Recurrence relations of special functions and group representations

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It is shown that the recurrence relations satisfied by several special functions can be related to representations of Lie algebras of dimension three or four. It is also shown that in some cases these recurrence relations can be related to the isometries of constant-curvature two-dimensional manifolds.

Keywords: Special functions; representations of Lie algebras.

Se muestra que las relaciones de recurrencia satisfechas por varias funciones especiales pueden relacionarse con representaciones de álgebras de Lie de dimensión tres o cuatro. Se muestra también que en algunos casos estas relaciones de recurrencia pueden relacionarse con las isometrías de variedades de dimensión dos con curvatura constante.

Descriptores: Funciones especiales; representaciones de álgebras de Lie.

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1. Introduction

As is well-known, the spherical harmonics of a given order form a representation space for the rotation group SO(3) and some recurrence relations satisfied by the spherical harmonics or the associated Legendre functions can be readily obtained making use of the commutation relations of the generators of the group; in fact, the explicit form of the spherical harmonics is frequently obtained using the generators of the group. In a similar manner, the Bessel functions of integral order are related to the Euclidean group of the plane and in both cases one can establish addition theorems making use of the unitarity of the representation. Other special functions are also related to Lie groups in various ways (see, *e.g.*, Refs. 1, 2 and the references cited therein) and in many cases the relationship arises naturally in some problems of quantum mechanics.

In this paper we show that if a family of special functions satisfies recurrence relations of a certain type, then the products of these functions with exponentials form bases for irreducible representations of Lie algebras of dimension three or four. In some cases the transformations generated by these Lie algebras are isometries of two-dimensional manifolds; in particular, we find that the recurrence relations for the Chebyshev polynomials are connected with the isometries of an hyperboloid in three-dimensional Minkowski space. It is also shown that by defining appropriately the inner product, one obtains a unitary representation for the corresponding Lie groups.

2. Recurrence relations and related Lie algebras

Several special functions of mathematical physics depend at least on one parameter, n, and obey recurrence relations of the form

$$\begin{aligned} \left(a(x)\frac{d}{dx}+b(x)+nr(x)\right)f_n(x) &= \lambda_n f_{n+1}(x), \\ \left(a(x)\frac{d}{dx}+c(x)-nr(x)\right)f_n(x) &= \mu_n f_{n-1}(x), \end{aligned} \tag{1}$$

where a, b, c, r are real-valued functions of x that do not contain n but may depend on other parameters and λ_n, μ_n do not depend on x. Some examples are given by the Bessel functions, the associated Legendre functions, the associated Laguerre polynomials, the Gegenbauer, Hermite and Chebyshev polynomials [1,3–6] (see Table I).

As we shall show, it is convenient to define the operators

$$\begin{aligned} T_+ &\equiv e^{iy}\left(a(x)\frac{\partial}{\partial x}+b(x)-ir(x)\frac{\partial}{\partial y}\right), \\ T_- &\equiv -e^{-iy}\left(a(x)\frac{\partial}{\partial x}+c(x)+ir(x)\frac{\partial}{\partial y}\right), \\ T_0 &\equiv -i\frac{\partial}{\partial y}, \end{aligned} \tag{2}$$

where y is a new variable, and the functions

$$F_n(x,y)\equiv f_n(x)e^{iny}. \tag{3}$$

Then Eqs. (1) are equivalent to

$$T_+F_n=\lambda_n F_{n+1}, \qquad T_-F_n=-\mu_n F_{n-1} \tag{4}$$

and

$$T_0F_n=nF_n. \tag{5}$$

Making use of the definitions (2) and (3) one finds that

$$\begin{aligned} T_-T_+F_n = -e^{iny}\{&a^2f_n''+a(a'+b+c-r)f_n' \\ &+[ab'+bc+nar'+ncr-(n+1)br \\ &-n(n+1)r^2]f_n\} \end{aligned}$$

TABLE I. Explicit expressions of the functions appearing in the recurrence relations (1)

Functions	f_n	$a(x)$	$b(x)$	$c(x)$	$r(x)$	λ_n	μ_n
Associated Legendre	P_m^n	$\sqrt{1-x^2}$	0	0	$\frac{x}{\sqrt{1-x^2}}$	$n-m$	$n+m$
Gegenbauer	C_n^α	x^2-1	$2\alpha x$	0	x	$n+1$	$-n-2\alpha+1$
Legendre	P_n	x^2-1	x	0	x	$n+1$	$-n$
Chebyshev	T_n	x^2-1	0	0	x	n	$-n$
Laguerre	L_n^α	x	$\alpha+1-x$	0	1	$n+1$	$-n-\alpha$
Bessel	J_n	1	0	0	$-1/x$	-1	1
Hermite	H_n	1	$-2x$	0	0	-1	$2n$

which, according to Eqs. (4), must coincide with

$$-\lambda_n \mu_{n+1} e^{iny} f_n,$$

thus

$$a^2 f_n'' + a(a' + b + c - r) f_n' + [ab' + bc + nar' + ncr - (n+1)br - n(n+1)r^2] f_n = \lambda_n \mu_{n+1} f_n. \quad (6)$$

Similarly, one finds that

$$T_+ T_- F_n = e^{iny}\{a^2 f_n'' + a(a' + b + c - r) f_n' + [ac' + bc - nar' - nbr + (n-1)cr - n(n-1)r^2] f_n\}$$

and therefore

$$a^2 f_n'' + a(a' + b + c - r) f_n' + [ac' + bc - nar' - nbr + (n-1)cr - n(n-1)r^2] f_n = \mu_n \lambda_{n-1} f_n. \quad (7)$$

By comparing Eqs. (6) and (7) it follows that

$$a(b' - c') - r(b - c) = K, \quad (8)$$

and

$$2(ar' - r^2) = N, \quad (9)$$

where K and N are independent of x and n, and

$$\lambda_n \mu_{n+1} - \mu_n \lambda_{n-1} = Nn + K. \quad (10)$$

Thus, the operators (2) must obey the commutation relations

$$[T_+, T_-] = NT_0 + K, \qquad [T_0, T_\pm] = \pm T_\pm, \quad (11)$$

where the constants N and K can be calculated by means of Eqs. (8) and (9) or (10). The values of N and K for the functions contained in Table I are given in Table II.

The Lie algebra generated by T_+, T_-, T_0 and, possibly, the identity, depends on the values of K and N. When N is different from zero, it is convenient to introduce the operators

$$\widetilde{T}_\pm \equiv e^{-in_0 y} T_\pm e^{in_0 y} = T_\pm + n_0 r(x) e^{\pm iy}, \quad (12)$$

where n_0 is a constant. From the commutation relations (11) one finds that

$$[\widetilde{T}_+, \widetilde{T}_-] = e^{-in_0 y}(NT_0 + K)e^{in_0 y} = NT_0 + (Nn_0 + K). \quad (13)$$

Therefore, choosing $n_0 = -K/N$, we obtain

$$[\widetilde{T}_+, \widetilde{T}_-] = NT_0, \qquad [T_0, \widetilde{T}_\pm] = \pm \widetilde{T}_\pm. \quad (14)$$

On the other hand, Eqs. (4), (5), and (12) give

$$\widetilde{T}_+(e^{-in_0 y} F_n) = \lambda_n (e^{-in_0 y} F_{n+1}),$$
$$\widetilde{T}_-(e^{-in_0 y} F_n) = -\mu_n (e^{-in_0 y} F_{n-1}) \quad (15)$$

and

$$T_0(e^{-in_0 y} F_n) = (n - n_0)(e^{-in_0 y} F_n). \quad (16)$$

By combining the commutation relations (14) one finds that

$$C_1 \equiv \widetilde{T}_+ \widetilde{T}_- + \tfrac{1}{2} N (T_0^2 - T_0) \quad (17)$$

commutes with $\widetilde{T}_\pm$ and T_0, which means that C_1 is a Casimir operator. In a similar manner one finds that when $N = 0$, the operator

$$C_2 \equiv T_+ T_- + K T_0 \quad (18)$$

commutes with $T_\pm$ and T_0. The eigenvalues of C_1 and C_2, denoted by κ_1 and κ_2, respectively, corresponding to the functions given in Table I are listed in Table II.

According to the preceding discussion, we have four different cases.

(i) $N > 0$. When N is positive, the Lie algebra generated by $\widetilde{T}_\pm$ and T_0 is isomorphic to $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$. This case contains the associated Legendre functions (see Table II).

(ii) $N < 0$. When N is negative, $\widetilde{T}_\pm$ and T_0 generate a Lie algebra isomorphic to $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2,\mathbb{R})$. This case contains the Gegenbauer, Legendre, Chebyshev, and Laguerre polynomials.

TABLE II. Values of the constants appearing in the commutation relations (11) and (14).

Functions	N	K	n_0	κ_1	κ_2	$w(x)$
Associated Legendre	2	0	0	$m(m+1)$	$\cdots$	1
Gegenbauer	-2	-2α	$-\alpha$	$\alpha(1-\alpha)$	$\cdots$	$\|1-x^2\|^{\alpha-3/2}$
Legendre	-2	-1	$-1/2$	$1/4$	$\cdots$	$\|1-x^2\|^{-1}$
Chebyshev	-2	0	0	0	$\cdots$	$\|1-x^2\|^{-3/2}$
Laguerre	-2	$-\alpha-1$	$-(\alpha+1)/2$	$(1-\alpha^2)/4$	$\cdots$	$x^{\alpha-1}e^{-x}$
Bessel	0	0	$\cdots$	1	$\cdots$	x
Hermite	0	-2	$\cdots$	$\cdots$	0	e^{-x^2}

(iii) $N=0$, $K=0$. When N and K vanish, the algebra generated by $T_\pm$ and T_0 is isomorphic to that of the Euclidean group of the plane. This case contains the Bessel functions.

(iv) $N=0$, $K\neq 0$. When N is equal to zero but K is different from zero, the algebra generated by $T_\pm$, T_0, and the identity is a central extension of the algebra of the Euclidean group of the plane. This case contains the Hermite polynomials.

3. Unitarity of the group actions

We introduce the inner product

$$(f,g)=\int_{x_1}^{x_2}dx\int_0^{2\pi}dy\,w(x,y)\,\overline{f(x,y)}\,g(x,y) \qquad (19)$$

where $w(x,y)$ is some weight function; the function w and the limits of integration are to be chosen in such a way that, for a space of functions obeying the appropriate boundary conditions, $T_0^\dagger=T_0$ and $T_+^\dagger=T_-$. The self-adjointness of T_0 requires that w be a function of x only, while the condition $T_+^\dagger=T_-$ amounts to $a'+aw'/w-b+r=c$, *i.e.*,

$$\ln|aw|=\int\frac{b+c-r}{a}dx. \qquad (20)$$

Then, $\widetilde{T}_+^\dagger=\widetilde{T}_-$, and the operators C_1 and C_2 are self-adjoint. The weight functions corresponding to the families of special functions appearing in Table I are listed in Table II.

Differentiating Eq. (9) it follows that $2r/a=r''/r'+a'/a$; therefore, when $b+c=0$, from Eq. (20) we find that, apart from an irrelevant constant factor, the weight function can be taken as

$$w=|a^3r'|^{-1/2}. \qquad (21)$$

By virtue of Eqs. (4), (5), (15), and (16), the functions $f_n(x)e^{iny}$ or $f_n(x)e^{i(n-n_0)y}$ are mapped into linear combinations of themselves under the transformations generated by the Lie algebra (11) or (14), respectively. Letting, for instance,

$$\Phi_n(x,y)\equiv N_nf_n(x)e^{iny},$$

where N_n is a normalization constant such that the functions Φ_n are normalized with respect to the inner product (19), for any transformation, $\mathcal{T}$, generated by the Lie algebra (11),

$$\mathcal{T}\Phi_n=\sum_{n'}\mathcal{T}_{n'n}\Phi_{n'},$$

where the matrix $(\mathcal{T}_{n'n})$ is unitary. (Recall that in some cases the functions cannot be normalized in the strict sense, *e.g.*,

$$\int_0^\infty J_n(\alpha x)J_n(\alpha' x)x\,dx=\alpha^{-1}\delta(\alpha-\alpha').)$$

Therefore, if (x,y) and (x',y') are two arbitrary points, we have

$$\sum_n\overline{\Phi_n(x,y)}\,\Phi_n(x',y') =\sum_n\overline{(\mathcal{T}\Phi_n)(x,y)}\,(\mathcal{T}\Phi_n)(x',y'). \qquad (22)$$

If there exists a transformation $\mathcal{T}$ in the group generated by the Lie algebra (11) such that $(\mathcal{T}\Phi_n)(x',y')$ does not vanish for only one value of n (as in the case where the f_n are the associated Legendre functions or the Bessel functions of integral order, since $P_m^n(1)=\delta_{n0}$ and $J_n(0)=\delta_{n0}$), then the right-hand side of Eq. (22) reduces to a single term and Eq. (22) becomes an addition theorem for the functions f_n.

4. Constant-curvature manifolds

When $b(x)$ and $c(x)$ vanish, the operators

$$T_0,\quad T_1=\frac{(T_++T_-)}{2},\quad \text{and}\quad T_2=\frac{(T_+-T_-)}{2i}$$

can be considered as vector fields on a two-dimensional manifold, M. This happens, for instance, for the associated Legendre functions, the Bessel functions and the Chebyshev polynomials (see Table I). In the case of the associated Legendre functions, taking M as the sphere with $x=\cos\theta$ and $y=\phi$, where θ and ϕ are the usual polar and azimuth angles, respectively, the operators T_i ($i = 0, 1, 2$) generate the rotations of the sphere and the functions (3) (which, apart from a constant factor, are the spherical harmonics) are the separable eigenfunctions of the Laplace–Beltrami operator of the sphere. Analogously, in the case of the Bessel functions of integral order, taking M as the Euclidean plane with $x = r$ and $y = \phi$, where r and ϕ are the usual polar coordinates, the operators T_i generate the orientation-preserving isometries of the plane and the products $J_n(x)e^{iny}$ are separable eigenfunctions of the Laplace operator. As we shall show, the Chebyshev polynomials are also related to the Laplace–Beltrami operator of a constant-curvature manifold.

In the case of the Chebyshev polynomials, the operators (2) are explicitly given by

$$T_0 = -i\frac{\partial}{\partial y}, \quad T_\pm = \pm\, e^{\pm iy}\left((x^2-1)\frac{\partial}{\partial x} \mp ix\frac{\partial}{\partial y}\right). \tag{23}$$

We look for a metric, $g_{\mu\nu}dx^\mu dx^\nu$, with

$$\mu, \nu = 1, 2, \qquad x^1 = x, \qquad x^2 = y,$$

which is invariant under the transformations generated by (23), *i.e.*,

$$X^\mu \partial g_{\nu\rho}/\partial x^\mu + g_{\mu\nu}\partial X^\mu/\partial x^\rho + g_{\mu\rho}\partial X^\mu/\partial x^\nu = 0,$$

where $X^\mu \partial/\partial x^\mu$ is any of the vector fields (23). We find that, up to an overall constant factor,

$$g_{\mu\nu}dx^\mu dx^\nu = \frac{dx^2}{(1-x^2)^2} - \frac{dy^2}{1-x^2}, \tag{24}$$

which has signature $(+-)$ for $-1 < x < 1$. A straightforward computation shows that (24) is the metric induced by the Lorentzian metric $-dX^2 - dY^2 + dZ^2$ on the surface $X^2 + Y^2 - Z^2 = 1$ parametrized by

$$X = \frac{\cos y}{\sqrt{1-x^2}}, \qquad Y = \frac{\sin y}{\sqrt{1-x^2}}, \qquad Z = \frac{x}{\sqrt{1-x^2}}.$$

It can be readily seen, making use of Eq. (9), that the vector fields T_i with $b(x) = c(x) = 0$ generate the orientation-preserving isometries of

$$g_{\mu\nu}dx^\mu dx^\nu = \frac{dx^2}{a^2(x)} + \frac{dy^2}{a(x)r'(x)}. \tag{25}$$

Using Eq. (9) again, one finds that the Gaussian curvature of the metric (25) is equal to $N/2$. Furthermore, letting $g \equiv \det(g_{\mu\nu})$, one has $\sqrt{|g|} = |a^3 r'|^{-1/2}$, which coincides with the weight function obtained in the preceding section [Eq. (21)]; thus, in the case under consideration, the area element appearing in the inner product (19) is the area element defined by the metric (25).

Making use of the general expression

$$\nabla^2 f = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu}\left(\sqrt{|g|}\, g^{\mu\nu}\frac{\partial f}{\partial x^\nu}\right)$$

for the Laplace–Beltrami operator, where $(g^{\mu\nu})$ is the inverse of $(g_{\mu\nu})$, Eq. (24) yields

$$\nabla^2 f = (1-x^2)^2\partial_x^2 f - (1-x^2)x\partial_x f - (1-x^2)\partial_y^2 f. \tag{26}$$

Therefore, the functions $F_n(x,y) = T_n(x)e^{iny}$ are eigenfunctions of ∇^2 with eigenvalue equal to zero. It may be noticed that

$$\nabla^2 = T_+T_- - T_0^2 + T_0 = T_1^2 + T_2^2 - T_0^2.$$

A family of functions, $T_n^\alpha(x)$, which contains the Chebyshev polynomials for $\alpha = 0$, is obtained looking for separable eigenfunctions of the Laplacian operator (26); if

$$\nabla^2(T_n^\alpha(x)e^{iny}) = \alpha T_n^\alpha(x)e^{iny},$$

one obtains

$$(1-x^2)T_n^{\alpha\prime\prime} - xT_n^{\alpha\prime} + \left(n^2 - \frac{\alpha}{1-x^2}\right)T_n^\alpha = 0. \tag{27}$$

The functions $T_n^\alpha(x)$ share the same raising and lowering operators (1) for all values of α and, for a fixed value of α, the functions $T_n^\alpha(x)e^{iny}$ form a basis for an irreducible representation of SO_0(2,1).

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