

The Casimir operator of SO(1,2) and the Pöschl-Teller potential: an AdS approach

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We present and discuss some features of the anti-de Sitter spacetime, that is jointly with de Sitter and Minkowski is only, the unique maximal isotropic manifold. Among all possible lorentzian manifolds, we restrict our attention to the anti-de Sitter (AdS) spacetime, with metric $\text{diag}(1, -1, -1)$. We start by presenting the conformal time metric on AdS and we then show how we can obtain the Schrödinger formalism [1]. The Lie algebra $\mathfrak{so}(1,2)$ is introduced and used to construct spin and ladder operators. After presenting the unitary representations, the AdS(1,2) spacetime is suitably parametrized and a representation of SO(1,2) is obtained, from which the Schrödinger equation with Pöschl-Teller potential is immediately deduced. Finally, we discuss some relations between the relativistic harmonic oscillator and the Klein-Gordon equation, using the AdS(1,2) static frame. Possible applications of the presented formalism are provided.

Keywords: Schrödinger equation; Pöschl-Teller potential; Casimir; spin and ladder operators; Cartan form; unitary representations; anti-de Sitter spacetime; hyperbolic coordinates; quantum mechanics.

Presentamos el espacio-tiempo de anti-de Sitter, el cual junto con los espacio-tiempos de Minkowsky y de Sitter, es la única variedad isotrópica maximal. Dentro de todas las variedades lorentzianas, restringimos nuestra atención al espacio-tiempo AdS con una métrica diagonal $(1, -1, -1)$. Después de presentar la métrica tiempo-conforme en AdS, usamos otro enfoque para mostrar como es posible obtener el formalismo de Schrödinger. Introducimos también el álgebra de Lie $\mathfrak{so}(1,2)$ y construimos los operadores de *spin* y de escalera (*ladder*) a partir de los generadores de esta álgebra. Después de mostrar la representación unitaria, parametrizamos adecuadamente el espacio-tiempo AdS(1,2) y deducimos la construcción de una representación de SO(1,2), de la cual obtenemos la ecuación de Schrödinger asociada al potencial de Pöschl-Teller. Finalmente discutimos algunas relaciones entre un oscilador armónico relativista y la ecuación de Klein-Gordon, usando el referencial estático AdS(1,2). Son presentadas posibles aplicaciones de este formalismo.

Descriptores: Ecuación de Schrödinger; potencial de Pöschl-Teller; operadores de Casimir, de *spin* y de escalera; representaciones unitarias; espacio-tiempo de anti-de Sitter; mecánica cuántica.

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1. Introduction

Extra dimensions were introduced in theoretical physics as an attempt to unify the four fundamental forces. Kaluza [2] and Klein [3] tried to join electromagnetism and gravitation in a theory formulated in a 5-dimensional space. It is well known that 4-dimensional physics is retrieved if the fifth dimension is compactified on a manifold of small size, much smaller than the classical radius of an electron.

AdS and de Sitter (dS) spacetimes are widely used in conformal field theories and applications in superstrings and supergravity theories [4–6]. It is known that, out of the Friedmann models that describe our universe, the Minkowski, dS and AdS spacetimes are the only maximal isotropic ones, so they allow a maximal number of conservation laws and also a maximal number of Killing vectors. dS and AdS spacetimes [7–10], are respectively solutions of Einstein equations with cosmological constant $\Lambda = \pm 3/R^2$, where R is the radius of the AdS universe ($R > 0$), and curvature given by the Ricci tensor $R_{\mu\nu} = \Lambda g_{\mu\nu}$. These manifolds are suitable as

geometric arenas to describe conformal field theories [11]. The AdS n -spacetime symmetry group is SO(1, n) and its metric is given by $ds^2 = -d\xi_0^2 + d\xi_i d\xi^i$ ($i = 1, \dots, n$), where $\{\xi_0, \xi_i\}$ are coordinates in AdS spacetime. The metric can still be written as

$$ds^2 = -dt^2 + R^2 \cosh^2(t/R) d\Gamma^2 \quad (1)$$

where $d\Gamma^2$ is the metric on the hypersphere S^{n-1} . The topology of AdS spacetime is given by $\mathbb{R} \times S^{n-1}$ [6].

In order to link the theory of hyperbolic spacetimes with the quantum-mechanical formalism, we consider the (1,2) anti-de Sitter universe, AdS(1,2) viewed as (1+2)-lorentzian manifold. The importance of (1+2)-spacetime theories has increased since Witten [4] proved that (1+2)-dimensional gravity is shown to be exactly soluble at the classical and quantum levels and has a straightforward renormalized perturbation expansion. We use AdS instead of dS spacetime, since dS Chern-Simmons theory [12] does not have a supersymmetric extension [13]. We also intend to introduce and discuss some important results concerning the group SO(1,2),

which has infinite-dimensional representations, and to link it to quantum mechanics. If we assume that the Casimir operator has eigenvectors in the $\mathcal{D}^{j,m}$ representation space, we can deduce the Schrödinger equation with Pöschl-Teller potential [14].

This paper is organized as follows: in Sec. 2 we present and discuss the AdS conformal time metric; and in Sec. 3, the Schrödinger description of AdS(1,2) using the static frame is presented. In Sec.4 three the Lie algebra $\mathfrak{so}(1,2)$ is defined and we introduce unitary representations with suitable quantum numbers as eigenvalues of the Casimir and spin operators. In Sec. 5 the spin and ladder operators are obtained in hyperbolic coordinates, which best fits the geometry and the topology of AdS(1,2) spacetime, and the Schrödinger equation is derived from the eigenvalue equation of the Casimir operator. It is shown that the eigenenergies associated with this equation are all positive, and the Schrödinger equation with Pöschl-Teller potential is obtained. And finally we comment possible extensions of the theory presented.

2. The conformal time metric in AdS(1, 2)

AdS(1, 2) spacetime is formally defined by the equation

$$-\xi_0^2 + \xi_1^2 + \xi_2^2 = R^2 \quad (2)$$

where ξ_0, ξ_1, ξ_2 are arbitrary coordinates in AdS(1, 2). We can parametrize these coordinates as trigonometric and hyperbolic functions of space (x) and time (t) as follows:

$$\xi_0 = R \sinh(t/R) \quad (3)$$

$$\xi_1 = R \cosh(t/R) \sin x \quad (4)$$

$$\xi_2 = R \cosh(t/R) \cos x \quad (5)$$

where R is the radius of the AdS(1,2) universe. It follows that the squared line element ds^2 can be written as

$$ds^2 = R^2 \cosh^2(t/R) dx^2 - dt^2, \quad (6)$$

which is a particular case of eq.(1).

It is well known that an observer in AdS spacetime can only realize a portion of space. This is because of the exponential inflation that occurs in the future. Therefore space expands so fast that light rays do not propagate all the way around it. To make the causal structure of AdS spacetime clear, one can introduce a new time coordinate, the so-called conformal time (τ), as follows:

$$\tau = 2 \arctan(e^t) \quad (7)$$

where $-\infty < t < \infty$ and $0 < \tau < \pi$. As conformal time has been introduced, Eq. (6) can be written as

$$\begin{aligned} ds^2 &= R^2 \cosh^2(t/R) (dx^2 - d\tau^2) \\ &= \frac{R^2}{\sin^2 \tau} (dx^2 - d\tau^2) \end{aligned} \quad (8)$$

since $\sin \tau = \cos^{-1}(t/R)$. We can prove that in the AdS(1, n) spacetime, Eq. (8) can be generalized as

$$ds^2 = \frac{R^2}{\sin^2 \tau} (d\Gamma^2 - d\tau^2), \quad (9)$$

where $d\Gamma^2$ is the metric on the hypersphere S^{n-1} .

3. The reduced anti-de Sitter spacetime

In this section we sketch an alternative approach through which Schrödinger has described [1] AdS(1,2) spacetime. Again the AdS manifold is provided with lorentzian metric implicitly given by $ds^2 = d\xi_1^2 + d\xi_2^2 - d\xi_0^2$, and it can be described by Eq. (2). Using pseudospherical coordinates, the coordinates ξ_0, ξ_1, ξ_2 are parameterized as follows:

$$\xi_1 = R \cos \chi \cosh(t/R)$$

$$\xi_2 = R \sin \chi \cosh(t/R)$$

$$\xi_0 = R \sinh(t/R), \quad -\infty < t < \infty, \quad 0 \leq \chi < 2\pi. \quad (10)$$

The above map is nowhere singular. The line element is given by

$$ds^2 = -R^2 \cosh^2 d\chi^2 + R^2 dt^2. \quad (11)$$

We observe that the new time t varies less rapidly than ξ_0 . In addition, instead of choosing χ as above, if the relation $\sin \chi = \xi_1/R$ is introduced, another map can be defined as follows:

$$\xi_1 = R \sin \chi$$

$$\xi_2 = R \cos \chi \cosh(t/R)$$

$$\xi_0 = R \cos \chi \sinh(t/R).$$

Another set of pseudopolar angles (χ, t) is defined on the 2-hyperboloid. The line element relative to this parameterization is $ds^2 = -R^2 d\chi^2 + R^2 \cos^2 \chi dt^2$. This is the so-called *static frame* of the de Sitter metric [1]. In a more familiar way, we introduce the coordinates (ρ, η) parameterizing AdS(1,2) as

$$\rho = R \sin \chi, \quad \eta = Rt, \quad (12)$$

which gives

$$ds^2 = -(1 - \rho^2/R^2)^{-1} d\rho^2 + (1 - \rho^2/R^2) d\eta^2. \quad (13)$$

It is easily seen that when R goes to infinity, $ds^2 \rightarrow -d\rho^2 + d\eta^2$. The metric implicitly given by this relation is lorentzian, which shows the well-known result that AdS spacetime goes to Minkowski one in the limit $R \rightarrow \infty$.

4. The Lie group SO(1,2)

In this section we reproduce a construction [14] of the Lie group SO(1,2). This group is generated by three (angular momentum) operators L_{12}, L_{23}, L_{31} and their respective commutation relations:

$$\begin{aligned} [L_{31}, L_{23}] &= iL_{12}, \\ [L_{12}, L_{13}] &= -iL_{23}, \\ [L_{12}, L_{23}] &= iL_{13}. \end{aligned} \quad (14)$$

If the coordinate representation is used, defining $J_x = -L_{23}$, $J_y = L_{13}$ and $J_z = -L_{12}$, we have

$$\begin{aligned} J_x &= -i(y\partial_z + z\partial_y) \\ J_y &= i(x\partial_z + z\partial_x) \\ J_z &= -i(x\partial_y + y\partial_x). \end{aligned} \quad (15)$$

It follows immediately from the commutation rules, Eq. (14), that

$$[J_x, J_y] = -iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y. \quad (16)$$

It is recalling that SO(1,2) is a non-compact Lie group.

4.1. The Casimir operator associated to SO(1,2)

The Casimir operator is given by $C = \theta^{ij} J_i J_j$, where θ^{ij} is the inverse of the Cartan form [15] and by implicitly defining the structure constants c_{pq}^r by $[J_p, J_q] = c_{pq}^r J_r$, we obtain $\theta_{ij} = c_{ik}^l c_{jl}^k$. We find $C_{123} = -i$, $C_{132} = -i$ and $C_{231} = i$, where all the others C_{ijk} are null. Using these results, we have

$$C = -J_x^2 - J_y^2 + J_z^2 = J_z(J_z + 1) - J_- J_+ \quad (17)$$

where the ladder operators [14]

$$J_{\pm} = J_x \pm iJ_y, \quad (18)$$

were introduced.

4.2. Unitary representations

As in [14], we choose the states associated to the unitary representations for C and J_z as their eigenstates:

$$C|ab\rangle = a|ab\rangle, \quad J_z|ab\rangle = b|ab\rangle, \quad a, b \in \mathbb{R}. \quad (19)$$

It is clear from Eq. (18) that $J_- = J_+^\dagger$ (here the operator \dagger denotes hermitian conjugation) and it implies that $J_+ J_-$ is a positive definite operator. Considering real eigenstates $|c\rangle$ of the operator $J_+ J_-$, $(J_+ J_-)|c\rangle = c|c\rangle$, we have corresponding positive eigenvalues, since

$$\langle c|J_+ J_+^\dagger|c\rangle = c\langle c|c\rangle \Rightarrow c > 0. \quad (20)$$

We first find that

$$\begin{aligned} J_+ J_- |ab\rangle &= [J_z(J_z + 1) - C]|ab\rangle \\ &= [-a + b(b - 1)]|ab\rangle \end{aligned} \quad (21)$$

and it follows that

$$[-a + b(b - 1)] \geq 0. \quad (22)$$

Analogously, using the operator $J_- J_+$, we obtain the relation

$$[-a + b(b + 1)] \leq 0. \quad (23)$$

For all real values for a we have no upper bound for $|b|$. When $a < -1/4$, considering $a = j(j + 1)$, this clearly forces $(j + 1/2)^2 < 0$. We choose $j = -1/2 + ik$, where, as shall be proved in the next section, k is the positive square root of the energy associated with a Schrödinger equation.

5. Representation of SO(1,2) in AdS(1,2) spacetime and the Schrödinger equation with Pöschl-Teller potential

In this section, a representation of SO(1,2) in AdS(1,2) spacetime is obtained. We can use the parametrization given by Eq. (10) (defining $\alpha = t/R$, where t is the time coordinate and R is the radius of AdS(1,2))

$$\begin{aligned} x &= R \cosh \alpha \cos \phi, \\ y &= R \cosh \alpha \sin \phi, \\ z &= R \sinh \alpha, \end{aligned} \quad (24)$$

$0 < \phi \leq 2\pi$, $0 \leq \alpha < \infty$ and $R \geq 0$. By the above parameterization, Eq. (15) and Eq. (18) are used to prove the following expressions:

$$\begin{aligned} J_{\pm} &= (-i \tanh \alpha \partial_\phi \mp \partial_\alpha) \exp(\pm i\phi) \\ J_z &= -i\partial_\phi. \end{aligned} \quad (25)$$

Under similarity maps, the coordinate representation of the eigenvalue equation

$$C|jm\rangle = j(j + 1)|jm\rangle. \quad (26)$$

can be found. A particular similarity map $U = \cosh^{1/2} \alpha$ is chosen [14]. Under this map we obtain

$$\partial_\alpha \mapsto -\frac{1}{2} \tanh \alpha + \partial_\alpha, \quad (27)$$

$$J_{\pm} \mapsto \exp(\pm i\phi) [\mp \partial_\alpha + \tanh \alpha (\pm 1/2 - i\partial_\phi)]. \quad (28)$$

Then

$$\begin{aligned} C &= -J_+ J_- + J_z(J_z - 1) \\ &= \partial_\alpha^2 - \cosh^{-2} \alpha (\partial_\phi^2 + 1/4) - 1/4. \end{aligned} \quad (29)$$

If the supposition $\langle \alpha\phi | jm \rangle = A_{jm}(\alpha) \exp(im\phi)$ is made, from Eq. (26) the Schrödinger equation with the Pöschl-Teller potential (SEPTP) is written [16]:

$$-\left[\partial_\alpha^2 + \cosh^{-2} \alpha (m^2 - 1/4)\right] A_{jm}(\alpha) = -(j + 1/2)^2 A_{jm}(\alpha) \quad (30)$$

The right side term of the above equation is the eigenenergy $E_j = -(j + 1/2)^2 = k^2 > 0$ (This potential is generally given by $V(\alpha) = -V_0(\cosh \alpha)^{-2}$).

The solutions of SEPTP are closely related to the AdS static frame. The static frame, described in section two, is fundamental in order to establish a connection between the classical equations of motion of the harmonic oscillator and the Eq. (30), since the solution of the Klein-Gordon equation of harmonic oscillator coincides with the solution of Eq. (30) (with potential multiplied by $2m$) [17]. This equivalence is stated in Ref. 17 where it can be shown that there always exist an AdS static frame which reproduces the classical equations of the non-relativistic harmonic oscillator.

6. Concluding Remarks

After Witten proved [4] that general relativity is a renormalizable quantum system in (1+2) dimensions, it is possible to point out a few interesting motivations to investigate the AdS spacetime. Many attempts have been made to generalize the

gauge theory of gravity in (1+2) dimensions to higher ones. The first attempt was to enlarge the Poincaré group of symmetries, supposing an AdS group symmetry [19], which contains the Poincaré group. Also, the AdS/CFT correspondence asserts that a maximal supersymmetric Yang-Mills theory in 4-dimensional Minkowski-spacetime is equivalent to a type IIB closed superstring theory [20]. The 10-dimensional arena for type IIB superstring theory is described by the product manifold $S^5 \times AdS$, an impressive consequence that justifies the investigations into AdS spacetime.

We have interpreted the SO(1,2) group in the light of AdS(1,2), obtaining the spin (J_z) and ladder (J_\pm) operators using the AdS(1,2) spacetime parametrization. It induces a link between the Schrödinger equation (with Pöschl-Teller potential) and the Dirac theory of spin-1/2 fermions. An analytical expression for the Dirac spinor in AdS(1,2) spacetime is obtained in [21]. Some recent applications relating AdS spacetimes and Pöschl-Teller relativistic systems are formulated using an analogous formalism [17], and possible supersymmetric extensions [17] will be discussed in a forthcoming paper.

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