Wigner functions of free "Schrödinger cat" states

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We study the evolution of a free "Schrödinger cat" state (a superposition of two coherent correlated states moving in opposite directions) using the formalism of the Wigner function. Two possible mechanisms to spatially separate the two states are considered: a "quantum sling" and a Paul trap that produces unstable motion. The numerical analysis shows how the two superposed states move away of each other, while keeping an interference term between them that is typical of quantum phenomena.

Keywords: Wigner functions, phase space, mom-classical states.

Estudiamos la evolución de un estado libre de "gato de Schrödinger" (una superposición de dos estados coherentes moviéndose en direcciones opuestas) utilizando el formalismo de la función de Wigner. Se toman en cuenta dos posibles mecanismos para separar especialmente los dos estados: una "resortera cuántica" y una trampa de Paul que produce movimientos inestables. El análisis numérico muestra cómo los dos estados superpuestos se alejan uno de otro, conservando al mismo tiempo entre ellos los términos de interferencia que son típicos de los fenómenos cuánticos.

Descriptores: Funciones de Wigner, espacio fase, estados mo clásicos.

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1. Introduction

Coherent states [1] have the minimum dispersion in position and momentum allowed by the Heisenberg uncertainty principle; they are analogous, at the quantum level, to classical states. They can also be defined in many other ways; for instance, as eigenstates of the annihilation operator of a harmonic oscillator.

There are several interesting generalizations of coherent states. Thus, for instance, coherent correlated states [2] satisfy the minimum dispersions and correlations allowed by the more general Robertson-Schrödinger uncertainty relations. The even and odd coherent states for harmonic oscillators, introduced in the 1970s by Dodonov, Malkin and Man'ko [3] and later called "Schrödinger cat" states [4], are superpositions of coherent states; they are closely related to coherent correlated and squeezed states [5]. Since Schrödinger cat states have a wide class of application in quantum optics, they have been much studied in recent years and have been realized in laboratories: for ions in Paul trap by Monroe *et al.* [6], and by Brune *et al.* [7] for photons in microcavities.

Schrödinger cat states in time varying field have interesting properties of their own. In particular, the behavior of these states in a Paul trap, which has a time varying frequency, has been studied by Castaños *et al.* [8]. Another situation is the "quantum sling", proposed in Ref. 9, which consists of an abrupt release of a particle, initially bound to a harmonic oscillator, leading to a free particle state. The quantum sling effect for a Schrödinger cat state has interesting implications, since it describes a situation in which the two states of the "cat" move in opposite directions; thus, for instance, the model was applied by Dremin and Man'ko [10] to the study of particles emitted by nuclei.

A particularly convenient way to describe the quantum state of a system is through the use of the Wigner function, which permits to visualize the pseudo-probability density in phase-space (see, *e. g.*, Ref. 11). The Wigner function is analogous to a classical joint probability function, but it can take negative values, which is precisely the benchmark of quantum effects.

Wigner functions have been studied under several physical conditions. The Wigner function of a Schrödinger cat state exhibits typical Gaussian-like probabilities located at two different regions of phase-space, with an additional interference term that takes negative values.

The aim of the present paper is to study the behavior of an unbound or free Schrödinger cat state. We are particularly interested in describing the separation in phase space, of the two coherent correlated states and their interference. For this purpose, we use the Wigner function formalism.

In Sec. 2 we present the general description of a parametric oscillators, that is, an oscillator with time dependent frequency. The Wigner functions are introduced in Sec. 3, and the particular cases of a quantum sling and a Paul trap are considered; numerical analysis of a free Schrödinger cat state in these two situation is presented. Finally we present a summary of results in Sec. 4.

2. Parametric oscillator

The Schrödinger equation for a parametric oscillator,

$$i\hbar\frac{\partial}{\partial t}\Psi = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m}{2}\omega^2 x^2\right]\Psi, \qquad (1)$$

with time varying frequency, $\omega = \omega\,(t)$, admits a solution of the form

$$\psi = N\exp\left\{-\frac{m}{2i\hbar}\frac{\dot{f}}{f}x^2 + \sqrt{\frac{2m}{\hbar}}\frac{\alpha}{f}x\right\}, \qquad (2)$$

where N is a normalization factor; and the function f is a solution of the classical equation of motion

$$\ddot{f} + \omega^2(t)f = 0, \qquad (3)$$

with the Wronskian

$$\dot{f}f^* - f\dot{f}^* = 2i. \qquad (4)$$

It turns out that the operator

$$\hat{a} = \frac{-i}{(2\hbar m)^{\frac{1}{2}}}\left(m\dot{f}\hat{q} - f\hat{p}\right) \qquad (5)$$

is a constant of motion and can be identified with the annihilation operator (in the Schrödinger picture) [7]. Its eigenstate, defined as $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$, is, in coordinate representation, the state defined by Eq. (2). It is a coherent correlated state [7].

Now, an even or odd (+ or -) Schrödinger cat state can be defined as a superposition of two coherent states:

$$\Psi_\pm = N_\pm \exp\left\{-\frac{m}{2i\hbar}\frac{\dot{f}}{f}x^2\right\}\left(\exp\left\{\left(\frac{2m}{\hbar}\right)^{\frac{1}{2}}\frac{\alpha_0}{f}x\right\} \pm \exp\left\{\left(\frac{2m}{\hbar}\right)^{\frac{1}{2}}\frac{\alpha_0}{f}x\right\}\right), \qquad (6)$$

where

$$|N_\pm|^2 = 2\left(\frac{m}{\pi\hbar f^* f}\right)^{\frac{1}{2}}\frac{e^{-\Re(\alpha_0^2 f^*/f)}}{e^{2|\alpha_0|^2} \pm e^{-2|\alpha_0|^2}} \qquad (7)$$

are normalization factors.

Since

$$\langle\alpha_\pm|\,\hat{a}\,|\alpha_\pm\rangle = 0 = \langle\alpha_\pm|\,\hat{a}^\dagger\,|\alpha_\pm\rangle\,, \qquad (8)$$

the expectation values of position and momentum are zero: $< p >=< q >$=0. The dispersions for the cat state, as given from the above expressions, take the form

$$\left\langle x^2\right\rangle_\pm = \frac{\hbar}{2m} \times\left[f^{*2}\alpha^2 + f^2\alpha^{*2} + ff^*\left(2\frac{|N\pm|^2}{|N_\mp|^2}|\alpha|^2 + 1\right)\right], \qquad (9)$$

$$\left\langle p^2\right\rangle_\pm = \frac{\hbar m}{2} \times\left[\dot{f}^*\alpha^2 + \dot{f}^2\alpha^{*2} + \dot{f}\dot{f}^*\left(2\frac{|N_\pm|^2}{|N_\mp|^2}|\alpha|^2 + 1\right)\right], \qquad (10)$$

and the correlation is given by

$$\frac{1}{2}\langle xp + px\rangle_\pm = \frac{\hbar}{2}\left[f^*\dot{f}^*\alpha^2 + f\dot{f}\alpha^{*2} + \frac{1}{2}\left(f^*\dot{f} + f\dot{f}*\right)\left(2\frac{|N_\pm|^2}{|N_\mp|^2}|\alpha|^2 + 1\right)\right]. \qquad (11)$$

These dispersions and correlations satisfy the Robertson-Schrödinger uncertainty relations [9].

3. Wigner function

We now consider the Wigner function, which is defined for a pure state as

$$W(q,p) = \frac{1}{2\pi} \times\int_{-\infty}^{\infty} e^{-ipx/\hbar}\Psi^*\left(q - \frac{x}{2}\right)\Psi\left(q + \frac{x}{2}\right)dx. \qquad (12)$$

It can be interpreted as a pseudo-probability distribution in phase space, and satisfies the normalization condition

$$\int W\left(q,p\right)dpdq = 1. \qquad (13)$$

The Wigner function is a Gaussian in p and q for a coherent state. For a Schrödinger cat state, it is a superposition of two gaussians with an additional interference term. For an harmonic oscillator, the Gaussian rotates around de origin with the same frequency as the oscillator.

In this paper, we are particularly interested in studying a parametric oscillator with a time dependent frequency. The formalism of Sec. 2 can be used for this purpose. From the definition (5) for the annihilation operator, we write the p and q coordinates as:

$$q = \sqrt{\frac{\hbar}{2m}}\left(f^*\alpha + f\alpha^*\right),\ p = \sqrt{\frac{\hbar m}{2}}\left(\dot{f}^*\alpha + \dot{f}\alpha^*\right), \qquad (14)$$

from where it follows with some straightforward algebra, that the Wigner function of a Schrödinger cat states is given by

$$W_\pm(\alpha,\alpha*) = |N_\pm|^2|f|\left(\frac{\hbar}{m\pi}\right)^{\frac{1}{2}}\exp\left\{\Re\mathrm{e}\left(\alpha_0^2 f^*/f\right) + |\alpha_0|^2\right\}\left(e^{-2|\alpha-\alpha_0|^2} + e^{-2|\alpha+\alpha_0|^2} \pm 2e^{-2|\alpha|^2}\cos\left[2i\left(\alpha_0^*\alpha - \alpha_0\alpha^*\right)\right]\right). \qquad (15)$$

As expected, this function consists of three terms, the first two being Gaussians representing each state of the cat in phase space, and the third term being the interference between these two superposed states. For the usual harmonic oscillator of frequency ω_0, we simply have $f = \omega_0^{-1/2}\exp\{-i\omega_0 t\}$

The contrast I, defined as the ratio between the magnitudes of the interference and Gaussians terms, turns out to be just

$$I = exp\{4|\alpha_0|^2\}. \tag{16}$$

It depends only of the initial value α_0.

3.1. Wigner function for the quantum sling

As an application of the previous formalism, we now consider the example of a “quantum sling” as defined in Ref. 9. This model describes a particle in a harmonically oscillating potential that is released suddenly at time $t = 0$. In the notation of the previous section, it is described by the following function:

$$f(t) = \begin{cases} \omega_0^{-\frac{1}{2}} e^{i\omega t}, \text{ for } t < 0, \\ \omega_0^{-\frac{1}{2}}(1 + i\omega_0 t), \text{ for } t > 0. \end{cases} \tag{17}$$

In order to simplify the notation, we define dimensionless variables

$$\bar{q} = q\left(\frac{m\omega_0}{2\hbar}\right)^{\frac{1}{2}} \quad \bar{p} = p\left(\frac{1}{2m\omega_0\hbar}\right)^{\frac{1}{2}}, \tag{18}$$

such that

$$\alpha_0 = \bar{q}_0 + i\bar{p}_0, \tag{19}$$

and

$$\alpha(t) = e^{i\omega_0 t}\bar{q} + ie^{i\omega_0 t}\bar{p}. \tag{20}$$

In terms of these variables, the Wigner function given in Eq. (15) takes the form

$$W_\pm|_{t<0} = \frac{2e^{|\alpha_0|^2}}{\pi\left(e^{2|\alpha_0|^2} \pm e^{-2|\alpha_0|^2}\right)}\Big(\exp\left\{-2\left[e^{-i\omega t}\alpha_0 - (\bar{q}+i\bar{p})\right]^2\right\} + \exp\left\{-2\left|e^{-i\omega t}\alpha_0 + (\bar{q}+i\bar{p})\right|^2\right\}$$
$$\pm 2\exp\left\{-2\left|\bar{q}+i\bar{p}\right|^2\right\}\cos\left(-2i\left[e^{-i\omega t}\alpha_0\left(\bar{q}+i\bar{p}\right) - e^{i\omega t}\alpha_0^*\left(\bar{q}-i\bar{p}\right)\right]\right)\Big) \tag{21}$$

for the oscillatory regime before the release of the sling. As expected, there are two Gaussians rotating one around the other with an interference inbetween; this case has been analyzed previously by many authors.

After the sling is released, that is, for $t > 0$, we have $\alpha(t) = \bar{q} + (1 + i\omega_0 t)\bar{p}$. Then, the Wigner function takes the explicit form

$$W_\pm|_{t>0} = \frac{2e^{|\alpha_0|^2}}{\pi\left(e^{2|\alpha_0|^2} \pm e^{-2|\alpha_0|^2}\right)}\Big(\exp\left\{-2\left[(\bar{q}-\bar{q}_0-\omega t\bar{p})^2 + (\bar{p}-\bar{p}_0)^2\right]\right\}$$
$$+ \exp\left\{-2\left[(\bar{q}+\bar{q}_0-\omega t\bar{p})^2 + (\bar{p}+\bar{p}_0)^2\right]\right\} \pm 2\exp\left\{-2\left[(\bar{q}-\omega t\bar{p})^2 + \bar{p}^2\right]\right\}\cos[4(\bar{q}_0\bar{p} - \bar{p}_0\bar{q} - \bar{p}_0\omega t\bar{p})]\Big) \tag{22}$$

In this case we have two Gaussian states separating from each other at constant speed, with a very large interference term between them. This behaviour is shown in Fig. 1, where we present a particular case of the cat state after it has been released from the sling. The interference terms are shown in close-up in Fig. 2.

3.2. Paul trap

As a next example, we consider the motion of an ion in a Paul trap. The time varying potential has the form

$$\omega^2(t) = A + B\cos\omega_0 t, \tag{23}$$

where A and B are related to *dc* and *ac* potentials, respectively, and ω_0 is the radiofrequency in the trap [12].

This form of the potential leads directly to a classical equation of motion that is just the Mathieu equation

$$\frac{d^2}{d\tau^2}f + (a + 2q\cos 2\tau)f = 0, \tag{24}$$

where a and q are dimensionless parameters related to the physical parameters of the trap, and τ is a dimensionless time: $\tau = \omega_0 t/2$. It is well known from the theory of Mathieu functions (see, *e. g.*, Abramowitz [13]) that there are stable and unstable solutions depending on the combined values of the parameters a and q.

In the following, we present a numerical study of the evolution of a Schrödinger cat state in a Paul trap. Two particular cases, one stable and one unstable, are considered. We are particularly interested in the unstable case, since it correspond to an unbound Schrödinger cat state and, as such, has some similarities with the quantum sling described in the previous section.

For our numerical study, we have chosen parameters values: $a = -0.15$, $q = 0.80$ for the stable case. As for the unstable case, the following values were taken: a = -0.5, q = 0.3. Furthermore, we define the dimensionless function

$$\bar{f} = \omega_0^{1/2} f, \tag{25}$$

which satisfies the initial conditions

$$\bar{f}\,(0) = 1; \quad \dot{\bar{f}}\,(0) = 2i, \tag{26}$$

where derivatives are with respect to the dimensionless time τ.

The Mathieu functions corresponding to these values of the parameters are given in Fig. 3, both for the stable and the unstable case.

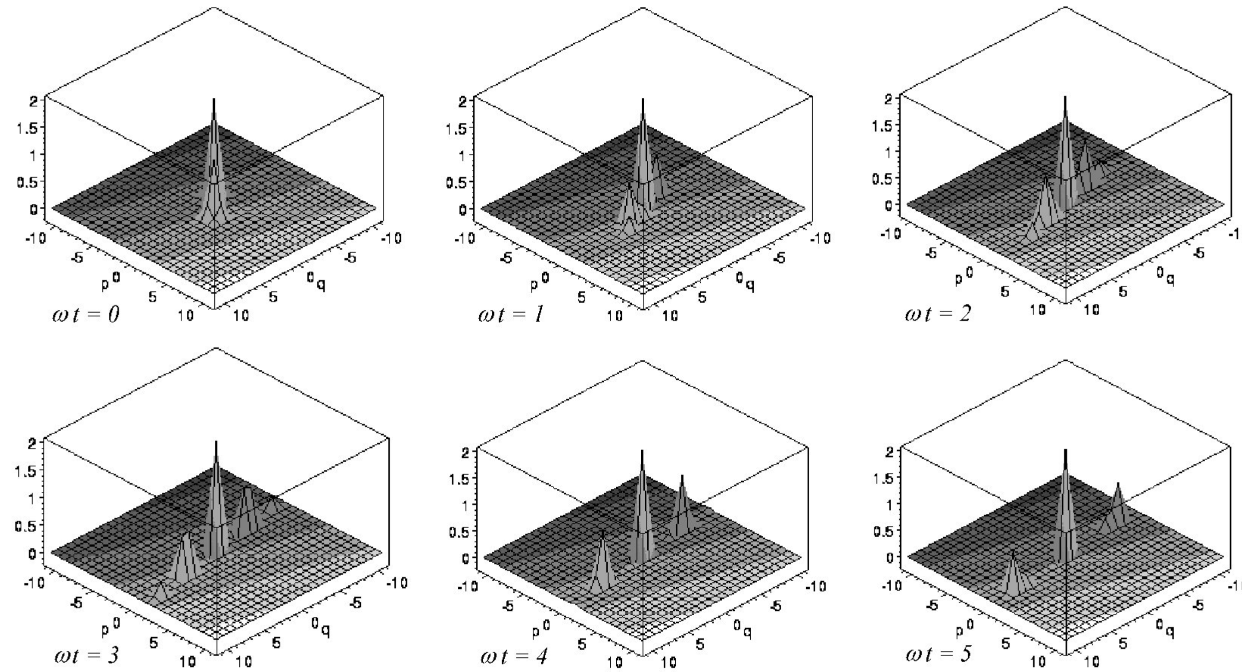

FIGURE 1. Wigner functions for a quantum sling, after release. The initial value is $\alpha_0 = 1 + i$.

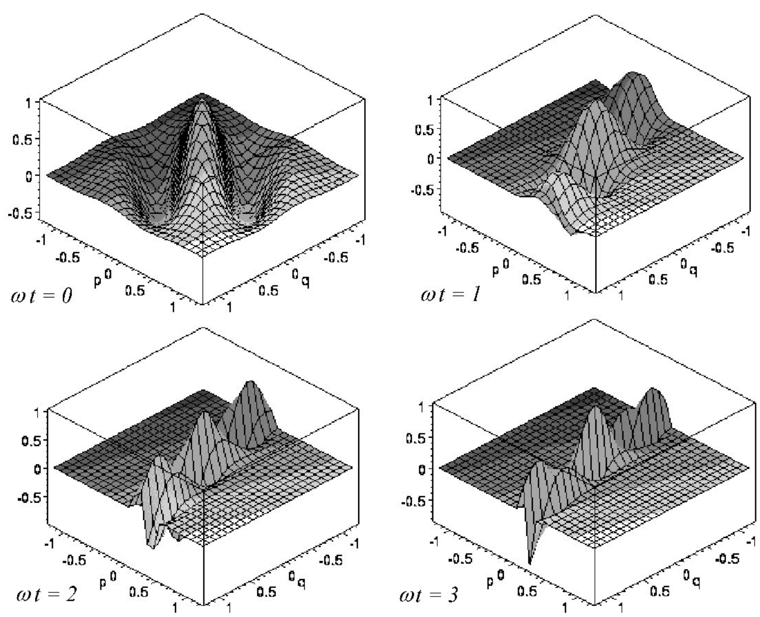

FIGURE 2. Close-up of the interference terms in Fig. 1

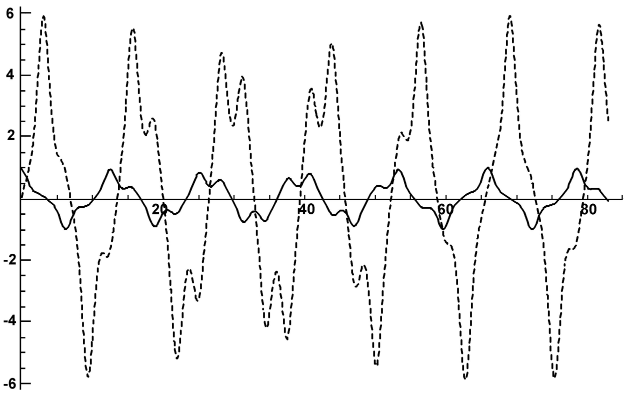

FIGURE 3a. Real and imaginary parts of the Mathieu function for the stable case: (a = -0.1, q = 0.75) - - - Im(f), —— Re(f)

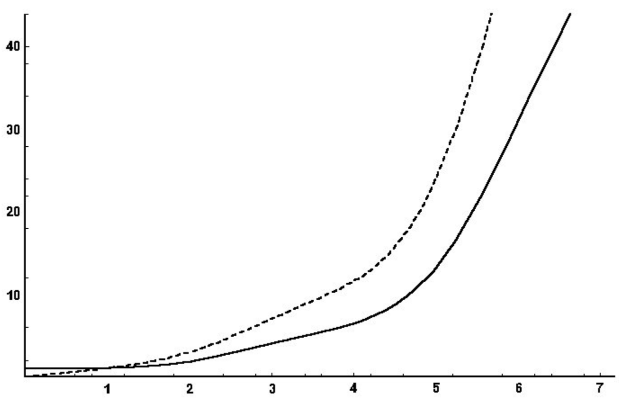

FIGURE 3b. Real and imaginary parts of the Mathieu function for the unstable case (a = -0.5, q = 0.3)

In Figs. 4 and 5, we present the dispersions of position and momentum for a cat state, together with the position-momentum correlation. The stable case, Fig. 4 is presented for the sake of completeness; as expected, the figures are very similar to those obtained by Castaños *et al.* [8] for similar values of the parameters.

Numerical results for the dispersions and correlations in the unstable case are shown in Fig. 5. In this case, it is seen that all these functions grow rapidly after a certain time, which reflects the unstability of the system.

In Fig. 6, we show the Wigner functions in the stable case as it evolves in the phase space plane. We used the fact that $\alpha = -i\left(\dot{f}\bar{q} - f\bar{p}\right)$ in terms of dimensionless variables $\bar{q}$ and $\bar{p}$. From the figures, it can be seen that the dispersions are squeezed, the highest dispersions in x being in coincidence with the lowest dispersions in p, and viceversa; as for the correlation, it remains constant as expected.

The Wigner function for the unstable case is shown in Fig. 7. The dispersions spread with time and the two states separates with an increasing velocity. The strong interference term between the two states is clearly visible.

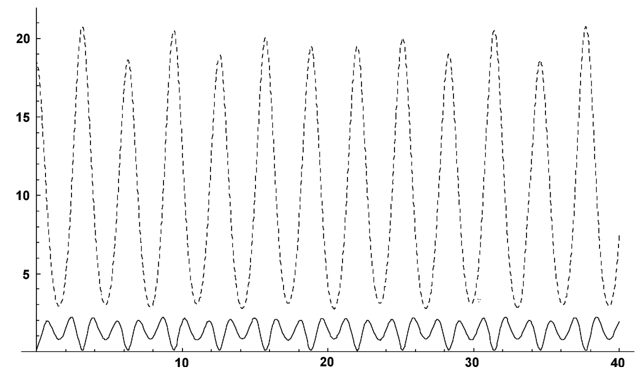

FIGURE 4a. Dispersion for Schrödinger cat state of $< x^2 >$ y $< p^2 >$. Stable case. - - - $< x^2 >$, —— $< p^2 >$

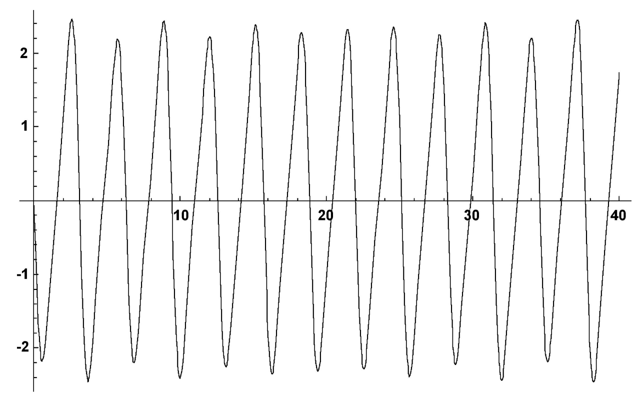

FIGURE 4b. Dispersion of $< xp + px >$. Stable case.

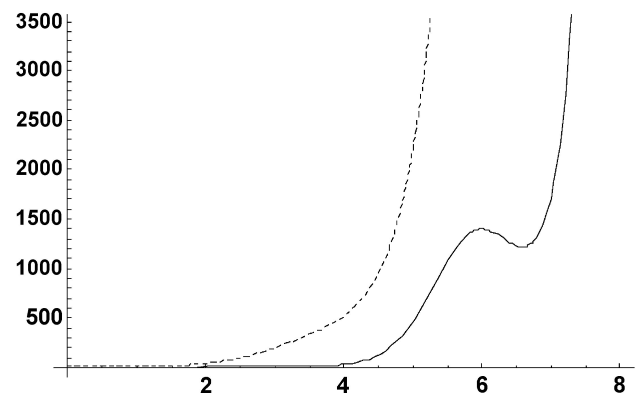

FIGURE 5a. Dispersion of Schrödinger cat state for $< x^2 >$ y $< p^2 >$. Unstable case. - - - $< x^2 >$, —— $< p^2 >$

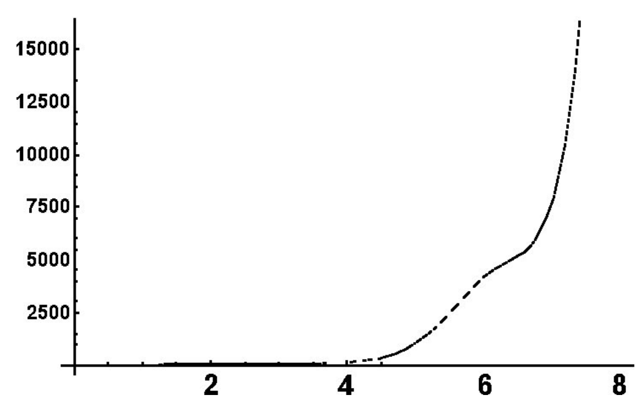

FIGURE 5b. Dispersion of $< xp + px >$. Unstable case.

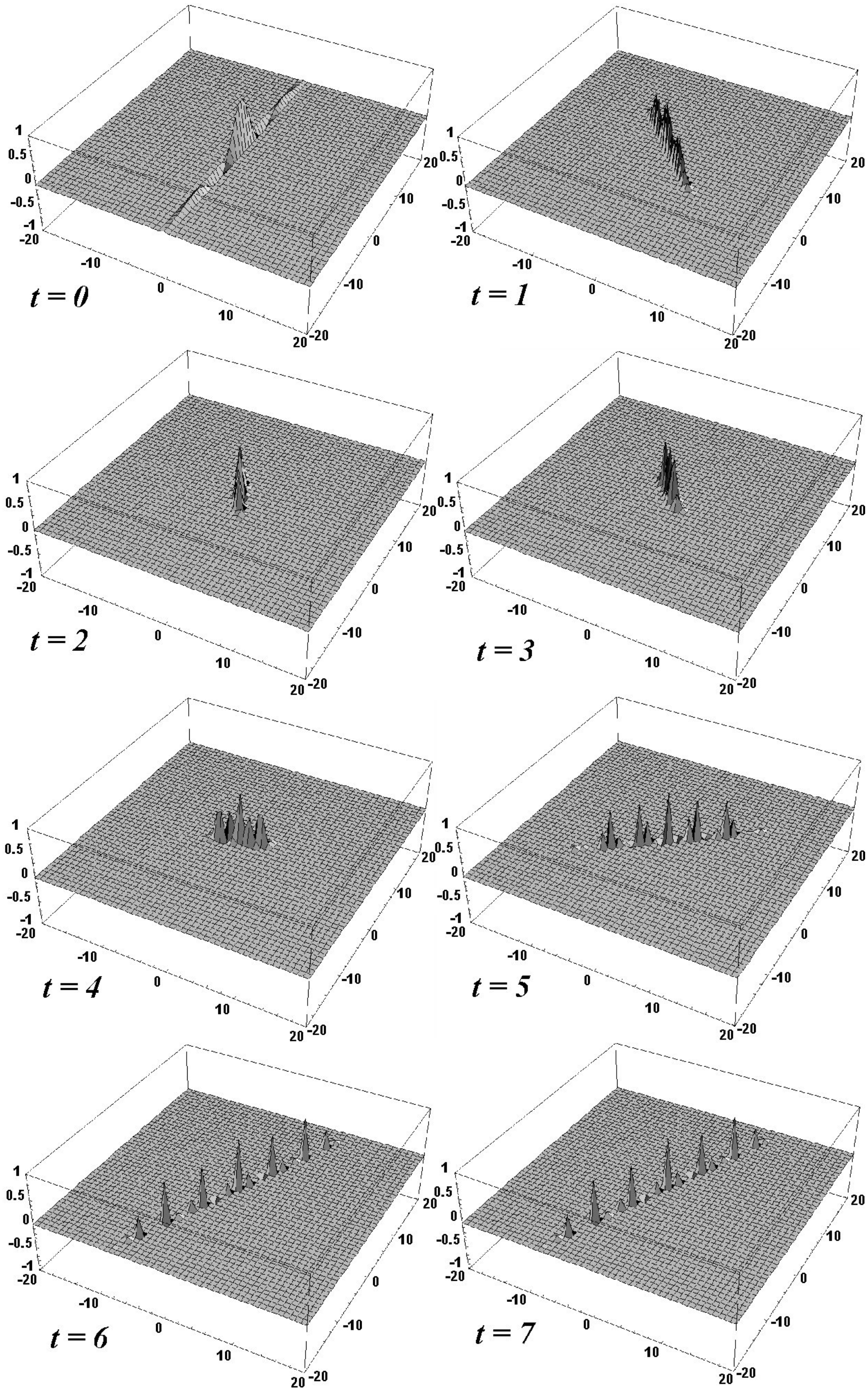

FIGURE 6. Stable case a = -0.15, q = 0.8. The interference terms are very prominent. The whole structure rotates with a variable angular velocity.

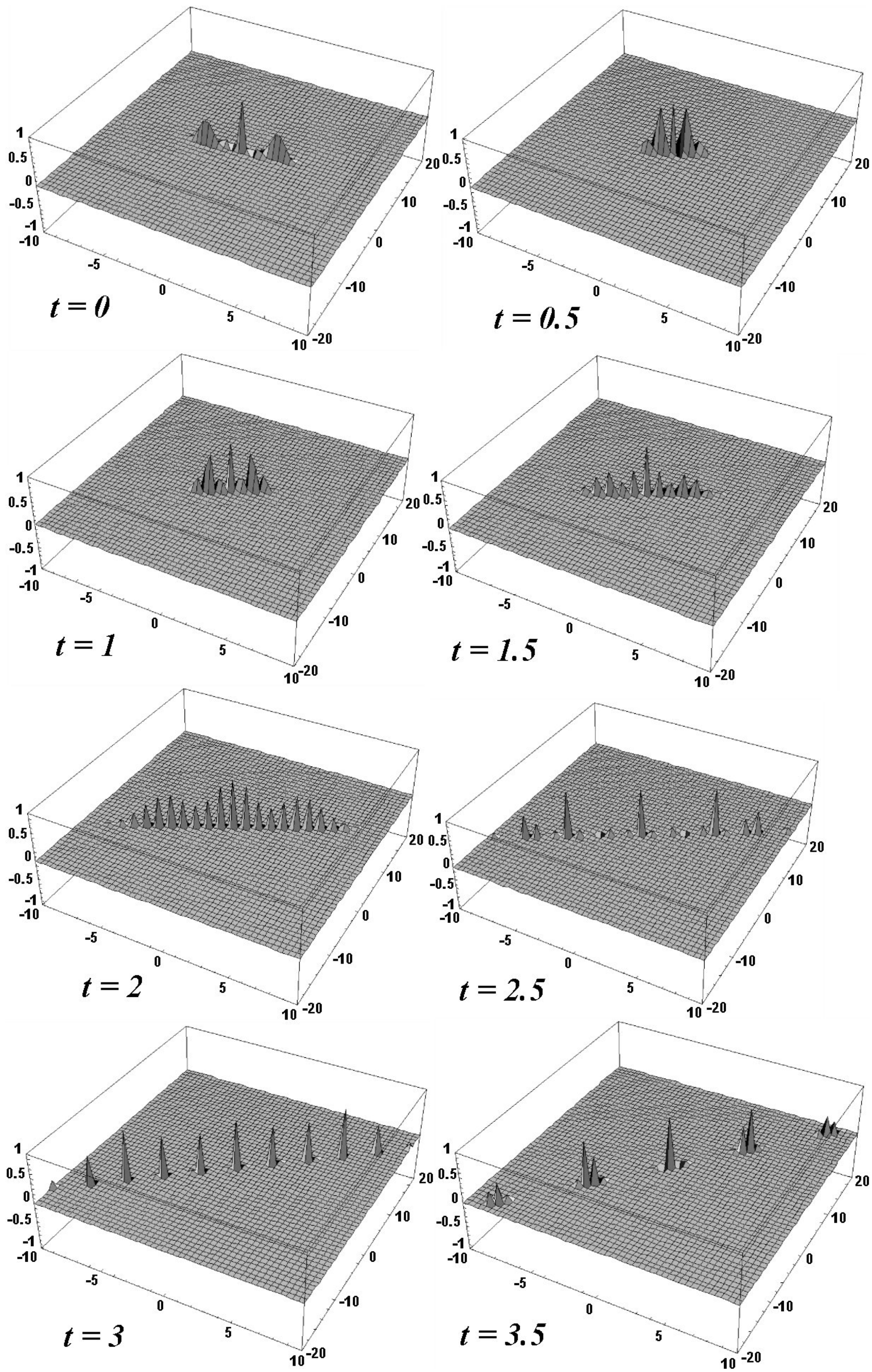

FIGURE 7. Unstable case, a = -0.5, q = 0.3. The gaussian states separate with incresing velocity.

4. Summary

We have obtained the graphical representations of the Wigner functions for a Schrödinger cat state in the case where there is spatial separation between the two superposed semiclassical states. As is clear from the figures, there is a strong interference term which takes negatives values in phase space, thus revealing the quantum nature of the state.

In the case of a quantum sling, the Schrödinger cat state is initially described in phase space, as expected, by two Gaussians rotating one around the other, with the usual interference term between them. After the release of the sling, the two Gaussians separate at constant speed, moving along the q coordinate. There is a squeezing of the state, since the dispersion of the position increases as t^2, while the dispersion of the momentum remains constant. The interference term does not vanish, but get squeezed in the p direction and spreads in the q direction.

A similar behavior also occurs for the unstable case of the parametric oscillator. The main difference is that the semiclassical states do not move along straight lines at constant speed. A squeezing of states is also present.

It is important to point out that, in a real situation, the interference will be lost by decoherence due to an interaction with the environment (see, *e. g.*, Hacyan [14], Barberis and Hacyan [15]). Our results show that a strong coherence between the two superposed states remains in an idealized situation. The conclusion, therefore, is that the interaction with the environment must be very strong in a realistic situation for the decoherence to take place rapidly. This process will be studied in a future work.

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