Remarks on noncommutative solitons

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In the first part of this work we consider an unstable non-BPS $Dp-\bar{D}p-$brane pair in Type II superstring theory. Turning on a background NS-NS $B-$field (constant and nonzero along two spatial directions), we show that the tachyon responsible for the unstability has a complex GMS solitonic solution, which is interpreted as the low energy remnant of the resulting $D(p-2)-$brane. In the second part, we apply these results to construct the noncommutative soliton analogous of Witten's superconducting string. This is done by considering the complex GMS soliton arising from the $D3-\bar{D}3-$brane annihilation in Type IIB superstring theory. In the presence of left-handed fermions, we apply the Weyl-Wigner-Moyal correspondence and the bosonization technique to show that this object behaves like a superconducting wire.

Keywords: Non-BPS branes, noncommutative soliton, superconducting string.

En la primera parte de este trabajo consideramos un par inestable de D-branas no-BPS, $Dp-\bar{D}p$, en la teoría de supercuerdas tipo II. Considerando un campo de fondo B NS-NS (constante y diferente de cero a lo largo de dos direcciones espaciales), mostramos que el taquión responsable de la inestabilidad corresponde a una solución solitónica compleja del tipo GMS, la cual puede ser interpretada como el remanente de bajas energías de la $D(p-2)$ resultante. En la segunda parte, aplicamos estos resultados para construir un solitón noconmutativo análogo a la cuerda superconductora de Witten. Esto se hace considerando un solitón complejo GMS que proviene de la aniquilación de un par $D3-\bar{D}3$ en la teoría de supercuerdas tipo IIB. En la presencia de fermiones izquierdos, aplicamos la correspondencia de Weyl-Wigner-Moyal y la técnica de bosonización para mostrar que este objeto se comporta como un alambre superconductor.

Descriptores: Branas no-BPS, solitones noconmutativos, cuerdas superconductoras.

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1. Introduction

Recent developments in string theory have not only proven to be insightful, but have rapidly overturned obsolete notions initially thought to be well established [1-3]. Particularly, the discovery of $D-$branes in the nonperturbative regime has revealed a deeper underlying structure, which might be a first glance of an ultimate theory [2,4,5].

On the other hand, topological defects (see for instance, Refs. 6 and 7) in field theory have been studied for a number of years. The traditional approach was to consider them as a consequence of the spontaneous breakdown of gauge symmetries. The spirit then was to explore the nonperturbative sector of the Standard Model and Grand Unified Theories containing large Lie groups. Also, a plethora of defects, ranging from monopoles and vortices to kinks and domain walls, have been obtained in both particle physics and condensed matter systems [8]. It is important to mention a very interesting example: the *superconducting string* found by Witten in Ref. [9]. The idea there is to consider a four-dimensional scalar theory with a $U(1)\times\widetilde{U}(1)$ gauge symmetry. The spontaneous breakdown of one of the $U(1)$'s yields a string-like solution; while the breaking of the remnant $U(1)$ in the core endows this string with superconductivity. In conclusion: if gauge symmetries are truly present in nature, such defects ought to exist and they should be found experimentally.

A more modern application of topological defects is in the understanding of $Dp-$brane anti-$Dp-$brane annihilations. Such configurations are non-BPS (for a review, see for instance Ref. 10), and they are unstable due to the presence of a tachyonic mode in their worldvolume. By finding a suitable vortex-like configuration for the tachyon field, it has been shown that the result of the above process is the emergence of a stable BPS $D(p-2)-$brane.

Another outstanding new trend is the study of $D-$branes in the presence of a NS-NS constant $B-$ field. In the low energy limit, its effect is the appearance of a Moyal $*-$product in the fields participating in the Operator Product Expansion (OPE), thence obtaining a noncommutative effective field theory (for a review, see Refs. 11 and 12). A relevant feature here is that we can associate fields in $\mathbb{R}^2_*$ to operators in the Hilbert space of a simple harmonic oscillator (SHO). This association is known as the Weyl-Wigner-Moyal (WWM) correspondence.

Harvey, Kraus, Larsen and Martinec (HKLM) used this approach to study the decay of a bosonic $D25-$brane into a $D23-$brane [13]. Considerations of the classical vacua and its implications were considered in Ref. 14. Finite noncommutative parameter corrections were performed in Ref. 15. Large non-commutativity parameter approximation is simpler than that for a finite parameter. In this paper we focus mainly in this approximation. Bosonic string theory has a tachyon mode, which makes $D-$branes of all dimensions unstable [10]. However, instead of searching a tachyonic vortex configuration, HKLM found a nontrivial solution by introducing a large $B-$field along two spatial directions in the $D25-$brane worldvolume. The solution is the real noncommutative soliton discovered by Gopakumar, Minwalla and Strominger (GMS) [16]. This object is identified with the remnant of the $D23-$brane. An extension of HKLM's work to Type II superstring theory is described in [17]. Further work on noncommutative tachyons in the large noncommutative parameter approximation worked out in Refs. 18–22. A K-theoretic description of noncommutative tachyons is done in Refs. 23 and 24. The description of tachyon condensation in orbifolds is discussed in Ref. 25. For a recent review on the subject, for instance, see Ref. 26.

The objective of the first part of this work is to apply HKLM's idea [13] and study the case of Type II superstring $Dp - \overline{Dp}$ annihilation in the presence of a large $B-$field along two spatial directions. Many techniques used by GMS in Ref. 16, such as that of identifying nontrivial solutions with projection operators, are applied in here as well. However, we now have a charged tachyon field under the Chan-Paton gauge symmetry $U(1) \times \widetilde{U}(1)$. The solution is a complex GMS soliton, and it is regarded as the low-energy remnant $D(p-2)-$brane. $Dp - \overline{Dp}$ pairs with $B-$field and non-zero magnetic fluxes were previously considered in Ref. 27 and further explored in Refs. 15 and 28. The generalization to nonabelian fluxes was studied at Ref. 29.

In the second part of this paper, we attempt to construct an object similar to Witten's superconducting string in the context of Noncommutative Solitons in string theory. The idea is to consider a non-BPS $D3 - \bar{D}3-$brane pair in Type IIB superstring theory in the presence of a large and constant NS-NS $B-$field turned on along only two worldvolume spatial directions. The fact that the tachyon is charged under a $U(1) \times \widetilde{U}(1)$ Chan-Paton gauge symmetry is a tantalizing similarity to Witten's original setup. Therefore, using the results obtained in the first part of this work, we identify the complex GMS solitonic solution to the tachyon with the low-energy remnant of the $D1-$brane (which is itself the product of the $D3 - \bar{D}3-$brane annihilation). We coin the term *noncommutative D-string* (or *noncommutative string*, for short) to describe this object. Like in Witten's *string*, there are left-handed fermions, which naturally arise from the supersymmetric spectrum of the open string attached to the $D1-$brane. We shall just consider the flat space case; so in the low-energy regime, these fermions live in a space $\mathcal{M}^{1+1} \times \mathbb{R}^2_*$ with the complex GMS soliton as a background field[a]. Integrating out the transverse noncommutative directions, this complex GMS soliton projects out most of the fermionic modes, leaving behind a simple two-dimensional theory. Happily, such theory can be exactly solved by the technique of bosonization, inspired by Witten's method for the case of the superconducting string [9]. Surprisingly, we find that the fermionic current along $z-$direction doesn't decay. Hence, our noncommutative string's behavior is similar to a superconducting wire.

This paper consists in two parts. The first one explains the construction of a complex GMS solitons from $Dp - \bar{D}p$ brane annihilation. The second part applies this idea to show the existence of a noncommutative version Witten's superconducting string in the context of superstring theory. In Sec. 2, we overview the basic properties of the unstable non-BPS $Dp - \overline{Dp}-$brane configuration in Type II superstring theory. Also, we explain how noncommutativity arises from the NS-NS $B-$field and introduce the WWM correspondence. In Sec. 3, we turn on a $B-$field on the $Dp - \overline{Dp}$ brane system and find a complex gauge-coupled GMS soliton, which we identify with a BPS $D(p-2)-$brane. In Sec. 4, we study the case where $p = 3$, and construct the noncommutative $D-$string. We couple the four-dimensional left-handed fermions (coming from the supersymmetric spectrum of the open string attached to the $D1-$brane) in the low energy regime to the background GMS soliton. Then, by integrating out the two noncommutative directions, we obtain a two-dimensional theory along the $D-$string. At the end we show, by applying the bosonization technique, that this object appears to be superconducting. Finall in Sec. 5 we give our final remarks.

2. Basic tools

The purpose of this section is to give the reader a brief overview of the tools and ideas necessary to address the problem of the $Dp - \overline{Dp}$ brane configuration with a $B-$field background. It must be pointed out, however, that our aim is not to provide an extensive review of noncommutative field theories. For a more complete discussion, see Refs. 11,12,26. For a very recent review on Weyl-Wigner-Moyal deformation quantization see Ref. 30.

2.1. $\mathbf{Dp - \overline{Dp}}-$brane annihilation

To begin with, consider a pair of parallel $Dp-$branes in Type II theory, with p even in the Type IIA and odd in the Type IIB theory. This system is stable and BPS, and has a $U(1) \times \widetilde{U}(1)$ Chan-Paton internal symmetry. Roughly speaking, we can turn one of the $Dp-$branes into a $\overline{Dp}-$brane by rotating it an angle π in the transverse directions [31]. A consequence of this is the reversal of the GSO projection, hence the occurrence of a tachyon along with the previously cancelled massive states. Thus, the $Dp - \overline{Dp}-$brane configuration obtained by rotating one of them is no longer BPS.

In general, the presence of a tachyon is a signal of instability. Under certain circumstances, such unstable non-BPS systems may decay into stable BPS $D-$branes of lower dimensions. In the case of $Dp - \overline{Dp}-$brane annihilation, this system may decay into a stable $D(p-2)-$brane [10].

The tachyon field T in the $Dp - \overline{Dp}-$brane worldvolume is charged $(-1, +1)$ under the gauge symmetry $U(1) \times \widetilde{U}(1)$. Therefore, the tachyonic lagrangian $\mathcal{L}_t$ is given by

$$\mathcal{L}_t = \overline{D_\mu T} D^\mu T - V(T), \tag{1}$$

where the covariant derivative is

$$D_\mu T = (\partial_\mu - iA_\mu + i\widetilde{A_\mu})T, \tag{2}$$

while A_μ and $\widetilde{A_\mu}$ are real functions and they are respectively the gauge fields of $U(1)$ and $\widetilde{U}(1)$.

The traditional method to find a stable $D(p-2)-$brane is as follows. First, we parametrize the original $(p+1)-$dimensional worldvolume by the coordinates $(r, \theta, \widetilde{x}^a)$, where $\widetilde{x}^a$ are longitudinal spacetime coordinates to the $D(p-2)-$brane. One must find a cylindrically symmetric vortex configuration localized in the vicinity of $r = 0$ for the tachyon [10]. Such a configuration is required to describe a pure vacuum for large r in a topologically nontrivial way. This is achieved by imposing the following asymptotic behavior $(r \to \infty)$:

$$T \sim T_{min} e^{i\theta}, A_\theta - \widetilde{A_\theta} \sim 1, \tag{3}$$

where T_{min} is the value in which $V(T)$ is minimized.

These conditions (3) ensure that for large r, $D_\mu T \to 0$ and $V(T) \to V(T_{min})$, leaving a soliton placed in the small r region. Notice that this soliton is independent of $\widetilde{x}^a$. This is a vortex string[b], and we identify it with a stable BPS $D(p-2)-$brane.

Nevertheless, imposing vortex-like asymptotic conditions as in (3) is not the only method of obtaining stable nontrivial solutions of the tachyon field. A few months back, it was found that the tachyon allows a different type of solutions if some directions are noncommutative [16].

2.2. Noncommutativity from string theory: the B−field

In string theory, the conventional low energy limit is to take $\alpha' \to 0$. The result of this is the inevitable decoupling of the massive modes from the effective theory.

If we additionally turn on a constant NS-NS $B-$field, we still decouple the massive modes from the theory. However, it turns out that one obtains a nonlocal deformation of the field theory due to noncommutativity. This is a stringy effect which helps display D$p-$branes as noncommutative solitons.

Recall that Type II theories have a massless NS-NS symmetric background field $g_{\mu\nu}$ with $\mu, \nu = 0, 1, \cdots 9$, which we shall interpret as the background metric. Likewise, these theories contain an antisymmetric field $B_{\mu\nu}$ in the massless NS-NS spectrum. These theories also admit R-R charged $Dp-$branes with open strings attached to them. In this case, the OPE is merely

$$e^{ik_1 \cdot X}_{(\tau)} e^{ik_2 \cdot X}_{(\tau')} \sim (\tau - \tau')^{2\alpha' g^{\mu\nu} k_{1\mu} \cdot k_{2\nu}} \times e^{i(k_1+k_2)\cdot X} + \cdots .$$

Turning on this $B-$field, the OPE becomes

$$e^{ik_1 \cdot X}(\tau) e^{ik_2 \cdot X}(\tau') \sim (\tau - \tau')^{2\alpha' G^{\mu\nu} k_{1\mu} k_{2\nu}} \times [e^{+\frac{i}{2}\Theta^{\mu\nu} k_{1\mu} k_{2\nu}}] \times e^{i(k_1+k_2)\cdot X} + \cdots ,$$

where

$$G^{\mu\nu} = (\frac{1}{g + 2\pi\alpha' B} g \frac{1}{g - 2\pi\alpha' B})^{\mu\nu}$$

is the effective metric seen by the open string modes, and

$$\Theta^{\mu\nu} = -(2\pi\alpha')^2 (\frac{1}{g + 2\pi\alpha' B} B \frac{1}{g - 2\pi\alpha' B})^{\mu\nu}$$

is known as the noncommutativity parameter matrix.

The above calculations were carried over by Seiberg and Witten in Ref. [32](see also references therein), but the noncommutativity interpretation was first given by Schomerus [33].

The configuration space counterpart of the extra factor appearing in the OPE, *i.e.* $e^{\frac{i}{2}\Theta^{\mu\nu}\partial_\mu \partial'_\nu}$, has a rather peculiar interpretation. This factor gives rise precisely to the Moyal $*-$product (a not commutative, but still associative product), which has been studied for a number of years as a key feature in an alternative description of quantum mechanics known as Deformation Quantization. Recently, this description was applied to the quantization of bosonic strings [34].

The presence of the Moyal $*-$product in the OPE means that fields in the effective theory get multiplied as follows:

$$(f * g)(x) = f(x) e^{\frac{i}{2}\Theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial'}_\nu} g(x')|_{x=x'} \neq (g * f)(x); \tag{4}$$

whereas in the absence of $B-$field, they were simply multiplied as

$$(f \cdot g)(x) = f(x) g(x) = g(x) f(x) = (g \cdot f)(x).$$

In conclusion, we can fix our $B-$field in any convenient way to obtain a desired effective theory with the characteristic that along those directions where $B \neq 0$, the worldvolume of the $D-$brane is noncommutative and fields are $*-$multiplied.

2.3. The Weyl-Wigner-Moyal correspondence

The simplest configuration is when the constant $B-$field is nonzero only along two spatial directions. Let's choose these to be x and y and call the noncommutative $x - y$ plane $\mathbb{R}^2_*$. Therefore

$$B_{\mu\nu} = \begin{pmatrix} 0 & B & \cdots & 0 \\ -B & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \tag{5}$$

where $B = B_{12} = -B_{21}$. As a result, we obtain noncommutativity along the $x - y$ plane:

$$[x, y]_* = i\Theta, \tag{6}$$

where $\Theta = \Theta^{12} = -\Theta^{21}$ is the noncommutativity parameter and $[x, y]_* \equiv x * y - y * x$ is the Moyal bracket.

Rescaling the coordinates to

$$x \to \frac{x}{\sqrt{\Theta}} \quad \text{and} \quad y \to \frac{y}{\sqrt{\Theta}},$$

we obtain the following commutator:

$$[x, y]_* = i, \tag{7}$$

which is very similar to $[\widehat{q}, \widehat{p}] = i$, the position and momentum commutator of a quantum particle in one spatial dimension. With this identification, calculations along noncommutating directions are straightforward.

As in deformation quantization, we can associate fields $f(x, y)$ in the noncommutative plane $\mathbb{R}^2_*$ to operators $\widehat{\mathcal{O}_f}(\widehat{q}, \widehat{p})$ in the quantum particle's Hilbert space $\mathcal{H}(\widehat{q}, \widehat{p})$. The common identification is performed by using the WWM prescription [26,30]:

$$\widehat{O_f}(\widehat{q}, \widehat{p}) = \frac{1}{(2\pi)^2} \int dk_q dk_p \widetilde{f}(k_q, k_p) \widehat{U}(\widehat{q}, \widehat{p}), \tag{8}$$

where

$$\widehat{U}(\widehat{q}, \widehat{p}) = e^{-i(k_q \widehat{q} + k_p \widehat{p})}$$

is a unitary operator, and the Fourier transform is just

$$\widetilde{f}(k_q, k_p) = \int dq dp f(q, p) e^{i(k_q q + k_p p)}.$$

Therefore, we can write the operator

$$\widehat{O_f}(\widehat{q}, \widehat{p}) = \frac{1}{(2\pi)^2} \int dq dp f(q, p) \widehat{\Omega}(\widehat{q}, \widehat{p}), \tag{9}$$

where

$$\widehat{\Omega}(\widehat{q}, \widehat{p}) = \int dk_q dk_p e^{i(k_q q + k_p p)} \widehat{U}(\widehat{q}, \widehat{p}) \tag{10}$$

is known as the Stratonovich-Weyl quantizer [34].

A major consequence of this correspondence is that now it is easier to perform integrations along $\mathbb{R}^2_*$. Thus, with the above prescription, it can be shown that

$$\frac{1}{2\pi\Theta} \int dq dp \, f(q, p) = Tr_{\mathcal{H}}\Big(\widehat{O_f}(\widehat{p}, \widehat{q})\Big). \tag{11}$$

Furthermore, another property is that in general for any complex field ϑ living in $\mathbb{R}^2_*$,

$$\int dx dy \, \overline{\vartheta} * \vartheta = \int dx dy \, \overline{\vartheta} \vartheta. \tag{12}$$

There are other convenient ways to work with fields in a noncommutative space. Let's parametrize $\mathbb{R}^2_*$ with complex coordinates

$$w = \frac{1}{\sqrt{2}}(x + iy) \quad \text{and} \quad \overline{w} = \frac{1}{\sqrt{2}}(x - iy)$$

and rescale them, so we are left with the following commutator:

$$[w, \overline{w}]_* = 1. \tag{13}$$

Notice that this is the analogous to the quantum SHO commutator: $[\widehat{a}, \widehat{a}^\dagger] = 1$, where $\widehat{a}$ is the annihilation operator and $\widehat{a}^\dagger$ the creation operator.

The above results can be easily generalized to the case where there are n pairs of noncommuting coordinates. In general, a field φ in

$$\mathcal{G}^{(p-2n)+1} \times \mathbb{R}^{2n}_*$$

can be expressed as[c]

$$\varphi(x^\mu) = \sum_{m,n} \varphi_{mn}(x^a) \Phi_{mn}(x^i), \tag{14}$$

where the $\Phi_{mn}(x^i)$ in $\mathbb{R}^{2n}_*$ are related to $|m\rangle \langle n|$ in

$$\mathcal{H}_n = \mathcal{H}(\widehat{q}_1, \widehat{p}_1) \oplus \cdots \oplus \mathcal{H}(\widehat{q}_n, \widehat{p}_n) \qquad (see [35]).$$

Further generalizations are overwhelmingly challenging, and beyond the scope of this work.

3. The $\mathrm{D(p-2)}-$brane as a noncommutative soliton

In Ref. 13 HLKM studied a process where a bosonic $D25-$brane decays into a $D23-$brane in the presence of a large $B-$field. They found a nontrivial solution to the real tachyon in the $D25-$brane. It was precisely the real GMS soliton [16] which they identified with the remnant of the $D23-$brane in the low energy limit. In this section, we will apply this idea to the complex gauge-coupled tachyon in the Type II $Dp - \overline{Dp}-$brane system with a constant and large background $B-$field [26].

3.1. $\mathrm{Dp} - \overline{\mathrm{Dp}}-$brane annihilation in the presence of a $\mathrm{B}-$field

Recall that the non-BPS $Dp - \overline{Dp}$ configuration is unstable because of the presence of a tachyon in its $(p+1)-$dimensional worldvolume. This tachyonic field has charge $+1$ under the group $U(1)$ with gauge fields A_μ and charge -1 under $\widetilde{U}(1)$ with gauge fields $\widetilde{A_\mu}$. Consider a constant background $B-$field of the form given in (5), so the worldvolume is $\Sigma_{p+1} = \mathcal{G}^{(p-2)+1} \times \mathbb{R}^2_*$. We will just focus on the case when the worldvolume metric is flat $G_{\mu\nu} = \eta_{\mu\nu}$, thus $\Sigma_{p+1} = \mathcal{M}^{(p-2)+1} \times \mathbb{R}^2_*$.

Parametrizing the $x - y$ plane with the complex variables $(w, \overline{w})$, and the commutative coordinates $\widetilde{x}^a$, the $(p+1)-$dimensional action is

$$S_t^{(p+1)} = \int_{\Sigma_{p+1}} d^{p-1}\widetilde{x} d^2 w \times \Big(\overline{D^\mu T} * D_\mu T - V_*(T, \overline{T})\Big), \tag{15}$$

where the covariant derivative is

$$D_\mu T = \partial_\mu T - i A_\mu * T + i \widetilde{A_\mu} * T.$$

Denoting $R_\mu = A_\mu - \widetilde{A_\mu}$, we are left with

$$S_t^{(p+1)} = \int_{\Sigma_{p+1}} d^{p-1}\widetilde{x} d^2 w \biggl((\partial^\mu \overline{T} + iR^\mu * \overline{T}) * (\partial_\mu T - iR_\mu * T) - V_*(T, \overline{T}) \biggr). \tag{16}$$

Our considerations apply to generic potentials $V(T, \bar{T})$, but we will, for definiteness, mostly discuss those of polynomial form

$$V_*(T, \overline{T}) = V_*(\overline{T} * T) = \sum_{k=1}^{n} a_k (\overline{T} * T)^k, \tag{17}$$

where, of course, we have abbreviated

$$(\bar{T} * T)^k = (\bar{T} * T) * (\bar{T} * T) * \cdots * (\bar{T} * T), \qquad (k - \text{times}) \tag{18}$$

and k is a positive integer.

Also, as in Ref. 16, in order for nontrivial solutions to exist, our potential $V_*(T, \bar{T})$ must have at least two local minima.

Let's now proceed and construct a simple solution. Recall that, in the absence of noncommutativity, we found a vortex solution (3) independent of $\widetilde{x}^a$. In the next section, we will be interested in a solution with the same spacetime dependence (as in the vortex case):

$$T = T(w, \overline{w}), \qquad \overline{T} = \overline{T}(w, \overline{w}). \tag{19}$$

The action is given by

$$S_t^{(p+1)} = \int_{\Sigma_{p+1}} d^{p-1}\widetilde{x} d^2 w \biggl(-\frac{1}{\Theta} \partial^{\overline{w}} \overline{T} * \partial_w T + \frac{i}{\sqrt{\Theta}} (R^\mu * \overline{T} * \partial_\mu T - \partial^\mu \overline{T} * R_\mu * T) + R^\mu * \overline{T} * R_\mu * T - V_* \biggr). \tag{20}$$

From now on, we will focus exclusively on the case of infinite noncommutativity, $\Theta \to \infty$, since in this limit, the kinetic term along the noncommutative plane vanishes. We could eventually introduce finite-Θ kinetic contributions in terms of a $\frac{1}{\Theta}-$expansion. However, for the time being, we want to keep the theory simple and focus more on the properties of the solitonic solutions that depend on the potential $V_*(\bar{T}, T)$. Thus, we are left with

$$S_t^{(p+1)} = \int_{\Sigma_{p+1}} d^{p-1}\widetilde{x} d^2 w \biggl(R^\mu * \overline{T} * R_\mu * T - V_*(\overline{T} * T) \biggr). \tag{21}$$

Defining the potential

$$\widetilde{V_*}(\overline{T}, T) = \overline{R^\mu T} R_\mu T - V_*(\overline{T} * T), \tag{22}$$

the following simple action in $\mathcal{M}^{(p-2)+1} \times \mathbb{R}^2_*$ is obtained:

$$S_t^{(p+1)} = \int_{\Sigma_{p+1}} d^{p-1}\widetilde{x} d^2 w \widetilde{V_*}(\overline{T}, T). \tag{23}$$

Notice that in the case when the gauge field and the tachyon $*-$commute, $\widetilde{V_*}(\bar{T}, T)$ reduces to a polynomial in $\bar{T} * T$. This is equivalent to assuming that the gauge fields are independent of the noncommutative coordinates, from which we can deduce that

$$R^\mu * \overline{T} * R_\mu * T = R^\mu \overline{T} * R_\mu T. \tag{24}$$

We will stick to this result in order to avoid complications, since we are interested on how noncommutativity affects the tachyon, not the gauge fields.

3.2. The complex GMS soliton

Now, we are ready to move on and find a nontrivial solution to the complex tachyon of the form (19). Let's rewrite (23) as

$$S_t^{(p+1)} = \int_{\mathcal{M}^{(p-2)+1}} d^{p-1}\widetilde{x} S_t^{(*)}, \tag{25}$$

where

$$S_t^{(*)} = \int_{\mathbb{R}^2_*} d^2 w \widetilde{V_*}(\overline{T}, T) \tag{26}$$

is the action along the noncommutative plane.

The equations of motion in $\mathbb{R}^2_*$ the above action yields are

$$\frac{\partial \widetilde{V_*}(\overline{T}, T)}{\partial T} = 0, \qquad \frac{\partial \widetilde{V_*}(\overline{T}, T)}{\partial \overline{T}} = 0. \tag{27}$$

We cannot use the same solution as in HKLM's real bosonic case because the tachyon is now charged [13]. Notice that, in the case of $\Theta = 0$, the solutions would simply solve to the following algebraic equations:

$$\frac{\partial \widetilde{V}(\bar{t}, t)}{\partial T} = R^\mu \bar{t} R_\mu + \sum_{k=2}^{n} k a_k (\bar{t} t)^{k-1} \bar{t} = 0,$$
$$\frac{\partial \widetilde{V}(\bar{t}, t)}{\partial \bar{t}} = R^\mu R_\mu t + \sum_{k=2}^{n} k a_k t (\bar{t} t)^{k-1} = 0, \tag{28}$$

where t is a scalar complex field. Such solutions are just constants in the commutative plane, $\mathbb{R}^2$.

We know that the introduction of noncommutativity allows more interesting solutions [16]. From (27), we see that the equations of motion in $\mathbb{R}^2_*$ are

$$\frac{\partial\widetilde{V_*}(\overline{T},T)}{\partial T}=R^\mu\overline{T}R_\mu+\sum_{k=2}^{n}ka_k(\overline{T}*T)^{k-1}*\overline{T}=0,\quad \frac{\partial\widetilde{V_*}(\overline{T},T)}{\partial \overline{T}}=R^\mu R_\mu T+\sum_{k=2}^{n}ka_k T*(\overline{T}*T)^{k-1}=0. \tag{29}$$

Let's construct a simple complex solution of the form

$$T=t_*T_0,\quad \overline{T}=\overline{t_*T_0}, \tag{30}$$

where T_0 and $\overline{T_0}$ have the following property:

$$(\overline{T_0}*T_0)*(\overline{T_0}*T_0)=(\overline{T_0}*T_0). \tag{31}$$

In the commutative case, we would not be able to construct a nontrivial function $\bar{T}_0*T_0$ that squares to itself. This is just one of the many amazing properties the $*-$product has. Let's now see what happens when we insert solution (30) into the equations of motion (29):

$$R^\mu(\overline{t_*T_0})R_\mu+\sum_{k=2}^{n}ka_k\Big((\overline{t_*T_0})*(t_*T_0)\Big)^{k-1}*(\overline{t_*T_0})=0,\ R^\mu R_\mu(t_*T_0)+\sum_{k=2}^{n}ka_k(t_*T_0)*\Big((\overline{t_*T_0})*(t_*T_0)\Big)^{k-1}=0. \tag{32}$$

Now, since t_* and $\overline{t_*}$ are simply constants under the $*-$product, they can be carried around the equations. Therefore, Eq. (32) can be rewritten as

$$R^\mu\overline{t_*}R_\mu\overline{T_0}+\sum_{k=2}^{n}ka_k(\overline{t_*}t_*)^{k-1}(\overline{T_0}*T_0)^{k-1}*(\overline{t_*T_0})=0,\ T_0R^\mu R_\mu t_*+\sum_{k=2}^{n}ka_kt_*(\overline{t_*}t_*)^{k-1}T_0*(\overline{T_0}*T_0)^{k-1}=0. \tag{33}$$

Next, $*-$multiply the first equation by T_0 on the right, and the second by $\overline{T_0}$ on the left, thereby obtaining

$$R^\mu\overline{t_*}R_\mu\overline{T_0}*T_0+\sum_{k=2}^{n}ka_k(\overline{t_*}t_*)^{k-1}(\overline{T_0}*T_0)^k\overline{t_*}=0,\ \overline{T_0}*T_0R^\mu R_\mu t_*+\sum_{k=2}^{n}ka_kt_*(\overline{t_*}t_*)^{k-1}(\overline{T_0}*T_0)^k=0. \tag{34}$$

Notice that from the property (31) we can deduce by iteration that

$$(\overline{T_0}*T_0)^k=(\overline{T_0}*T_0), \tag{35}$$

where k is a positive integer (*i.e.* the term $\bar{T}_0*T_0$ behaves like a projection operator).

Therefore, using the above result, we can rewrite (33) as

$$\Big(R^\mu\overline{t_*}R_\mu+\sum_{k=2}^{n}ka_k(\overline{t_*}t_*)^{k-1}\overline{t_*}\Big)\overline{T_0}*T_0=0,$$

$$\overline{T_0}*T_0\Big(R^\mu R_\mu t_*+\sum_{k=2}^{n}ka_kt_*(\overline{t_*}t_*)^{k-1}\Big)=0. \tag{36}$$

Since we are searching for nontrivial solutions, we know that when $\bar{T}_0*T_0\neq 0$, the following equations hold:

$$R^\mu\overline{t_*}R_\mu+\sum_{k=2}^{n}ka_k(\overline{t_*}t_*)^{k-1}\overline{t_*}=0,$$

$$R^\mu R_\mu t_*+\sum_{k=2}^{n}ka_kt_*(\overline{t_*}t_*)^{k-1}=0. \tag{37}$$

These are precisely the algebraic equations of motion (28) in the case of absent noncommutativity, with $t=t_*$ and $\bar{t}=\overline{t_*}$.

In summary, we found that the coefficients t_* and $\bar{t_*}$ of the solution that we have constructed are themselves solutions to the algebraic (commutative) equations:

$$\frac{\partial\widetilde{V}(\bar{t},t)}{\partial t}=0,\qquad \frac{\partial\widetilde{V}(\bar{t},t)}{\partial \bar{t}}=0. \tag{38}$$

Our task now is to find T_0 and $\overline{T_0}$ such that they satisfy $(\overline{T_0}*T_0)*(\overline{T_0}*T_0)=(\overline{T_0}*T_0)$. Notice that, via the WWM correspondence, we can associate the fields T_0 and $\overline{T_0}$ to the operators

$$\widehat{T_0}=i\widehat{P},\qquad \widehat{\overline{T_0}}=-i\widehat{P}, \tag{39}$$

in $\mathcal{H}(\widehat{a},\widehat{a}^\dagger)$, where $\widehat{P}$ is the projection operator $\widehat{P}=\widehat{P}^2$. In the SHO basis any projection operator may be expressed as $\widehat{P}=|n\rangle\langle n|$.

According to the WWM correspondence, the operator $|n\rangle\langle n|$ in $\mathcal{H}(\widehat{a},\widehat{a}^\dagger)$ is related in to the Wigner function $2(-1)^ne^{-r^2}L_n(2r^2)$ in $\mathbb{R}^2_*$, where $L_n(s)$ is a Laguerre polynomial [16]. It can be shown that the general solution is a linear combination of projection operators *(i.e.,* Wigner functions) with complex coefficients that minimize the commutative potential $\widetilde{V}(t,\bar{t})$. However, for the time being, we will only focus on the state in the lowest energy which is given by the Gaussian packet $T_0(r^2)=2e^{-r^2}$, where

$$r^2=x^2+y^2=w\overline{w}+\overline{w}w\quad\text{and}\quad L_0(s)=1.$$

Summarizing, from the complex tachyon in $\mathcal{M}^{(p-2)+1} \times \mathbb{R}^2_*$ we have a complex GMS soliton of the form

$$T(w,\overline{w}) = 2it_* e^{-r^2}, \overline{T}\,(w,\overline{w}) = -2i\overline{t_*} e^{-r^2}, \qquad (40)$$

where t_* and $\overline{t_*}$ minimize the algebraic equation

$$\widetilde{V}\,(t_*, \bar{t}_*) = 0. \qquad (41)$$

It is remarkable that the only information we need to know about the potential $\widetilde{V}$ is that it possesses at least two local minima, and the values of T and $\overline{T}$ for which these would be minimized if noncommutativity is absent.

This object may be interpreted as the low energy remnant of a stable $D(p-2)-$brane arising from the annihilation of the unstable non-BPS $Dp-\overline{Dp}-$brane pair in Type II theory.

In the case $p = 3$ in Type IIB theory, we coin the term noncommutative string for the resulting complex GMS soliton (*which is itself the low energy remnant of the $D-$string*).

In the following section, we are going to apply all the $\mathbb{R}^2_* \leftrightarrow \mathcal{H}(\widehat{p},\widehat{q})$ technology to obtain an effective theory along the noncommutative string from a theory with left-handed fermions in $\mathcal{M}^{1+1} \times \mathbb{R}^2_*$, and show that the conductivity on this object persists.

4. The noncommutative string in the presence of Fermions

In this second part of this work, we will show the existence of an analog to Witten's original superconducting string [9] in the context of noncommutative solitons and $D-$brane annihilations in string theory.

The idea is to begin with a $D3 - \bar{D}3-$brane configuration in Type IIB superstring theory and in the presence of a large background $B-$field turned on along the $x - y$ plane in the worldvolume. Such system is unstable and decays into a $D1-$brane, which has our complex GMS soliton as its low-energy remnant. The open string attached to the $D1-$brane has chiral fermions in its supersymmetric spectrum. This is because this spectrum is induced from the ten-dimensional Type IIB theory, which is chiral. Such fermions see the complex noncommutative soliton has a background field. By applying the WWM correspondence, we will integrate out the noncommutative coordinates and find an effective two-dimensional worldsheet theory along our $D-$string. In Sec. 4.1, for the sake of simplicity, we will first integrate the case where the gauge fields R_μ are absent. In Sec. 4.2 we introduce the gauge fields R_μ, which appear as a "mass" term in the effective theory. The bosonization technique is used in Sec. 4.3 to display superconductivity.

4.1. Free Fermions in $\mathcal{M}^{1+1} \times \mathbb{R}^2_*$

In $\mathcal{M}^{1+1} \times \mathbb{R}^2_*$ we can express left-handed Dirac spinors as

$$\Psi = \begin{pmatrix} 0 \\ \psi_L, \end{pmatrix} \qquad (42)$$

where ψ_L is a two-component spinor obeying the Weyl equation $\overrightarrow{\sigma} \cdot \widehat{p}\psi_L = -\psi_L$. In the above equations, $\widehat{p} = \overrightarrow{p}/|\overrightarrow{p}|$, where $\overrightarrow{p}$ is the spatial part of the fermion's momentum. Also, $\overrightarrow{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$, where σ^i are the well-known Pauli matrices. Thus,

$$\overrightarrow{\sigma} \cdot \overrightarrow{p} = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}. \qquad (43)$$

In four dimensions, the free fermions satisfy the massless Dirac equation

$$i\partial\!\!\!/\Psi = 0, \qquad (44)$$

where

$$\partial\!\!\!/ = \gamma^\mu \partial_\mu, \text{ and } \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \; \gamma^0 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix},$$

are the Dirac matrices and where σ^0 is a 2×2 unit matrix[d]. These matrices satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

Define the chirality operator

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}.$$

We can define a left-handed spinor

$$\Psi_L = \frac{1}{2}(\widehat{1} - \gamma^5)\Psi = \begin{pmatrix} 0 \\ \psi_L \end{pmatrix},$$

where $\widehat{1}$ is the 4×4 unit matrix and ψ_L obeys the chirality equation: $\gamma^5\psi_L = -\psi_L$. With this background, we now are ready to introduce the noncommutative string defined in (40).

The action for fermions in the presence of this object has the following generic form of Yukawa couplings [35]

$$S_f^{(4)} = \int dtdzd^2w \bigg(f(\overline{T}) * \overline{\Psi} * g(T) * \gamma^\mu \partial_\mu \Psi \bigg), \qquad (45)$$

where f and g are polynomials similar to (17), which play the role of fermion-soliton coupling. Therefore, using (31), we find that

$$f(\overline{T}) = f(\overline{t_*})\overline{T_0}, \qquad g(T) = g(t_*)T_0. \qquad (46)$$

Now, we know

$$\overline{\Psi} = \Psi^\dagger\gamma^0 = \begin{pmatrix} 0 \\ \psi_L \end{pmatrix}^\dagger \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix} = (\overline{\psi_L}, 0).$$

Thus, the action (45) can be reexpressed as

$$S_f^{(4)} = \int dtdzd^2w f(\overline{t_*})g(t_*) \times \bigg(\overline{T_0} * (\overline{\psi_L}, 0) * T_0 * \gamma^\mu \partial_\mu \begin{pmatrix} 0 \\ \psi_L \end{pmatrix} \bigg). \qquad (47)$$

In rescaled units of the noncommutativity parameter Θ, Dirac operator is written

$$\gamma^\mu \partial_\mu = \gamma^a \partial_a - \frac{1}{\sqrt{\Theta}} \gamma^\alpha \partial_\alpha. \qquad (48)$$

In the limit $\Theta \to \infty$, we get

$$\gamma^\mu \partial_\mu = \gamma^a \partial_a, \tag{49}$$

and

$$S_f^{(4)} = \int dtdzd^2w f(\overline{t_*})g(t_*) \times \left[\overline{T_0} * (\psi_L, 0) * T_0 * (\gamma^0\partial_0 - \gamma^3\partial_3) \begin{pmatrix} 0 \\ \psi_L \end{pmatrix}\right].$$

Applying the WWM correspondence and recalling the trace formula (11), let's rewrite the action above as

$$S_f^{(4)} = 2\pi\Theta f(\overline{t_*})g(t_*) \int dtdz S_f^{(*)}, \tag{50}$$

where the action along the noncommutative coordinate plane (written in terms of the two-component spinors) is

$$S_f^{(*)} = Tr\left\{\widehat{\overline{T_0}}(\widehat{\overline{\psi_L}}, 0)\widehat{T_0}\left[\begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \partial_0\widehat{\psi_L} \end{pmatrix} + \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \partial_3\widehat{\psi_L} \end{pmatrix}\right]\right\}$$
$$= Tr\left[(\widehat{\overline{T_0}}\,\widehat{\overline{\psi_L}}, \widehat{\overline{T_0}}\,\widehat{\overline{\psi_R}})\begin{pmatrix} \widehat{T_0}\sigma^0\partial_0\widehat{\psi_L} + \widehat{T_0}\sigma^3\partial_3\widehat{\psi_L} \\ 0 \end{pmatrix}\right]. \tag{51}$$

This may be rewritten as

$$S_f^{(*)} = Tr\left(\widehat{\overline{T_0}}\,\widehat{\overline{\psi_L}}\widehat{T_0}\sigma^0\partial_0\widehat{\psi_L} + \widehat{\overline{T_0}}\,\widehat{\overline{\psi_L}}\widehat{T_0}\sigma^3\partial_3\widehat{\psi_L}\right). \tag{52}$$

Now, using (14) and the WWM correspondence, in the SHO basis, we expand

$$\widehat{\psi}_L(x^\mu) = \sum_{m,n\geq 0} \psi^L_{mn}(z,t)\,|m\rangle\,\langle n|\,. \tag{53}$$

Indeed, having obtained this:

$$\widehat{T_0} = i\,|0\rangle\,\langle 0|\,, \qquad \widehat{\overline{T_0}} = -i\,|0\rangle\,\langle 0|\,, \tag{54}$$

we are in the position to calculate the trace of a generic term of the form $\widehat{\overline{T_0}}\widehat{\overline{\psi_L}}\widehat{T_0}D\widehat{\psi_L}$, where D is a 2×2 matrix differential operator. Thus,

$$Tr\left(\widehat{\overline{T_0}}\,\widehat{\overline{\psi_L}}\widehat{T_0}D\widehat{\psi_L}\right) = Tr\left[(-i\,|0\rangle\,\langle 0|)(\sum_{m,n\geq 0}\overline{\psi^L_{mn}}\,|m\rangle\,\langle n|)(i\,|0\rangle\,\langle 0|)D(\sum_{r,s\geq 0}\psi^L_{rs}\,|r\rangle\,\langle s|)\right]$$
$$= Tr\left(|0\rangle \sum_{m,n,r,s\geq 0}(\overline{\psi^L_{mn}}D\psi^L_{rs})(\langle 0\mid m\rangle\,\langle n\mid 0\rangle\,\langle 0\mid r\rangle\,\langle s|)\right). \tag{55}$$

In the process, we have used the fact that the kets $|n\rangle$ form a complete orthonormal basis which, by definition, satisfy $\langle m\mid n\rangle = \delta_{mn}$. Also, each ket $|n\rangle$ is applied into a one-dimensional subspace of the Hilbert space. This means that $Tr_{\mathcal{H}}(|m\rangle\,\langle n|) = \delta_{mn}$. Applying these facts, we deduce that

$$Tr\left(\widehat{\overline{T_0}}\,\widehat{\overline{\psi}}_L\widehat{T_0}D\widehat{\psi_L}\right) = \sum_{m,n,r,s\geq 0}\left((\overline{\psi^L_{mn}}D\psi^L_{rs}\delta_{0m}\delta_{n0}\delta_{0r})Tr(|0\rangle\,\langle s|)\right) = \overline{\psi^L_{00}}D\sum_{s\geq 0}\psi^L_{0s}\delta_{0s} = \overline{\psi^L_{00}}D\psi^L_{00}. \tag{56}$$

With this result, the action on the noncommutative plane is

$$S_f^{(*)} = \overline{\psi^L_{00}}\sigma^0\partial_0\psi^L_{00} + \overline{\psi^L_{00}}\sigma^3\partial_3\psi^L_{00}. \tag{57}$$

In the performing of the trace, we actually integrated out w and $\overline{w}$. Also, notice how the properties of the projection operators T_0 and $\bar{T}_0$ have "projected out" most of the $\psi^L_{mn}(z,t)'s$, leaving behind just the $\psi^L_{00}(z,t)$ term in the effective two-dimensional theory along the noncommutative $D-$string. Therefore, the left-handed fermionic action along the noncommutative string is

$$S_f^{(ncs)} = S_f^{(4)} = 2\pi\Theta f(\overline{t_*})g(t_*) \times \int dtdz(\overline{\psi^L_{00}}\sigma^0\partial_0\psi^L_{00} + \overline{\psi^L_{00}}\sigma^3\partial_3\psi^L_{00}). \tag{58}$$

This is precisely the localization of chiral fermions on the $D1-$string, done with the techniques utilized in Ref. 35. In the present case the chiral fermions are localized on the noncommutative $D-$string.

From this point on, we shall avoid the use of unnecessary subindices, since these yield no information when the effective theory on $\mathcal{M}^{1+1}$ is studied. Thus, we will simply use

$$\psi_{00}^L(z,t) = \psi^L(z,t). \tag{59}$$

Thus, the effective action for left-handed fermions along the string is

$$S_f^{(ncs)} = 2\pi\Theta f(\overline{t_*})g(t_*)\int dtdz\big(\overline{\psi}^L\sigma^a\partial_a\psi^L\big). \tag{60}$$

It is time to move on and generalize this result to the case when gauge fields are turned on.

4.2. $\mathbf{U(1)}\times\widetilde{\mathbf{U}}(\mathbf{1})$ Gauge-coupled Fermions in $\mathcal{M}^{\mathbf{1+1}}\times\mathbb{R}^{\mathbf{2}}_*$

The appearance of gauge fields arising from the Chan-Paton factors should supply with further properties characterizing the noncommutative string. The introduction of this gauge field merely amounts, as usual, to minimally coupling fermions to

$$R^\mu : \partial_\mu\Psi \rightarrow D_\mu\Psi = (\partial_\mu - iR_\mu)\Psi.$$

So, the four-dimensional action for gauge-coupled fermions in the presence of the noncommutative string is

$$S_{gauge}^{(ncs)} = \int dtdzd^2w\Big(f(\overline{T})*\overline{\Psi}*g(T)\not{D}\Psi\Big) = \int dtdzd^2w\Big(f(\overline{T})*\overline{\Psi}*g(T)*\gamma^\mu(\partial_\mu\Psi - iR_\mu*\Psi)\Big), \tag{61}$$

which may be written as

$$S_{gauge}^{(ncs)} = S_{f1}^{(ncs)} + S_{f2}^{(ncs)}, \tag{62}$$

where $S_{f1}^{(ncs)}$ is given by (45) and

$$S_{f2}^{(ncs)} = -i\int dtdzd^2w\Big(\gamma^\mu R_\mu f(\overline{T})*\overline{\Psi}*g(T)*\Psi\Big) \tag{63}$$

is the contribution due to the presence of the gauge fields[e].

Applying the WWM correspondence, the action (63) is written as

$$\begin{aligned} S_{f2}^{(ncs)} &= -2\pi i\Theta f(\bar{t}_*)g(t_*)\int dtdzTr\Big(\widehat{\overline{T_0}}\,\widehat{\overline{\Psi}}\widehat{T_0}\gamma^\mu R_\mu\widehat{\Psi}\Big) \\ &= -2\pi\Theta if(\overline{t_*})g(t_*)\int dtdzS_{f2}^{(*)}, \end{aligned} \tag{64}$$

where

$$S_{f2}^{(*)} = Tr\Big(\widehat{\overline{T_0}}\,\widehat{\overline{\Psi}}\widehat{T_0}\gamma^\mu R_\mu\widehat{\Psi}\Big) \tag{65}$$

is the gauge-field contribution to the action on the noncommutative plane.

Using the relation

$$\gamma^\mu R_\mu = \begin{pmatrix} 0 & \sigma^0 R_0 \\ \sigma^0 R_0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^i R_i \\ -\sigma^i R_i & 0 \end{pmatrix}, \tag{66}$$

we rewrite (65) as

$$\begin{aligned} S_{f2}^{(*)} &= Tr\left\{\widehat{\overline{T_0}}(\widehat{\overline{\psi_L}},0)\left[\begin{pmatrix} 0 & \widehat{T_0}\sigma^0R_0 \\ \widehat{T_0}\sigma^0R_0 & 0\end{pmatrix} - \begin{pmatrix} 0 & \widehat{T_0}\sigma^iR_i \\ -\widehat{T_0}\sigma^iR_i & 0\end{pmatrix}\right]\begin{pmatrix} 0 \\ \widehat{\psi_L}\end{pmatrix}\right\} \\ &= Tr\left[(\widehat{\overline{T_0}}\widehat{\overline{\psi_L}},\widehat{\overline{T_0}}0)\begin{pmatrix} \widehat{T_0}\sigma^0R_0\widehat{\psi_L} - \widehat{T_0}\sigma^iR_i\widehat{\psi_L} \\ 0\end{pmatrix}\right], \end{aligned} \tag{67}$$

or equivalently as

$$S_{f2}^{(*)} = Tr\Big(\widehat{\overline{T_0}}\widehat{\overline{\psi_L}}\widehat{T_0}\sigma^0R_0\widehat{\psi_L} - \widehat{\overline{T_0}}\widehat{\overline{\psi_L}}\widehat{T_0}\sigma^iR_i\widehat{\psi_L}\Big). \tag{68}$$

Recalling (53) and (39), let's now calculate the trace of a generic term of the form $\widehat{\overline{T_0}}\widehat{\overline{\psi}}_L\widehat{T_0}R\widehat{\psi}_L$, where R is a 2×2 matrix field independent of w and $\overline{w}$.

After some computations similar to those of the previous section, we are left with

$$Tr\Big(\widehat{\overline{T_0}}\,\widehat{\overline{\psi}}_L\widehat{T_0}R\widehat{\psi}_L\Big) = \overline{\psi_{00}^L}R\psi_{00.}^L$$

Therefore, the gauge field contribution to the action along the noncommutative plane is

$$S_{f2}^{(*)} = \overline{\psi_{00}^L}\sigma^0R_0\psi_{00}^L - \overline{\psi_{00}^L}\sigma^iR_i\psi_{00}^L. \tag{69}$$

Having integrated out the coordinates w and $\overline{w}$, the gauge-field contribution to the action along the noncommutative string is

$$S_{f2}^{(ncs)} = -2\pi i\Theta f(\overline{t_*})g(t_*) \times \int dtdz(\overline{\psi_{00}^L}\sigma^0 R_0\psi_{00}^L - \overline{\psi_{00}^L}\sigma^i R_i\psi_{00}^L). \quad (70)$$

Getting rid of unnecessary subindices, we have

$$S_{f2}^{(ncs)} = -2\pi i\Theta f(\overline{t_*})g(t_*)\int dtdz\big[\overline{\psi}^L\sigma^\mu R_\mu\psi^L\big]. \quad (71)$$

Since $\sigma^\mu R_\mu = \sigma^a R_a - \sigma^\alpha R_\alpha$ and we may define the matrix mass parameter $m(z,t) \equiv \sigma^\alpha R_\alpha$. Therefore, Eq. (71) can be expressed as

$$S_{f2}^{(ncs)} = -2\pi\Theta i f(\overline{t_*})g(t_*) \times \int dtdz\bigg(i\overline{\psi}^L\sigma^a R_a\psi^L - \overline{\psi}^L m\psi^L\bigg). \quad (72)$$

In conclusion, the complete gauge-coupled action along the string is

$$S_{gauge}^{(ncs)} = -2\pi\Theta i f(\overline{t_*})g(t_*) \times \int dtdz\bigg(i\overline{\psi}^L\sigma^a D_a\psi^L - \overline{\psi}^L m\,\psi^L\bigg), \quad (73)$$

where

$$D_a\psi_L(z,t) = (\partial_a - iR_a)\psi_L(z,t).$$

With all these tools, we are ready to calculate the current along this object.

4.3. The Bosonization technique: superconductivity

For the time being, we will focus in massless case; *i.e.*,

$$R_1 = R_2 = 0 \quad \text{and} \quad R_a \neq R_a(z), \quad a = 0,3.$$

First of all, let's rescale the action of gauge-coupled fermions along the noncommutative string such that the coefficient outside the integral is set equals to one. So, upon reintroducing the gauge potential

$$S_{gauge}^{(ncs)} = \int dzdt\bigg(i\overline{\psi_L}\sigma^a D_a\psi_L\bigg). \quad (74)$$

In any theory with fermions in two dimensions, we can equivalently use bosons or fermions by applying the technique of bosonization. The idea is to introduce a scalar field $\zeta(z,t)$ living on the noncommutative string:

$$\overline{\psi_L}\sigma^a\psi_L = \frac{1}{\sqrt{\pi}}\varepsilon^{ab}\partial_b\zeta. \quad (75)$$

Thus according to Ref. 9, a two-dimensional kinetic term is

$$i\overline{\psi_L}\sigma^a D_a\psi_L = \frac{1}{2}(\partial_a\zeta)(\partial^a\zeta) - \frac{1}{\sqrt{\pi}}R_a\varepsilon^{ab}\partial_b\zeta, \quad (76)$$

which yield the following equation of motion:

$$\partial_a\partial^a\zeta + \frac{1}{\sqrt{\pi}}E = 0, \quad (77)$$

where $E = \varepsilon^{ab}\partial_a R_b$ is the electric field in two dimensions.

Now, the conserved current is just $J^a = -\overline{\psi_L}\sigma^a\psi_L$, which means that from Eq. (75) that $J^3 = -(1/\sqrt{\pi})\dot{\zeta}$. From (77) and the $z-$independence of R_a we see that $\ddot{\zeta} = -(1/\sqrt{\pi})E$. Thus, we get for J^3 (the current along the string) that

$$\frac{dJ^3(z,t)}{dt} = \frac{1}{\pi}E. \quad (78)$$

This equation means that the string is superconducting. If an electric field E is applied for some time T a current ET/π remains even is the electric field is turned off after time T.

For a regular wire of finite conductivity σ, the current is $J^3 = \sigma E^3$ (where E^3 the component of the electric field along the string) and vanishes after a certain characteristic time if E^3 is turned off. The situation for our noncommutative string is quite similar to the Witten's superconducting string.

Conservation of the fermionic current could be related to the conservation of some fermionic numbers, such as the lepton and baryon numbers of the theory on the brane. It would be very interesting to construct specific brane configurations of intersecting branes which reproduces Standard Model and some GUT's with superconducting noncommutative $D-$strings. Here the fermionic conserved current will be directly related to the fermionic quantum numbers of baryon and lepton numbers of the underlying reproduced models. This will be reported in a forthcoming communication.

5. Final remarks

In the present paper, $D-$brane annihilation and noncommutativity were merged together to obtain a new object: a noncommutative string with nondecaying conductivity.

The necessary constituents to construct this entity were all present in the Type IIB superstring theory. By rotating one of them an angle π in the transverse directions, we turned it into a $\overline{Dp}-$brane. The result was a non-BPS $Dp-\overline{Dp}-$brane system, which is unstable due to the presence of a tachyon in its worldvolume. On the other hand, the NS-NS sector gave rise to the ubiquitous background $B-$field, which played a pivotal role in the introduction of noncommutativity.

The predominant approach to such an annihilation has been to find a vortex-like configuration of the tachyon field, thereby obtaining a stable BPS $D(p-2)-$brane as the result. The tachyon in the $Dp-\overline{Dp}-$brane worldvolume is charged under the gauge group $U(1)\times\widetilde{U}(1)$ arising from the Chan-Paton factors on each $D-$brane. Assuming we have a flat metric, we introduce a constant $B-$field along two spatial directions. In the low-energy limit one obtain an effective noncommutative theory where the fields are Moyal

$*-$multiplied. From here on, we generalize the work of GMS to the case where the field is complex and gauge-coupled.

For definiteness, we only discuss potentials of polynomial form (17). Also, since we are only interested in how noncommutativity acts on the tachyon, we assume that it does not affect the gauge fields and that Eq. (24) is satisfied. This merely amounts to redefining the potential to

$$\widetilde{V}(T,\bar{T}) = R^{\mu}T * R_{\mu}T,$$

which is itself also a polynomial in $\bar{T} * T$.

With this result, we show the natural existence of an object analogous of Witten's superconducting string [9], in the context of noncommutative soliton theory. By making use of the WWM correspondence, we find that the noncommutative $D-$string in the large noncommutativity limit ($\Theta \to \infty$) is completely specified by Eq. (40).

Starting with Type IIB theory $D3 - \overline{D3}$ annihilation with a $B-$field turned on along the $x-y$ plane, the complex GMS is the remnant of a BPS $D-$string. From the localization of the chiral fermions ψ_L in the supersymmetric spectrum of the open sector (in the sense of Ref. 35), we may construct a two-dimensional effective description of the fermionic degrees of freedom along the commutative coordinates (z,t). This is done by integrating out the two noncommutative transverse coordinates $(w,\overline{w})$ and exploring the soliton's projector properties. Although we could have calculated the current directly, we used the bosonization technique for simplicity. The open string sector allows fermionic states in the worldvolume $\mathcal{M}^{1+1} \times \mathbb{R}^2_*$. We find that, by obtaining an equation of the type (78), the conserved current is a persistent one.

Future subsequent work might include the use of the bosonization technique to explore more types of phenomena, such as light scattering by the noncommutative $D-$string (see [9]). Also we are interested in the construction of intersecting brane configurations, thereby reproducing the Standard Model and some GUT's containing noncommutative superconducting $D-$strings. In such cases, it might be possible to identify the existence of conserved fermionic superconducting current with that of conserved lepton and baryon quantum numbers. Likewise, we could make some progress in including finite-Θ effects and generalizing to the case when the gauge fields get affected by noncommutativity. Another issue to be consider is to explore the stability of our solution. Some of these issues are currently under investigation.

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a. In the general case $\mathcal{M}^{n+1}$ denotes a Minkowski space with one timelike and n spacelike dimensions, while $\mathbb{R}^s_*$ denotes an $s-$dimensional noncommutative space.

b. A similar situation is studied in Ref. 9. However, in that work $U(1)$ is spontaneously broken to give rise to the string and the other fields in $\widetilde{U}(1)$ make the string superconducting.

c. x^a live in $\mathcal{G}^{q+1}$ which is a manifold that reduces to the Minkowski space $\mathcal{M}^{q+1}$ as the metric G_{ab} goes to a flat metric η_{ab}. Likewise, x^i live in a $2n-$dimensional noncommutative space $\mathbb{R}^{2n}_* = \mathbb{R}^2_* \times \cdots \times \mathbb{R}^2_*$ ($n-$times).

d. Notation: In this section, we will denote the indices $\mu,\nu,\cdots = 0,1,2,3$; $i,j,\cdots = 1,2,3$; $a,b,\cdots = 0,3$ (commuting coordinates) and $\alpha,\beta,\cdots = 1,2$ (noncommuting coordinates).

e. Unlike 53, there is no need to expand R_μ, because it is constant on the noncommutative plane. This condition is equivalent to saying that R_μ and the tachyon commute [see Eq. 24].

1. M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory*, vols I and II (Cambridge: Cambridge University Press, 1987).
2. J. Polchinski, *String Theory*, vols I and II (Cambridge: Cambridge University Press, 1998).
3. D. Lüst and S. Theisen, *Lectures on String Theory* (Berlin: Springer-Verlag, 1989).
4. J. Polchinski, "TASI Lectures on D-Branes," hep-th/9611050.
5. C.V. Johnson, "D-Brane Primer," hep-th/0007170.
6. A. Vilenkin and E.P.S. Shellard,*Cosmic Strings and Other Topological Defects* (Cambridge: Cambridge University Press 1994).
7. G. t' Hooft, "Monopoles, Instantons and Confinement," hep-th/0010225.
8. H. Kleinert, *Gauge Fields in Condensed Matter* (World Scientific: 1989).
9. E. Witten, *Nucl. Phys. B* **249** (1985) 557.
10. A. Sen, "Non-BPS States and Branes in String Theory", hep-th/9904207; J.H. Schwarz, "TASI Lectures on Non-BPS D-Brane Systems", hep-th/9908144; A. Lerda and R. Russo, "Stable non-BPS States in String Theory, hep-th/9905006.
11. M.R. Douglas and N.A. Nekrasov, "Noncommutative Field Theory", hep-th/0106048.
12. R.J. Szabo, "Quantum Field Theory and Noncommutative Spaces", hep-th/0109162.
13. J. Harvey, P. Kraus, F. Larsen, and E.J. Martinec, *JHEP* **0007** (2000) 042 (hep-th/0005031).
14. A.S. Gorsky, Y.M. Makeenko and K.G. Selivanov, *Phys. Lett. B* **492** (2000) 344 (hep-th/0007247).
15. J. Harvey, P. Kraus and F. Larsen, *JHEP* **0012** (2000) 024 (hep-th/0010060).
16. R. Gopakumar, S. Minwalla, and A. Strominger, "Noncommutative Solitons," JHEP **0005** (2000) 020, hep-th/0003160.
17. K. Dasgupta, S. Mukhi and G. Rajesh, *JHEP* **0006** (2000) 022 (hep-th/0005006).

18. E. Witten, "Noncommutative Tachyons and String Field Theory", hep-th/0006071.
19. C. Sochichiu, *JHEP* **08** (2000) 026 (hep-th/0007217).
20. R. Gopakumar, S. Minwalla and A. Strominger, "Symmetry Restoration and Tachyon Condensation in Open String Theory", hep-th/0007226.
21. N. Seiberg, *JHEP* **09** (2000) 003 (hep-th/0008013).
22. A. Sen, "Some Issues in Noncommutative Tachyon Condensation", hep-th/0009038.
23. E. Witten, "Overview of K-Theory Applied to Strings", hep-th/0007175.
24. J.A. Harvey and G. Moore, "Noncommutative Tachyons and K-Theory", hep-th/0009030.
25. E.J. Martinec and G. Moore, "Noncommutative Tachyons on Orbifolds", hep-th/0101199.
26. J.A. Harvey, "Komaba Lectures on Noncommutative Solitons and D-branes", hep-th/0102076.
27. R. Tatar, "A Note on Noncommutative Field Theory and Stability of Brane-Antibrane Systems", hep-th/0009213.
28. P. Kraus, A. Rajaraman and S. Shenker, *Nucl. Phys. B* **598** (2001) 169 (hep-th/0010016).
29. K. Dasgupta and Z. Yin, "Nonabelian Geometry", hep-th/0011034.
30. C.M. Zachos, "Deformation Quantization: Quantum Mechanics Lives and Works in Phase-Space", hep-th/0110114.
31. S. de Alwis, *Phys. Lett. B* **461** (1999) 329 (hep-th/9905080).
32. N. Seiberg and E. Witten, *JHEP* **09** (1999) 032 (hep-th/9908142).
33. V. Schomerus, *JHEP* **9906** (1999) 030 (hep-th/9903205).
34. H. García-Compeán, J.F. Plebański, M. Przanowski, and F.J. Turrubiates, *J. Phys. A* **33** (2000) 7935 (hep-th/0002212).
35. L. Pilo and A. Riotto, *JHEP* **03** (2001) 015 (hep-ph/0012174).