Electroweak radiative corrections to semileptonic $\tau$ decays

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I present an update on the electroweak radiative correction factor to semileptonic $\tau$ decays, including a next-to-leading order resummation of large logarithms. My result differs both qualitatively and quantitatively from the one recently obtained by Davier et al. As two consequences, (i) the discrepancy between the predictions for the muon $g - 2$ based on $\tau$ decay data and $e^+e^-$ annihilation data increases, and (ii) the $g - 2$ prediction based on $\tau$ decay data appears to be consistent (within about one standard deviation) with the experimental result from BNL.

Keywords: Electroweak radiative corrections; tau decay; perturbative QCD.

Presentamos una actualización del factor de corrección radiativa electrodébil al decaimiento semileptonico del $\tau$, incluyendo una resumación de segundo orden de logaritmos grandes. Nuestro resultado difiere cualitativa y cuantitativamente del recientemente obtenido por Davier et al. Tenemos dos resultados, (i) la discrepancia entre la predicción para el $g-2$ del muon basado en los datos del decaimiento del $\tau$ y los datos de la aniquilación $e^+e^-$, se incrementa y (ii) la predicción de $g-2$ basada en los datos del decaimiento del $\tau$ parece ser consistente (dentro de una desviación estandar) con el resultado experimental de BNL.

Descriptores: Correcciones radiativas electrodébiles; decaimiento del $\tau$; QCD perturbativa.


The largest theoretical uncertainty in the Standard Model prediction of the anomalous magnetic moment of the muon is due to the hadronic two-loop vacuum polarization contribution, $\Delta a_\mu^{had,2}$. This contribution is two orders of magnitude larger than the ultimate experimental error anticipated by the Muon $g - 2$ Collaboration at BNL [2], so it needs to be controlled at the 1% level or better. While non-perturbative QCD effects prevent a first principles calculation, $\Delta a_\mu^{had,2}$ can be rigorously obtained experimentally from a dispersion relation which relates it to an integral over $e^+e^-$ annihilation cross sections. Using the conserved vector current (CVC) hypothesis one can obtain additional information by studying the invariant mass distribution of $\tau$ decay hadronic final states. This necessitates a careful assessment of CVC breaking effects, which was done in a recent article by Davier et al. [1]. In this note, I present an update of the short distance electroweak radiative corrections to $\tau$ decays, representing a particular CVC breaking effect. This update is motivated by two mistakes in one of the formulas of Ref. [1]. Numerically, the corresponding shifts are modest, but not negligible, and have the same sign.

The leading electroweak radiative corrections to $\tau$ decays are enhanced by a large logarithm [3, 4],

$$S_{EW} = 1 + \frac{3\alpha}{4\pi}(1 + 2Q) \ln \frac{M_\tau^2}{m_\tau^2} = 1.01878 \quad (1)$$

where $M_Z = 91.1876(2)$ GeV [5] is the $Z$ boson mass, and $\alpha = \alpha(m_\tau) = 1/133.50(2)$ [6] is the QED coupling at the $\tau$ lepton mass, $m_\tau = 1776.99(3)$ MeV [7], evaluated in the MS renormalization scheme. $\bar{Q}$ is the hypercharge of the weak doublet produced in the final state. Therefore, $\bar{Q} = 1/6$ for semileptonic decays, $\tau^- \rightarrow \nu_\tau \bar{\nu}(s)$. Since $\bar{Q} = -1/2$ for leptons, there are no large logarithms for leptonic $\tau$ decays.

The remaining (not logarithmically enhanced) corrections at $\mathcal{O}(\alpha)$ have been obtained in Ref. [8] (final state fermion masses are neglected throughout). In the notation of Eq. (17) of Ref. [1] they are,

$$S_{EW}^{\text{sub}, \text{had}} = 1 + \frac{\alpha(m_\tau)}{\pi} \left(\frac{85}{24} - \frac{\pi^2}{2}\right), \quad (2)$$

$$S_{EW}^{\text{sub}, \text{lep}} = 1 + \frac{\alpha(m_\tau)}{\pi} \left(\frac{25}{8} - \frac{\pi^2}{2}\right), \quad (3)$$

for semileptonic and lepton decays, respectively. In Ref. [1], however, $S_{EW}^{\text{sub}, \text{had}}$ was erroneously identified with the ratio,

$$S_{EW}^{\text{sub}, \text{lep}} = 1 + \frac{5}{12} \frac{\alpha(m_\tau)}{\pi} = 1.00999. \quad (4)$$

This amounts to a double counting of the correction $S_{EW}^{\text{sub}, \text{lep}} - 1 = -0.00432$: the hadronic spectral functions are normalized relative to the leptonic branching ratio (see Eq. (10) of Ref. [1]) so that the ratio (4) must be included, but it is incorrect to perform an additional division by $S_{EW}^{\text{sub}, \text{lep}}$.

Since numerically,

$$\frac{\alpha_s}{\pi} \ln \frac{M_Z^2}{m_\tau^2} \sim \mathcal{O}(1), \quad (5)$$

short distance QCD effects are of similar size as the $\mathcal{O}(\alpha)$ corrections discussed in the previous paragraph. They have been computed in Ref. [9] and modify Eq. (1),

$$S_{EW} = 1 + \frac{3\alpha}{4\pi} \ln \frac{M_\tau^2}{m_\tau^2} \left[1 + 2\bar{Q} - 2\bar{Q} \frac{\alpha_s}{\pi}\right]. \quad (6)$$
The short distance QCD correction corresponding to the term proportional to the strong coupling constant, \( \alpha_s \), has been approximated \([9]\) in order to obtain an analytic result. I have checked that this approximation reproduces the exact \( \mathcal{O}(\alpha_s \ln M_Z^2) \) result within about 1%. Since two scales enter Eq. (6), it is clear that a next-to-leading order renormalization group analysis is in order.

Resummation of the leading order logarithms in Eq. (1) is done using the renormalization group equation (RGE) \([10]\),

\[
\left[ \mu^2 \frac{\partial}{\partial \mu^2} + \beta_0^{(1)} \frac{\alpha}{\pi} \right] S(\mu_0, \mu) = 0, \tag{7}
\]

where \( \beta_0^{(1)} \) is the lowest order QED \( \beta \)-function coefficient. This RGE is subject to the initial condition, \( S(\mu_0, \mu_0) = 1 \), and its solution is given by,

\[
S(\mu_0, \mu) = \left[ 1 - \frac{\alpha(\mu_0)}{\pi} \beta_0^{(1)} \ln \frac{\mu^2}{\mu_0^2} \right] \tag{8}
\]

Applied to the case at hand, this is often written as,

\[
S(m_\tau, M_Z) = \left[ \frac{\alpha(m_b)}{\alpha(m_\tau)} \right] \left[ \frac{\alpha(M_Z)}{\alpha(m_b)} \right] \left[ \frac{\alpha(M_Z)}{\alpha(M_W)} \right] = 1.01937, \tag{9}
\]

where the solution to the one-loop RGE of QED,

\[
\mu^2 \frac{d}{d\mu^2} \alpha(\mu) = \beta_0^{(1)} \frac{\alpha^2(\mu)}{\pi}, \tag{10}
\]

has been employed. It should be stressed, that consistency with the RGE demands one-loop evolution of \( \alpha(\mu) \) within each of the factors in Eq. (9). On the other hand, the values used across the various factors, may be related to each other either by one-loop evolution or including higher order running effects, since the difference is of higher order in the RGE (7). The increase of \( S(m_\tau, M_Z) \) in Eq. (9) relative to Eq. (1) due to the summation of \( \mathcal{O}(\alpha^3 \ln^2 M_Z^2) \) effects is about 3% of the non-resummed correction.

I will now extend the RGE analysis of the previous paragraph to properly sum up all logarithms[We neglect non-logarithmic and therefore non-enhanced terms of \( \mathcal{O}(\alpha_s \alpha) \). This is in accordance with common practice where solutions of an \( n \)-loop RGE are supplemented by \((n-1)\)-loop threshold (matching) terms of non-logarithmic nature.] of \( \mathcal{O}(\alpha_s \ln M_Z^2) \). Eq. (7) is to be replaced by,

\[
\left[ \mu^2 \frac{d}{d\mu^2} + \beta_0^{(1)} \frac{\alpha}{\pi} \left( 1 - \frac{\alpha_s}{4\pi} \right) \right] S(\mu_0, \mu) = \left[ \mu^2 \frac{\partial}{\partial \mu^2} + \beta_0^{(1)} \frac{\alpha}{\pi} \right] S = 0, \tag{11}
\]

where \( \beta_0^{(3)} \) is the lowest order QCD \( \beta \)-function coefficient.

With the definitions,

\[
\eta_s = \frac{\alpha_s(m_\tau)}{4\pi} \left[ 1 + \frac{75}{76} \frac{\alpha_s(m_\tau)}{\alpha_s(m_\tau)} \right]^{-1}, \tag{12}
\]

Eq. (11) is solved by,

\[
S(m_\tau, M_Z) = \left[ \frac{\alpha(m_\tau)}{\alpha(m_b)} \right] \left[ \frac{\alpha(M_W)}{\alpha(m_b)} \right] \left[ \frac{\alpha(M_Z)}{\alpha(M_W)} \right] = 1.01907 \pm 0.00001, \tag{13}
\]

where I used the solutions to the one-loop RGE of QCD,

\[
\mu^2 \frac{d}{d\mu^2} \alpha(\mu) = -\beta_0^{(3)} \frac{\alpha^2(\mu)}{\pi}, \tag{14}
\]

and of QED(QCD corrections to Eq. (10) are suppressed by an additional factor \( \alpha_s/\pi \). Their inclusion gives rise to the summation of \( \mathcal{O}(\alpha_s \alpha \ln^2 M_Z^2) \) effects, but the integration cannot be performed analytically. Numerically this summation affects the result at the 10^{-5} level which can safely be neglected.]. The shift, \(-0.00030\), between Eqs. (9) and (13) is somewhat larger than the shift, \(-0.00022\), obtained in Ref. [4], which is in part due to the summation, but mainly due to the inputs. The uncertainty in Eq. (13) is from the current uncertainty in \( \alpha_s = 0.120 \pm 0.002 \), while other parametric uncertainties are minuscule[What enters Eqs. (9) and (13) is the \( \overline{\text{MS}} \) b-quark definition, which is free of renormalon ambiguities and therefore much better known (see Ref. [11] for a recent sub-percent determination) than the b-quark pole mass. Higher order matching corrections are also smaller if one uses the \( \overline{\text{MS}} \) mass definition.].

Neglecting two-loop \( \mathcal{O}(\alpha^3 \ln M_Z^2) \) effects, Eq. (13) simplifies,

\[
S(m_\tau, M_Z) = \left[ \frac{\alpha(m_b)}{\alpha(m_\tau)} \right] \left[ \frac{\alpha(M_W)}{\alpha(m_b)} \right] \left[ \frac{\alpha(M_Z)}{\alpha(M_W)} \right] = 0.01907 \pm 0.00001, \tag{15}
\]

which differs by only \( \approx 3 \times 10^{-6} \). Neglecting further the numerically similar three-loop \( \mathcal{O}(\alpha^3 \ln M_Z^2) \) effects, one can expand the second line in Eq. (15) to linear order in \( \alpha_s \). If one then writes the resulting expression in terms of QCD scale parameters, \( \Lambda_{\text{QCD}} \), one encounters the double-logarithmic form originally obtained 20 years ago [9].

To summarize, next-to-leading order effects reduce the leading order summation by about 50%, \( \text{i.e.} \), they are numerically of the same order. Both effects are in turn numerically.
of order $\alpha_s$, so they must be included for a complete $O(\alpha)$ evaluation. Unknown higher orders are suppressed by at least a factor of $\alpha_s/\pi$ relative to any of the effects mentioned before. Thus, the uncertainty due to higher order effects is of order $O(\alpha_s^2) \sim 0.0003$.

Following Ref. [1] one can write,

$$S_{\text{EW}} \equiv S(m_{\tau}, M_Z) \frac{\text{ren,had}}{\text{num,lep}} = 1.0201 \pm 0.0003, \quad (16)$$

but there $S_{\text{EW}} = 1.0267 \pm 0.0027$ is quoted instead. Almost 2/3 of the difference is due to the error pointed out after Eq. (4), and about 5% due to neglecting next-to-leading order contributions to $S(m_{\tau}, M_Z)$. Another 15% difference is likely due to applying Eq. (9) incorrectly (as discussed above). The remaining 15% can perhaps be traced to use of the on-shell definition of $\alpha_s$ in place of the $\overline{\text{MS}}$ definition as used in the present work. Note, that the derivation of Eq. (9) assumes a mass-independent renormalization scheme (such as the $\overline{\text{MS}}$ scheme), in which at each fermion threshold the $\beta$-function coefficients change by a finite amount: this is the origin of the product form of Eq. (9). Thus, the solution (9) cannot be applied to mass-dependent schemes, such as the on-shell renormalization scheme. It is emphasized again, that the numerical difference to Ref. [1] should not be viewed as a scheme-dependence and thus as an estimate of uncalculated higher order corrections (which are much smaller as discussed above). On the contrary, one should expect that a self-consistent treatment within the on-shell scheme will reproduce the result of the present work.

As far as the $\tau$-based analysis of Ref. [1] is concerned, about 77% of the data is affected by $S_{\text{EW}}$. Since $S_{\text{EW}}$ obtained in this paper differs by about 0.65% from the one in Ref. [1], one expects a 0.5% shift in the extracted $\Delta a_{\mu}^{\text{had}}(2)$. Including an update of the CKM matrix element $|V_{ud}|$ entering the analysis (the value, $|V_{ud}| = 0.9752 \pm 0.0007$, is replaced by, $|V_{ud}| = 0.97485 \pm 0.00046$, from the fit result of Ref. [7]), this amounts to about one half of the current experimental uncertainty of 0.8 parts per billion [2] for the muon magnetic moment. The $\tau$-based Standard Model prediction would then be consistent with the measurement [2] within about one standard deviation. The discrepancy to the $e^+e^-$ based analysis of Ref. [1] would correspondingly be larger. Furthermore, the smaller errors in Eq. (16) and in $|V_{ud}|$ compared to Ref. [1] should lead to a slight reduction of the overall uncertainty of the $\tau$-based result. As a final remark, the recent determination [11] of $\alpha_s$ from the $\tau$ lifetime, when updated with the present next-to-leading order analysis, increases $\alpha_s(M_Z)$ by less than 0.001.

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