \Psi(x', 0) \quad (1)$$

where the Green's function or time-propagator is given by [12, 13]

$$G(x, x', t, 0) = \left(\frac{m}{2\pi i \hbar \alpha_0 \hat{z}} \right)^{\frac{1}{2}} \exp \left\{ \frac{im}{2\hbar} \left[\frac{\dot{\hat{z}}}{\hat{z}} x^2 - 2 \frac{x}{\hat{z}} \left(\frac{x'}{\alpha_0} + \frac{\hat{u}}{\hat{z}} \left(\frac{x'}{\alpha_0} \right)^2 \right) \right] \right\}. \quad (2)$$

This form might look more cumbersome than the one usually used for free motion, but it has the advantage that it also applies to other problems. Time-dependence enters via the time-dependent quantities $\hat{z}(t)$ and $\hat{u}(t)$ that have to fulfil the classical equations of motion of the corresponding system; so, for free motion,

$$\ddot{\hat{z}}(t) = 0 \quad \text{and} \quad \ddot{\hat{u}}(t) = 0, \quad (3)$$

and are interrelated via the conservation law

$$\dot{\hat{z}}\hat{u} - \dot{\hat{u}}\hat{z} = 1. \quad (4)$$

The constant quantity α_0 is connected with the initial position uncertainty $\langle \tilde{x}^2 \rangle_0 = \langle x^2 \rangle(t=0) - \langle x \rangle^2(t=0)$ via $\alpha_0^2 = (2m/\hbar)\langle \tilde{x}^2 \rangle_0$, where $\langle \dots \rangle = \int \Psi^* \dots \Psi dx$ denote mean values.

It should be mentioned that form (2) of our propagator is similar to the one used by Husimi [14]. There, however, the propagator is applied to describe the time-evolution of a Gaussian with a purely imaginary exponent, whereas the Gaussian WPs discussed in [12, 13] and in this paper have a complex exponent. In particular, the real part of this exponent is connected with the position uncertainty whose possible time-dependence is the origin of the typical quantum

mechanical effect discussed later in this section, which represents the characteristic difference with respect to the classical situation.

The occurrence of α_0 , which is proportional to the initial position uncertainty, should not lead erroneously to the conclusion that our propagator depends on a property of the initial wave function. This form is only used to show more closely the relation to the uncertainties and a non-vanishing term in the quantum mechanical continuity equation. Consideration of the definitions of \hat{u} and \hat{z} as given below [e.g. in Eq. (11)] shows that our Green's function (2) does not depend on the initial width of the WP.

For an initial Gaussian WP

$$\Psi_{WP}(x', 0) = \left(\frac{m\beta_0}{\pi\hbar} \right)^{1/4} \exp \left\{ \frac{im}{2\hbar} \left[i\beta_0 x'^2 + 2 \frac{p_0}{m} x' \right] \right\} \quad (5)$$

with mass m , initial momentum p_0 and $\beta_0 = 1/\alpha_0^2$, the time-dependent WP $\Psi_{WP}(x, t)$, obtained by integrating product of $\Psi_{WP}(x', 0)$ with the Green's function (2) can be written as

$$\Psi_{WP}(x, t) = \left(\frac{m}{\pi\hbar} \right)^{1/4} \left(\frac{1}{\hat{u} + i\hat{z}} \right)^{1/2} \times \exp \left\{ \frac{im}{2\hbar} \left[\frac{\dot{\hat{z}}}{\hat{z}} x^2 - \frac{(x - \frac{p_0 \alpha_0}{m} \hat{z})^2}{\hat{z}(\hat{u} + i\hat{z})} \right] \right\} \quad (6)$$

which can be simplified, using $\hat{z}\alpha_0 p_0/m = \langle x \rangle(t)$, to yield the more familiar form of a Gaussian WP,

$$\Psi_{WP}(x, t) = \left(\frac{m}{\pi\hbar} \right)^{1/4} \left(\frac{1}{\hat{u} + i\hat{z}} \right)^{1/2} \times \exp \left\{ \left[-\frac{m}{2\hbar} \left(\frac{1}{\hat{u}^2 + \hat{z}^2} - \frac{i}{2} \frac{\frac{d}{dt}(\hat{u}^2 + \hat{z}^2)}{\hat{u}^2 + \hat{z}^2} \right) \right] (x - \langle x \rangle)^2 + \frac{i}{\hbar} \langle p \rangle (x - \langle x \rangle) + \frac{i}{\hbar} \frac{\langle p \rangle \langle x \rangle}{2} \right\} \quad (7)$$

with a maximum at the classical position $\langle x \rangle(t)$, width connected with the position uncertainty

$$\langle \tilde{x}^2 \rangle(t) = \frac{\hbar}{2m} (\hat{u}^2 + \hat{z}^2),$$

and a phase depending on the classical momentum $\langle p \rangle$. Important for the following is that the phase can also be dependent on a term proportional to the relative change in time of the WP width:

$$\frac{\frac{d}{dt}(\hat{u}^2 + \hat{z}^2)}{(\hat{u}^2 + \hat{z}^2)} = \frac{\frac{d}{dt}\langle \tilde{x}^2 \rangle}{\langle \tilde{x}^2 \rangle}. \quad (8)$$

The corresponding density function

$$\varrho(x, t) = \Psi_{WP}^*(x, t) \Psi_{WP}(x, t)$$

fulfills the continuity equation

$$\frac{\partial}{\partial t} \varrho + \nabla_x (\varrho v) = \frac{\partial}{\partial t} \varrho + v \nabla_x \varrho + \varrho \nabla_x v = 0, \quad (9)$$

where the velocity field v in the probability current $j = \varrho v$ is defined as

$$v = \frac{\hbar}{2mi} \left(\frac{\nabla_x \Psi}{\Psi} - \frac{\nabla_x \Psi^*}{\Psi^*} \right) = \frac{\hbar}{2mi} \nabla_x \ln \frac{\Psi}{\Psi^*},$$

i.e. it depends on the phase of Ψ .

In the case of our WP solutions, we obtain for the quantum system

$$\nabla_x v = \frac{1}{2} \frac{\frac{d}{dt}(\hat{u}^2 + \hat{z}^2)}{\hat{u}^2 + \hat{z}^2} = \frac{1}{2} \frac{\frac{d}{dt}\langle \tilde{x}^2 \rangle}{\langle \tilde{x}^2 \rangle}. \quad (10)$$

This quantity only vanishes for WPs with constant width but, in particular for the free motion, the WP is spreading in time, i.e. $\nabla_x v \neq 0$. This is a typical quantum mechanical property of the system that should disappear when we make the transition to the classical world. In particular for the free motion WP, we can give the explicit analytical form of the time-dependence of $\nabla_x v$, using the fact that in this case

$$\hat{u} = \alpha_0 = \text{const}, \quad \hat{z} = \frac{1}{\alpha_0} \cdot t \quad (11)$$

is valid to obtain

$$\nabla_x v = \beta_0 \frac{(\beta_0 t)}{1 + (\beta_0 t)^2}. \quad (12)$$

For the classical situation, a continuity equation formally equivalent to (9) exists but, in this case, it is for the distribution of a virtual ensemble over the phase space described by a density function ϱ_{cl} of the generalized coordinates and momenta and of time, known as phase density.

The fundamental theorem of Liouville selects from the transformations of phase space those which represent possible motions of the phase density and can be given in several equivalent formulations [15]. From Hamilton's equations of motion, it follows that

$$\nabla v = 0, \quad (13)$$

(but now in phase space with ∇ representing derivatives with respect to position and momentum coordinates), i.e. the “liquid” formed from the phase points is incompressible. This becomes more obvious if we rearrange Eq. (9) according to

$$\frac{\partial}{\partial t} \varrho + v \nabla \varrho = -\varrho \nabla v = 0, \quad (14)$$

to yield the so-called Liouville equation.

Introducing the hydrodynamical or substantial derivative $D/Dt = \partial/\partial t + v\nabla$ (representing the change of a quantity in the immediate neighbourhood of a point moving with the “liquid”) and defining a “Volume” V as the inverse of the phase density ϱ_{cl} , $V = 1/\varrho_{cl}$, Eq.(14) can be rewritten as

$$\frac{\frac{D}{Dt} V}{V} = \nabla v = 0. \quad (15)$$

From (14) and (15) follows the principle of conservation of density in phase space and the principle that a phase extension always retains its volume during motion. Making use of the concepts of measure theory, these statements can be made even more precise [15], stating that the measure of a measurable point-set is an invariant of the time-evolution of the virtual ensemble.

Comparison of (15) with Eq. (10) shows that, in the quantum mechanical case, the WP-width (or the square root of $\langle \tilde{x}^2 \rangle$) plays a role comparable to the volume in phase space in the classical case. The major differences, comparing both situations, are that, apparently, the condition $\nabla v = 0$ already does not apply for free motion in the quantum mechanical case and that, in the classical case also the momentum variable contributes to the volume element. (In the case of quantum mechanical free motion, the momentum uncertainty $\langle \tilde{p}^2 \rangle$ is constant, so that its product with $\langle \tilde{x}^2 \rangle(t)$ still would not be constant, but growing in time.)

In order to include the momentum-aspect also in our quantum mechanical considerations, it should be mentioned that the time-dependence of the WP-width is also related to another quantum mechanical property, namely the correlation of position and momentum uncertainties via

$$\begin{aligned} \langle [\tilde{x}, \tilde{p}]_+ \rangle &= \langle [(x - \langle x \rangle), (p - \langle p \rangle)]_+ \rangle \\ &= \hbar(\dot{\hat{u}}\hat{u} + \dot{\hat{z}}\hat{z}) = m \frac{d}{dt} \langle \tilde{x}^2 \rangle. \end{aligned} \quad (16)$$

Using, again, (11) for the free motion, the correlation is given explicitly as

$$\langle [\tilde{x}, \tilde{p}]_+ \rangle = \langle \tilde{x}\tilde{p} + \tilde{p}\tilde{x} \rangle = \frac{1}{\alpha_0^2} t = \beta_0 t. \quad (17)$$

Since the uncertainty product of position and momentum, $U = \langle \tilde{x}^2 \rangle \langle \tilde{p}^2 \rangle$, can be expressed with $\langle [\tilde{x}, \tilde{p}]_+ \rangle$ as

$$U = \frac{\hbar^2}{4} + \frac{1}{4} \langle [\tilde{x}, \tilde{p}]_+ \rangle^2, \quad (18)$$

these correlations are responsible for the increasing uncertainty as time progresses. Thus, a system that is initially prepared in a state of minimum uncertainty, i.e. $U = \hbar^2/4$, which comes closest to a point in phase space, will not keep this minimal extension in position and momentum uncertainties if the correlations do not vanish, i.e. if $\nabla_x v \neq 0$. In the case of the free motion of an isolated system, U grows according to (18) and (17) with

$$U = \frac{\hbar^2}{4} \left[1 + (\beta_0 t)^2 \right] \quad (19)$$

quadratically in time to diverge for $t \rightarrow \infty$.

It should be mentioned that some authors [16] assume that the role of the phase space volume in quantum mechanics is played by the square root of the so-called “invariant

uncertainty product" $\langle \tilde{x}^2 \rangle \langle \tilde{p}^2 \rangle - \frac{1}{4} \langle [\tilde{x}, \tilde{p}]_+ \rangle^2 = \hbar^2/4$, which is definitely a constant of motion; but this only corresponds to the phase space volume at the *initial time* $t_0 = 0$, not for times $t > t_0$. This conserved quantity is more related with a kind of conservation of angular momentum (see, for example [13]) than to conservation of phase space volume. Again, it is just the fact that

$$\nabla_x v = \frac{1}{2} \frac{\frac{d}{dt} \langle \tilde{x}^2 \rangle}{\langle \tilde{x}^2 \rangle} = \frac{1}{2m} \frac{\langle [\tilde{x}, \tilde{p}]_+ \rangle}{\langle \tilde{x}^2 \rangle} \neq 0,$$

i.e., the WP width not being constant, that causes the introduction of a second term, proportional to $\langle [\tilde{x}, \tilde{p}]_+ \rangle$, into the expression for momentum uncertainty,

$$\langle \tilde{p}^2 \rangle = \langle p^2 \rangle - \langle p \rangle^2 = \frac{\hbar^2}{4 \langle \tilde{x}^2 \rangle} + \frac{\frac{1}{4} \langle [\tilde{x}, \tilde{p}]_+ \rangle^2}{\langle \tilde{x}^2 \rangle}, \quad (20)$$

which is responsible for the change-in-time of the initial phase space volume. Therefore, the subtraction of $(1/4) \langle [\tilde{x}, \tilde{p}]_+ \rangle^2$ from the uncertainty product $U = \langle \tilde{x}^2 \rangle \langle \tilde{p}^2 \rangle$ obviously removes the time-dependence and leaves only the initial uncertainty product $U_0 = \langle \tilde{x}^2 \rangle_0 \langle \tilde{p}^2 \rangle_0 = \hbar^2/4$. For $t > t_0$, the uncertainty product will grow in time if, as in our case, $\langle \tilde{x}^2 \rangle(t)$ is spreading and $\langle \tilde{p}^2 \rangle$ is constant (in our case, the two time-dependent terms on the lhs of Eq. (20) add up to a constant). This is precisely the difference between the classical situation and the quantum mechanical case - which is what we wish to point out in this paper.

The situation where the system interacts with an environment will be discussed in Sec. 4.

3. Green's function applied to the diffraction-in-time problem

The Green's function discussed in Section 2 can also be applied if the initial conditions are modified. One encounters such a situation when considering the problem of opening a completely absorbing shutter on which a stream of particles of definite velocity is impinged, which was solved long ago by one of the authors [11]. The solution was obtained in a form entirely analogous to the optical one of diffraction by a straight edge, and can be given as

$$M(x, k, t) = \exp \left[i(kx - \frac{1}{2}k^2 t) \right] \times \frac{1}{\sqrt{2}} \left\{ \left[\frac{1}{2} - C(w) \right] + i \left[\frac{1}{2} - S(w) \right] \right\}. \quad (21)$$

Since the argument

$$w = \sqrt{\frac{m}{2\hbar t}} \left(x - \frac{\hbar}{m} kt \right) \quad (22)$$

of the occurring Fresnel integrals

$$C(w) = \sqrt{\frac{2}{\pi}} \int_0^w \cos y^2 dy, \\ S(w) = \sqrt{\frac{2}{\pi}} \int_0^w \sin y^2 dy \quad (23)$$

is time-dependent, the problem was described as diffraction-in-time.

This problem, however, corresponds to the situation of a free particle propagation with the initial condition

$$\psi(x, 0) = \exp(ikx)\Theta(-x), \quad (24)$$

where $\Theta(x)$ is the step function given by

$$\Theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \quad (25)$$

and k in the plane wave is connected with the momentum via $p = \hbar k$.

The time-dependent solution can now be obtained by simply applying our Green's function of Eq. (2) to this initial condition, *i.e.*

$$M(x, k, t) = \int_{-\infty}^0 dx' G(x, x', t, 0) \exp(ikx'). \quad (26)$$

It is a straightforward matter to show that $M(x, k, t)$ can be written as

$$M(x, k, t) = \frac{1}{2\pi} \left(\frac{m}{i\hbar\alpha_0 \hat{z}} \right)^{1/2} \times \exp \left\{ i \left[(\dot{z}\alpha_0 k)x - \frac{\hbar}{2m} (\dot{z}\alpha_0 k)^2 \frac{\hat{z}}{\hat{z}} \right] \right\} \cdot I_M, \quad (27)$$

where the integral I_M ,

$$I_M = \int_{-\infty}^0 d\bar{x} \exp \left\{ - \frac{m}{i2\hbar} \frac{\hat{u}}{\hat{z}} \frac{\bar{x}^2}{\alpha_0} \right\}, \quad (28)$$

depends on the shifted variable

$$\bar{x} = x' - \left(\frac{\alpha_0}{\hat{u}} x - \frac{\hbar}{m} \frac{\hat{z}}{\hat{u}} \alpha_0^2 k \right), \quad (29)$$

with $d\bar{x} = dx'$.

Comparison with the Fresnel integrals in (21) shows that the argument w can now be expressed, using the time-dependent parameters of the Green's function, as

$$w = \sqrt{\frac{m}{2\hbar\hat{u}\hat{z}}} \left(x - \frac{\hbar}{m} \alpha_0 \hat{z} k \right). \quad (30)$$

Again, as already mentioned in Section 2, the occurrence of the "initial width" α_0 is only purely formal to show the similarities of the different quantum mechanical problems discussed in Secs. 2 and 3. With the definition of \hat{u} and

\hat{z} as given in Eq.(11), α_0 does not have any explicit influence on the quantities given in Eqs. (27)-(30).

So the solution (27) simplifies to

$$M(x, k, t) = \frac{1}{2\pi} \left(\frac{m}{i\hbar t} \right)^{1/2} \exp \left\{ i \left[kx - \frac{\hbar}{2m} k^2 t \right] \right\} \cdot I_M, \quad (31)$$

as expected, and w turns into the form of (22). However, comparison of (22) and (30) allows us now to express the time-dependence of w in terms of the quantities of G that also occur in the problem of the WP discussed before, in particular in $\nabla_x v$ and in $\langle [\tilde{x}, \tilde{p}]_+ \rangle$. Again using (11), we find $(1/\hbar)\langle [\tilde{x}, \tilde{p}]_+ \rangle = \dot{\tilde{z}}\tilde{z} = \frac{1}{\alpha_0}\tilde{z} = \beta_0 t$ and thus we can identify

$$t = \frac{1}{\hbar\beta_0} \langle [\tilde{x}, \tilde{p}]_+ \rangle. \quad (32)$$

For an easier determination of the classical limit, the solutions of the diffraction-in-time problem will be given in terms of Wigner distribution functions. As shown in [17], this give us

$$W(x, p; k, t) = \frac{\sin[g(\hbar k - p)]}{\pi(\hbar k - p)} \Theta \left(\frac{pt}{m} - x \right) \quad (33)$$

where

$$g \equiv \frac{2}{\hbar} \left(\frac{pt}{m} - x \right). \quad (34)$$

The time-dependence in $W(x, p; k, t)$ results from the time-dependence in $M(x, k, t)$ or $w(t)$, so that t occurring in (34) can be expressed by (32), and we obtain

$$g = \frac{2}{\hbar} \left(\frac{p}{\hbar m \beta_0} \langle [\tilde{x}, \tilde{p}]_+ \rangle - x \right). \quad (35)$$

From this expression, the time necessary for the non-classical oscillatory pattern to vanish can be estimated, as will be shown in Sec. 5.

4. Effect of a dissipative environment on the characteristic quantum phenomena

In recent papers, several authors [1–5] have claimed that interaction with an environment is the actual key for the disappearance of typical quantum mechanical properties such as coherence, entanglement, etc. Any kind of interaction with an external world - even a measurement whose result is not recorded [4] - is supposed to have the same effect on the quantum system, so the details of the environment, and of the interaction with it, should not matter. Therefore, in the following, we will treat the environment globally, *i.e.*, regarding only its effect on the quantum system under investigation, without specifying its details or the details of the interaction. For this purpose, we apply a method used earlier by us to describe dissipative systems. This approach can

use either non-unitary transformations [18, 19] or nonlinear modifications of the SE [20–22] and is physically equivalent to the system-plus-reservoir approach (for details see, for example [23, 24]). We shall show the influence of the dissipative environment on the occurrence, magnitude and time-scale of the quantum effects discussed in the previous sections.

The treatment of dissipative systems within the Hamiltonian formalism was previously about sixty years ago by Caldriola [25] and Kanai [26], but an apparent violation of the uncertainty principle after canonical quantization made it questionable and the target for criticisms for a long time [27–29]. The situation became even more paradoxical after Sun and Yu [23, 24] showed that the Hamiltonian operator corresponding to Kanai and Caldriola's explicitly time-dependent Hamiltonian function could be derived from the conventional system-plus-reservoir approach. The paradox was resolved by one of the authors [18] showing that the transformation of the operators must be accompanied by a non-unitary transformation of the wave functions. In particular, for a solution Ψ of the formal Caldriola-Kanai equation, the corresponding solution Ψ_{dis} for the physical dissipative system is given by $\ln \Psi_{dis} = e^{-\gamma t} \ln \Psi$. For the free motion WP, this transformation has already been applied in an earlier work [30] leading to the same WP that is also the solution of a nonlinear SE with logarithmic nonlinearity [22] describing the same situation of the usual physical, instead of a formal canonical, level. For our purpose, it is sufficient to state that, in this case, the continuity equation (9) is replaced by an irreversible Fokker-Planck-type equation of the form

$$\frac{\partial}{\partial t} \varrho + \nabla_x (\varrho v) - D \Delta_x \varrho = 0, \quad (36)$$

where $D = (\gamma/2)\langle \tilde{x}^2 \rangle$ and γ is the friction coefficient. (The diffusion coefficient D on the quantum mechanical coordinate variance in exactly the same way as it does depend on the classical coordinate variance in the Langevin/Fokker-Planck theory (see, for example [31]), and can, in particular in non-equilibrium situations, also be time-dependent.)

Now then, the total velocity field contains, in addition to the convective velocity field v , the diffusive velocity field $v_D = -D(\nabla \varrho / \varrho)$ with $-D \Delta_x \varrho = \nabla_x v_D$. For the total velocity field $v_{tot} = v + v_D$, again

$$\nabla_x v_{tot} = \nabla_x (v + v_D) = \frac{1}{2} \frac{d}{dt} \frac{\langle \tilde{x}^2 \rangle_{dis}}{\langle \tilde{x}^2 \rangle_{dis}} \quad (37)$$

applies, whereas, for the convective velocity alone,

$$\nabla_x v = \frac{1}{2} \frac{d}{dt} \frac{\langle \tilde{x}^2 \rangle_{dis}}{\langle \tilde{x}^2 \rangle_{dis}} - \frac{\gamma}{2} \quad (38)$$

holds. Since both expressions differ only by a constant, they both attain their maximum value for the same time t_{max} .

It should be mentioned that, in the dissipative case, two WP solutions exist for the damped free motion (see, *e.g.*, [32]), but, for purposes of comparison with the results

of the isolated system, we only discuss the solution that corresponds to a minimum-uncertainty WP for $t=0$, *i.e.* with $U(t=0)=\hbar^2/4$.

The WP width, or position uncertainty, has been given in [30] or [22], respectively, and has the form

$$\langle \tilde{x}^2 \rangle_{dis} = \langle \tilde{x}^2 \rangle_0 \left[e^{\gamma t} + \left(\frac{\beta_0}{\gamma/2} \right)^2 \sinh^2 \frac{\gamma}{2} t \right]. \quad (39)$$

Thus, $\nabla_x v$, corresponding to (12), and taking (38) into consideration, is given by

$$(\nabla_x v)_{dis} = \beta_0 e^{-\gamma t} \frac{\left(\frac{\beta_0}{\gamma} \right) (1 - e^{-\gamma t})}{1 + \left(\frac{\beta_0}{\gamma} \right)^2 (1 - e^{-\gamma t})^2}. \quad (40)$$

In this case too, the time-dependence of $\langle \tilde{x}^2 \rangle_{dis}$ is connected with the correlation of position and momentum uncertainties but now, as can be proven using the WP solution of the dissipative case, via

$$\langle [\tilde{x}, \tilde{p}]_+ \rangle_{dis} = m \left(\frac{d}{dt} \langle \tilde{x}^2 \rangle_{dis} - \gamma \langle \tilde{x}^2 \rangle_{dis} \right) \quad (41)$$

and the uncertainty product, expressed with the help of $\langle [\tilde{x}, \tilde{p}]_+ \rangle_{dis}$, is again given by

$$U_{dis} = \frac{\hbar^2}{4} + \frac{1}{4} \langle [\tilde{x}, \tilde{p}]_+ \rangle_{dis}^2. \quad (42)$$

For the case considered, *i.e.*, the damped free motion with minimum initial WP, U_{dis} can be written explicitly as

$$U_{dis} = \frac{\hbar^2}{4} \left\{ 1 + \left[\left(\frac{\beta_0}{\gamma} \right) (1 - e^{-\gamma t}) \right]^2 \right\} \geq \frac{\hbar^2}{4}, \quad (43)$$

with $U_{dis}(t=0) = \hbar^2/4$.

As mentioned at the end of Section 2, for the isolated system, U is diverging for $t \rightarrow \infty$. However, in the case of the interacting system, it follows from (43) that for $t \rightarrow \infty$, U_{dis} approaches a finite maximum value given by

$$U_{dis,\infty} = \frac{\hbar^2}{4} \left[1 + \left(\frac{\beta_0}{\gamma} \right)^2 \right] = \text{finite.} \quad (44)$$

From this expression, it follows that the maximum value of uncertainty decreases with increasing γ . A large value of γ , however, corresponds to a frequent interaction with the environment - where this interaction can also be interpreted as an observation or a measurement. In the limit of $\gamma \rightarrow \infty$, *i.e.* continual measurement, the maximum value of U_{dis} is identical to its initial value $U_{dis}(t=0) = \hbar^2/4$. This is, however, very similar to what is known as quantum Zeno effect - where continual measurement, described by a single parameter, causes a system to remain arbitrarily close to its initial state [33, 34]. But note that, even in the limit of "permanent interaction" or "continual measurement", *i.e.* for $\gamma \rightarrow \infty$, it is not possible to come below the lower bound of $U = \hbar^2/4$.

The effect of dissipation on the diffraction-in-time problem has already been discussed in detail in Ref. 35. There it was shown that the dissipation restricts the region into which the non-classical oscillatory pattern can proceed and also reduces the amplitude of the oscillations.

5. Discussion of the maximum and disappearance of the quantum mechanical effects

Considering, first, the $(\nabla_x v)$ -term in the continuity equation of the WP, it is obvious from Eq. (12) that in the so-called classical limit, *i.e.*, for $\hbar \rightarrow 0$, this quantity vanishes to yield the proper classical result - since β_0 is proportional to \hbar , $\beta_0 = \hbar/2m\langle \tilde{x}^2 \rangle_0$.

However, in nature, \hbar is not zero, but has a definite constant value ($\hbar = 1.043 \cdot 10^{-27} \text{ g cm}^2 \text{s}^{-1}$). Therefore, the questions arise: under what circumstances - in particular, on what time-scale - does $(\nabla_x v)$ attain its maximum value and, therefore, has the best chance of being observed? And how does the effect decrease to such small values that it does not show up on a macroscopic scale? For this purpose, it is necessary to determine the time when $(\nabla_x v)$ attains its maximum and calculate, or at least estimate, the corresponding value of $(\nabla_x v)$.

For the *isolated* system, it follows from Eq. (12) that the maximum is reached for

$$t_{\max} = \frac{1}{\beta_0} \quad (45)$$

with the maximum value of

$$(\nabla_x v)(t_{\max}) = \frac{1}{2} \beta_0. \quad (46)$$

For microscopic systems with $m \approx 10^{-27} \text{ g}$ and $\langle \tilde{x}^2 \rangle_0 \approx 10^{-16} \text{ cm}^2$, this leads to $\beta_0 \approx (1/2)10^{16} \text{ s}^{-1}$ and thus $t_{\max} \approx 2 \times 10^{-16} \text{ s}$, so the $\nabla_x v$ -term reaches a large maximum value of $(\nabla_x v) \approx (1/4)10^{16} \text{ s}^{-1}$. Note that t_{\max} is exactly in the time-range of several hundred attoseconds where the effects investigated by Dreismann et al [7, 8, 10] also become experimentally observable.

For macroscopic systems with $m \approx 1 \text{ g}$ and $\langle \tilde{x}^2 \rangle_0 \approx 1 \text{ cm}^2$, it follows that $t_{\max} \approx 2 \cdot 10^{27} \text{ s}$, *i.e.* about 10^{20} years and $(\nabla_x v) \approx (1/4)10^{-27} \text{ s}^{-1}$. So, even for a comparably small system on a macroscopic scale, the maximum "compressibility" of the density becomes negligibly small and reaches its maximum only after a time much longer than the age of the universe.

For times much smaller than t_{\max} , $(\nabla_x v)$ grows proportional to $\beta_0^2 t$; for times much larger than t_{\max} , *i.e.* $(\beta_0 t) \gg 1$, $(\nabla_x v)$ vanishes in proportion to $1/t$, which can also be expressed as

$$(\nabla_x v)_t = \frac{1}{t} = \frac{\hbar \beta_0}{\langle [\tilde{x}, \tilde{p}]_+ \rangle}. \quad (47)$$

Note that this quantity does not depend on any of the system's parameters, such as mass m or initial "dimensions" $\langle \tilde{x}^2 \rangle_0$, whereas, $t_{\max} = 1/\beta_0$ does.

So, for the isolated system, density ϱ - which corresponds to an initial WP that can be constructed by a coherent superposition of infinitely many plane waves - reaches, for microscopic systems, a large maximum of "compressibility" after a very short time of a few hundred attoseconds, after which the value of $(\nabla_x v)$ decreases in good approximation proportional to $1/t$; so, after one second for example, it has already dropped to the comparably small value of $\nabla_x v = 1 \text{ s}^{-1}$. For macroscopic systems, β_0 is so small that even the maximum effect is negligible and would not be reached within any observable time-scale. For the system *interacting with its environment*, the maximum of $(\nabla_x v)_{dis}$, as given in Eq. (40), and the corresponding time t_{max} must be determined. In this case, the maximum effect is reached at

$$t_{max} = \frac{1}{\gamma} \ln \left[1 + \sqrt{\frac{\gamma^2}{\gamma^2 + \beta_0^2}} \right]. \quad (48)$$

Here, t_{max} does not only depend on the parameter β_0 that characterizes the system, but also on the parameter γ , characterizing the interaction with the environment.

As we have seen in the isolated case, $\beta_0 \approx 10^{16} \text{ s}^{-1}$ holds, for microscopic systems; for macroscopic systems, however, $\beta_0 \approx 10^{-27} \text{ s}^{-1}$. If we assume that the friction coefficient γ corresponds to a collision frequency with surrounding molecules, in the gas phase, $\gamma \approx 10^{10} \text{ s}^{-1}$, and in condensed matter or liquids, $\gamma \approx 10^{13} \text{ s}^{-1}$ should be valid.

The situation $\gamma \ll \beta_0$ seems, therefore, realistic for microscopic collision processes. In this case, the square root in Eq. (48) can be well approximated by γ/β_0 . The logarithm in (48) is then of the form $\ln(1+X)$ with $X = \gamma/\beta_0$ and can be expanded according to [36]

$$\begin{aligned} \ln(1+X) &= X - \frac{1}{2}X^2 + \frac{1}{3}X^3 - \dots \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{X^k}{k}. \end{aligned} \quad (49)$$

Since X is of the order of 10^{-3} or smaller, X^2 and higher powers of X can be neglected and we obtain

$$t_{max} \approx \frac{1}{\gamma} \frac{\gamma}{\beta_0} = \beta_0^{-1}, \quad (50)$$

exactly as in the isolated case. Therefore, also including the environment, the maximum effect is reached in just the same attosecond region. Using the same argument as in the truncation of the expansion (49), $e^{-\gamma t_{max}} = e^{-(\gamma/\beta_0)}$ can be replaced by $1 - (\gamma/\beta_0)$, so that $(\nabla_x v)_{dis}$ of Eq. (40) reduces for $t = t_{max}$ to

$$(\nabla_x v)_{dis}(t_{max}) = \frac{1}{2} \beta_0 e^{-\gamma t_{max}} = \frac{1}{2} \beta_0 e^{-\frac{\gamma}{\beta_0}}. \quad (51)$$

For $\gamma/\beta_0 \approx 10^{-3}$, $e^{-(\gamma/\beta_0)}$ can be very well-approximated by 1, from whence it follows that also the maximum value of $(\nabla_x v)_{dis}(t_{max}) \approx (1/2)\beta_0$ is practically the same as in the isolated case.

The major difference, in comparing the two cases, is now, however, that, for the isolated system, the maximum value decays only in proportion to $1/t$, whereas, for the interacting system, it decays exponentially according to $e^{-\gamma t}$, i.e. faster and with stronger interaction.

Consideration of the case $\gamma \geq \beta_0$ requires, for the microscopic situation, a large value of γ , i.e., $\gamma \geq 10^{16} \text{ s}^{-1}$. In this case, the square root in (48) can be approximated by 1 so that the maximum effect is reached for

$$t_{max} = \frac{1}{\gamma} \ln 2 \approx \frac{1}{\gamma}. \quad (52)$$

Due to the large value of γ , t_{max} is also, at most, in the range of a few hundred attoseconds or even smaller. With the exact value of t_{max} , it follows that $e^{-\gamma t_{max}} = (1/2)$ and the expression (40) for $(\nabla_x v)$ reduces to

$$(\nabla_x v)(t_{max}) = \frac{1}{4} \frac{\beta_0^2}{\gamma} = \frac{1}{2} \beta_0 \left(\frac{1}{2} \frac{\beta_0}{\gamma} \right), \quad (53)$$

i.e. the value of the case $\gamma \ll \beta_0$ multiplied by a factor smaller than 1.

For $t > t_{max}$, $(\nabla_x v)$ decays exponentially according to $(\nabla_x v) \approx (\beta_0^2/\gamma) e^{-\gamma t}$.

Turning to macroscopic systems with $\beta_0 \approx 10^{-27} \text{ s}^{-1}$, the case $\gamma > \beta_0$ is easily fulfilled. The time when the maximum effect is obtained is given by (52), i.e., it is independent of the system parameter, depending only on the environmental interaction. For instance, for $\gamma = 10^{13} \text{ s}^{-1}$, t_{max} would be reached already after $t_{max} = 0.7 \times 10^{-13} \text{ s}$, i.e., by a factor 10^{40} faster than in the isolated situation with $t_{max} \approx 10^{27} \text{ s}$. However, for this value of γ , the maximum value of $(\nabla_x v)$ for the macroscopic system would be only $(\nabla_x v)(t_{max}) \approx 10^{-67} \text{ s}^{-1}$, i.e., totally negligible. On the other hand, if one wants to obtain a relevant value for $(\nabla_x v)_{max}$, γ must be smaller than 10^{-54} s^{-1} , which excludes any realistic interaction with the environment.

Considering now our second example, the diffraction in time, we can, again, look first at what we obtain in the classical limit. As can be seen from Eqs. (34) or (35) and (32), if we take $\hbar \rightarrow 0$, then $g \rightarrow +\infty$, as the step function takes on the value 1 only if $(pt/m) - x > 0$. Using one of the definitions of the δ -function [37],

$$\delta(\hbar k - p) = \lim_{g \rightarrow \infty} \frac{\sin[g(\hbar k - p)]}{\pi[\hbar k - p]}, \quad (54)$$

the classical limit of the Wigner distribution function (33) can be written as

$$W_{cl}(x, p; k, t) = \delta(\hbar k - p) \Theta\left(\frac{\hbar k}{m} t - x\right), \quad (55)$$

where we used the presence of $\delta(\hbar k - p)$ in Eq. (55) to replace p by $\hbar k$ in the step function. We find that the classical limit yields what we expect, since the only value possible for the momentum of the particle is $p = \hbar k$, and this value is only assumed when $x < (\hbar k t/m)$; as for $x > (\hbar k t/m)$,

the particle would not yet have arrived at point x . Furthermore, the typical wave aspect expressed by the oscillating sine-function in (33), corresponding to the diffraction pattern, is totally eliminated. Thus, the classical limit of the Wigner distribution function for the diffraction-in-time problem confirms our intuition.

Taking into account again that in nature $\hbar \neq 0$, the same limit $g \rightarrow \infty$ can also be reached, for $2pt/\hbar m \rightarrow \infty$, i.e. $t \rightarrow \infty$. To estimate the time that is needed for the oscillatory pattern to vanish, we need $gp \gg 1$, i.e.

$$gp = \frac{2}{\hbar} \frac{p^2 t}{m} = \frac{4}{\hbar} E \cdot t \gg 1 \quad (56)$$

or

$$t = \frac{\langle [\tilde{x}, \tilde{p}]_+ \rangle}{\hbar \beta_0} \gg \frac{\hbar}{4E}, \quad (57)$$

where E is the energy of the system.

If we assume that the energy of our WP hitting the shutter before it opens is comparable with the energies transferred in the scattering experiments by Dreismann [10], which are in the range $E \approx 2 - 30$ eV, or by Ikeda *et al.* [38–40], which are around 0.5 eV, we obtain for t in (57) values between 5 – 300 attoseconds which, again, are in the time-domain where, in the above-mentioned experiments, typical quantum entanglement effects were observed. Also recent results by Karlsson *et al.* [41–43] report characteristic quantum effects in scattering experiments for times smaller than 500 attoseconds.

The influence of an interacting environment in this case does not damp the duration in time of the quantum effect (as

in our first example) but the extension in space (and, also different from our first example) does not leave the magnitude of the effect untouched but reduces the amplitude of the oscillating diffraction pattern, as discussed in [35].

In conclusion, we can state that only for microscopic systems does the non-vanishing $\nabla_x v$ -term show a significant effect that attains its maximum on the time-scale of about a hundred attoseconds - which is the same time-scale where, also, the diffraction-of-time phenomenon still shows its typical quantum mechanical pattern. This is exactly the same time-scale where recent scattering experiments observed typical quantum mechanical properties for relatively massive objects like protons or neutrons, even in ambient conditions. Inclusion of the interaction with a dissipative environment does not change the time or the magnitude of the maximum of the $\nabla_x v$ -effect, only its temporal decay behaviour. For the diffraction-in-time problem, however, the environment restricts the spacial extension as well as the magnitude of the quantum effect.

Experiments attempting to observe and measure the effects discussed should have the greatest chance of success if they focus on microscopic systems in the attosecond time-regime since, particularly for non-isolated systems, the effects might decrease rapidly for longer times and are negligible for macroscopic systems in general.

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