Efficiency of a Curzon and Ahlborn engine with Dulong-Petit heat transfer law

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Using the maximization of the power output per cycle, the optimization of a thermal engine performing a Carnot-type cycle is considered. It is assumed that the heat transfer between the reservoirs and the engine occurs according to the Dulong and Petit's heat transfer law. It is found that the efficiency obtained with this heat transfer law can be written as a power series in the parameter $\lambda \sim 1/(\ln V_{max} - \ln V_{min})$, where V_{max} and V_{min} are the maximum volume and minimum volume spanned by the cycle, respectively. It is also shown that the calculated efficiency verifies the semi-sum property of the ecological efficiency.

Keywords: Finite time thermodynamics; optimization; power output.

Se considera la optimización de una máquina térmica por medio de un ciclo tipo Carnot, usando la maximización de la potencia de salida en el ciclo. Se supone que la transferencia de calor entre los almacenes y la máquina se realiza de acuerdo con la ley de transferencia de calor de Dulong y Petit. Se encuentra que la eficiencia obtenida con esta ley de transferencia de calor, se puede escribir como una serie de potencias del parámetro $\lambda \sim 1/(\ln V_{max} - \ln V_{min})$, donde V_{max} y V_{min} son los volúmenes máximo y mínimo subtendidos por el ciclo, respectivamente. También se muestra que la eficiencia calculada cumple la propiedad de semi-suma de la eficiencia ecológica.

Descriptores: Termodinámica de tiempos finitos; optimización; potencia de salida.

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1. Introduction

Classical equilibrium thermodynamics has been very important in the study of thermal engines; its main role in thermal engine analysis has consisted in providing upper bounds for process variables such as efficiency, work, heat and others. However, the classical equilibrium thermodynamics bounds are usually far away from typical real values. Moreover, the problem of taking a system from a given initial state to a given final state while producing a minimum of entropy or a minimum loss of availibility leads to reversible processes. These processes are equivalent to each other and have zero value of both entropy or loss availability, but need infinitely long process time. In many applications it is natural to introduce a constraint for the available process time. This approach is known as finite time thermodynamics and discussed, *e.g.*, [1,2,3]. In this contex, endoreversible processes [4] are generally considered, where the system internally is reversible and the production of entropy is caused by the transport to the system, or from the system. So that, endoreversible thermodynamics can be considered as an extension of classical equilibrium thermodynamics to include irreversible processes[1,2]. A typical endoreversible system is the so-called Curzon and Ahlborn engine[3] (see Fig. 1). This heat engine is a Carnot-type cycle in which there is no thermal equilibrium between the working fluid and the reservoirs, at the isothermal branches of the cycle; furthermore, there exists a finite time heat transfer given by Newton's heat transfer law. The engine is a non-null power model, in contrast with all reversible models which are zero-power models. The curzon and Ahlborn cycle is applied to thermal engines, to include the irreversible processes of the interaction of the engine with the reservoirs while it is maintained the thermodynamic equilibrium within the working substance. The expression for the power output associated with the Curzon and Ahlborn cycle corresponds to a convex function with only a maximum point into the interval $0 < \beta < 1$, with $\beta = T_2/T_1$; such that the efficiency of this cycle at maximum power regime is [3],

$$\eta_{CA} = 1 - \sqrt{\frac{T_2}{T_1}}\,, \tag{1}$$

where T_2 and T_1 are the temperatures of the cold and hot reservoirs respectively.

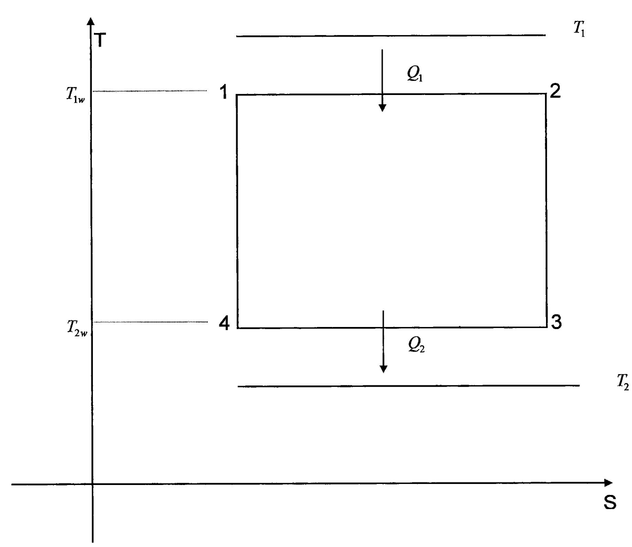

FIGURE 1. Curzon and Ahlborn Cycle in the entropy S and absolute temperature T plane. Q_1 represents the absorbed heat at T_1 and Q_2 represents the rejected heat at T_2.

The success of Eq. (1) lies in the fact that it gives a reasonable prediction for the reported experimental values of efficiency for a certain number of power plants (see table 1 of Ref. [3]), in contrast with the values obtained by the Carnot efficiency

$$\eta_C = 1 - \frac{T_2}{T_1}. \tag{2}$$

When the temperatures of the working substance in the isothermal branches (Fig. 1), T_{iw}, $i = 1, 2$, with $T_{2w} < T_{1w}$, are taken to be the same than the temperatures of the reservoirs, $T_{1w} = T_1$ and $T_{2w} = T_2$, the irreversibility of the heat transfer process between the engine and the reservoirs is ignored. Since the publication of the Curzon and Ahlborn paper, Eq. (1) has been obtained in several different ways[1,4,5,6], and other methods to qualify the performance of the Curzon and Ahlborn engine have been proposed. Particularly Angulo-Brown[7] advanced an optimization criterion that combines the power output of the cycle P and the total entropy production σ by the so-called ecological function $E = P - T_2\sigma$, obtaining the ecological eficiency η_E, namely,

$$\eta_E = 1 - \sqrt{\frac{\beta^2 + \beta}{2}},$$

which has the following property:

$$\eta_E \approx \frac{1}{2}(\eta_C + \eta_{CA}), \tag{3}$$

where η_{CA} and η_C are given by the Eq. (1) and (2), respectively.

On the other hand, Gutkowicks-Krusin, Procaccia and Ross[8] have derived Eq. (1) as the upper bound of the efficiency as a function of the ratio of the maximum volume V_{max} and the minimum volume V_{min} spanned by the heat engine through the quantity $\lambda \sim 1/(\ln V_{max} - \ln V_{min})$.

Similary, for the criterion of merit named ecological [7], Ladino-Luna and de la Selva[9] have shown that the ecological efficiency is also expressed as the upper bound of the power series of the same parameter λ of Ref. 8.

The Ref. 8 and 9 have assumed that the time for the adiabatic branches, in the Curzon and Ahlborn engine, can be written as a function of the time obtained from the duration of the isothermal branches by means of the heat transfer law used. In all of the above quoted calculations Newton's heat transfer law was considered in order to calculate the time of duration of the processes of heat transfer between the engine and the reservoirs. Using a non linear heat transfer law, particulary the Dulong and Petit's heat transfer law, Arias-Hernández and Angulo-Brown [10] and Angulo-Brown and Páez-Hernández [11] studied and obtained numerical results that suggest that expression (3) is a general property.

In the present work, with the optimization of the power output of a Curzon and Ahlborn engine, an approximate expression for the ecological efficiency by means of the Dulong and Petit's heat transfer law [12] is derived, corresponding to the zeroth order term in a power series in the parameter λ cited above. For this approximate expression, the equivalent expression to (3) is also derived. Furthermore, numerical results are compared with the numerical results of Refs. 10 and 11. The times for the isothermal branches and adiabatic branches are taking similary as the times in Refs. 8 and 9 to build the expression of power output.

2. Efficiency at maximum power output

From the definition of efficiency $\eta \equiv 1 - Q_2/Q_1$, the endoreversibility condition, namely $Q_1/T_{1w} = Q_2/T_{2w}$, makes the form of efficiency $\eta = 1 - T_{2w}/T_{1w}$ which is applicable to any endoreversible cycle. However, the ratio of temperatures changes with the maximized function and with the assumed heat transfer law in an endoreversible engine model. In the following three cases: the Carnot efficiency [Eq. (2)], the Curzon and Ahlborn efficiency [Eq. (1)] and the so-called ecological efficiency [Eq. (3)], the efficiency always depends on a function $z(\beta)$, with $\beta = T_2/T_1$, as

$$\eta = 1 - z(\beta). \tag{4}$$

Thus, the idea is to find the function $z(\beta)$ that follows from the maximization of the function that represents the power output of the engine, $P(\beta)$, and then to substitute it in Eq. (4) to obtain the corresponding efficiency for the heat tranfer law assumed. Let us consider a gas in a cylinder with a piston as the working substance that exchanges heat with the reservoirs, and let us use a heat transfer law of the form:

$$\frac{dQ}{dt} = \alpha(T_f - T_i)^k, \tag{5}$$

where $k > 1$, and α is the thermal conductance, dQ/dt is the rate of heat Q exchanged and T_f and T_i are the temperatures for the heat exchange process considered. The first law of thermodynamics in its local form, applied to the system, is

$$\frac{dE}{dt} = \frac{dQ}{dt} - p\frac{dV}{dt},$$

and assuming an ideal gas as the working substance of the cycle, one obtains

$$\frac{dQ}{dt} = p\frac{dV}{dt}, \quad \text{or} \quad \alpha(T_f - T_i)^k = \frac{RT_i}{V}\frac{dV}{dt},$$

where we have used Eq. (5).

Following now the procedure employed in Refs. 8 and 9, the times for the isothermal branches t_1 and t_3 are

$$t_1 = \frac{RT_{1w}}{\alpha(T_1 - T_{1w})^k}\ln\frac{V_2}{V_1}$$
$$t_3 = \frac{RT_{2w}}{\alpha(T_{2w} - T_2)^k}\ln\frac{V_3}{V_4}; \tag{6}$$

and the corresponding exchanged heats Q_1 and Q_2 are, respectively,

$$Q_1 = RT_{1w}\ln\frac{V_2}{V_1}, \quad Q_2 = RT_{2w}\ln\frac{V_4}{V_3}, \tag{7}$$

where R is the universal gas constant and V_1, V_2, V_3, V_4 are the corresponding volumes for the states 1,2,3,4 in Fig. 1, respectively. For the adiabatic processes using $\gamma \equiv C_p/C_v$, one finds the following times:

$$t_2 = \frac{RT_{1w}}{\alpha(T_1 - T_{1w})^k(\gamma-1)} \ln \frac{T_{1w}}{T_{2w}},$$
$$t_4 = -\frac{RT_{2w}}{\alpha(T_{2w} - T_2)^k(\gamma-1)} \ln \frac{T_{2w}}{T_{1w}}. \qquad (8)$$

Let us make now the following change of variables:

$$x = \frac{T_{1w}}{T_1}, \qquad z = \frac{T_{2w}}{T_{1w}}; \qquad (9)$$

the power output of the engine becomes

$$P \equiv \frac{W}{t_{tot}} = \frac{T_1^k \alpha (1-z)(1+\lambda \ln z)}{\dfrac{1}{(1-x)^k} + \dfrac{z}{(zx-\beta)^k}}, \qquad (10)$$

where λ denotes the parameter

$$\lambda = \frac{1}{(\gamma-1)\ln \frac{V_3}{V_1}}. \qquad (11)$$

Imposing the condition $\frac{\partial P}{\partial x} = 0$ one obtains $x = x(z,\beta)$ as

$$x = \frac{z^{\frac{2}{k+1}} + \beta}{z + z^{\frac{2}{k+1}}}, \qquad (12)$$

and from $\partial P/\partial z = 0$, one gets

$$\frac{-z(1+\lambda \ln z)(zx-\beta) + \lambda(1-z)(zx-\beta)}{z(1-z)(1+\lambda \ln z)(zx-\beta)} - \frac{xk(zx-\beta)^{k-1} + (1-x)^k}{(zx-\beta)^k + z(1-x)^k} = 0. \qquad (13)$$

Substituting the variable x in Eq. (13) with the help of Eq. (12), the resulting expression is the following one, which shows the implicit function $z = z(\lambda, \beta)$, for a given k:

$$z^{\frac{2}{k+1}}(z-\beta)(\lambda(1-z) - z(1+\lambda \ln z)) + zk(z^{\frac{2}{k+1}} + \beta)(1-z)(1+\lambda \ln z) - \frac{z(1-z)(1+\lambda \ln z)\left(z^2 + \beta z^{\frac{2k}{k+1}} + z^{\frac{2}{k+1}}(z-\beta)\right)}{(z^{\frac{2k}{k+1}} + z)} = 0. \qquad (14)$$

Since the solution of Eq. (14) is not analyticaly feasible when k is not an integer, the case discussed here is $k = 5/4$ (the Dulong and Petit's heat transfer law) [12]. One can then take the approximations only for the exponents,

$$\frac{2}{k+1} \sim 1, \qquad \frac{2k}{k+1} \sim 1, \qquad (15)$$

in Eq. (14) to obtain

$$(1+\lambda \ln z)((k\beta + zk)(1-z) - z(z-\beta)) + \lambda(1-z)(z-\beta) - (1+\lambda \ln z)(1-z)z = 0. \qquad (16)$$

The approximations (15) are both reasonable when $k = 5/4$.

Equation (16) allows to derive the following explicit expression for the function $z = z(\beta, k)$ in the limit $\lambda = 0$:

$$z = \frac{(1-\beta)(k-1) \pm \sqrt{(\beta-1)^2(1-k)^2 + 4k^2\beta}}{2k}. \qquad (17)$$

Taking now $k = 5/4$ in Eq. (17) one obtains the following value for the positive root $z_{PDP}(\beta)$:

$$z_{PDP} = \frac{1 - \beta + \sqrt{\beta^2 + 98\beta + 1}}{10}. \qquad (18)$$

The numerical results for $\eta_{PDP} \equiv 1 - z_{PDP}$ and the semi-sum η_{SDP} defined as[7]

$$\eta_{SDP} \approx \frac{1}{2}(\eta_C + \eta_{PDP}) \qquad (19)$$

are shown in Table I, compared with η_{MEDP} in references [10,11]. Figure 2 shows the comparison between η_{PDP} and η_{CA} with the temperatures of the reservoirs in real plants[10,11].

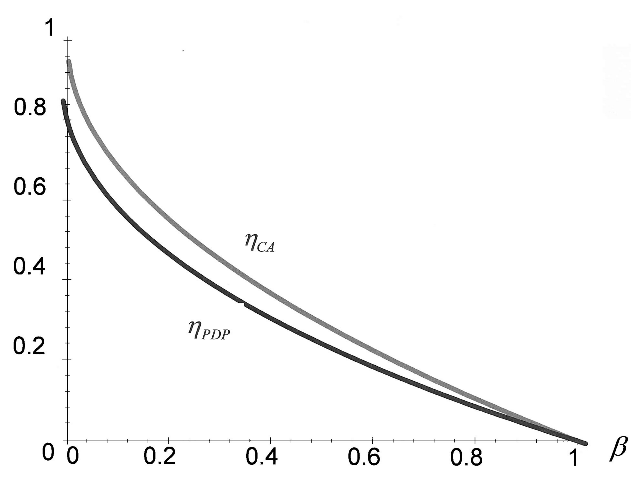

FIGURE 2. Comparison between the Curzon and Ahlborn efficiency, η_{CA}, and the efficiency η_{PDP} obtained here from the approximate (18) for the Dulong and Petit's heat transfer law.

TABLE I. Comparison of the numerically obtained efficiency by Arias-Hernández and Angulo-Brown (1994) and Angulo-Brown and Páez-Hernández (1993), η_{MEDP}, with the approximate efficiency η_{PDP} and the semi-sum also approximate efficiency η_{SDP} obtained here.

Power plant	T_2(K)	T_1(K)	η_{CA}	η_{MEDP}	η_{PDP}	η_{SDP}	η_{OBS}
1962, West Thurrock conventional coal fired steam plant.	298	838	0.403367	0.49	0.33577	0.49008	0.36
1964, Lardarello (Italy) geothermal steam plant.	353	523	0.1784	0.239	0.1453	0.23517	0.16
1956, steam power plant in the U.S.	298	923	0.4318	0.52	0.36006	0.5186	0.40
1949, combined-cycle (steam and mercury) plant in the U.S	298	783	0.3831	0.47	0 .31804	0.46872	0.34
1985, Doel 4 (Belgium)	283	566	0.2929	0.373	0.24113	0.37056	0.35

Now assuming that z obained from Eq. (16) can be expressed as a power series in the parameter λ, we have the following expresion for η_{DP}:

$$\begin{aligned}\eta_{DP} &= 1 - z_{DP}(\lambda, \beta) \\ &= 1 - z_{PDP}(1 + b_1(\beta)\lambda + b_2(\beta)\lambda^2 + O(\lambda^3)) \,, \quad (20)\end{aligned}$$

one can find the coefficients $b_i(\beta), i = 1, 2, ...,$ taking thevalue $z_0 = z_{PDP}(\beta)$, Eq. (18), by successively taking the derivative with respect to λ. The two first ones coefficients $b_1(\beta)$ and $b_2(\beta)$ are:

$$b_1(\beta) = \frac{16(1 - z_0)(\beta - z_0)}{z_0(5 - 4\beta - 40z_0)}, \quad (21)$$

and

$$b_2(\beta) = \frac{4(z_0 - 1)(z_0 - \beta)}{(1 + 9\beta - 10z_0)^2}\left\{\frac{[(-1 + \beta + 10z_0)\ln z_0 + 8z_0 - 4\beta - 4](\beta + 1 - 10z_0)}{1 + 9\beta - 10z_0} + \frac{40(z_0 - 1)(z_0 - \beta)}{1 + 9\beta - 10z_0} - [(9\beta + 1 - 10z_0)\ln z_0 + 4 + 4\beta - 8z_0]\right\}, \quad (22)$$

which are positives for β values in the interval $0 < \beta < 1$, as we can see in Figs. 3 and 4. The numerical results obtained from Eq. (19) are in good agreement with the previous reported values for η_{DP} calculated by means of other approaches (see Table I).

3. Conclusions

One of the main achievements of finite time thermodynamics has been to formulate heat engine models under more realistic condition than those of classical equilibrium thermodynamics. By means of finite time thermodynamics models, good agreement between theorethical values of process variables and experimental data has been obtained, [1,3,7,8]. In this contex, a first result in the present paper is embodied by Eq. (20) together with Eq. (21) and Eq. (22). It expresses the fact that the efficiency for a Carnot type engine depends on the size of the engine as represented by the parameter $\lambda \sim 1/(\ln V_3 - \ln V_1)$. The leading term in (20), corresponding to the exact numerical value calculated without taking into account explicitely the dependency on λ, is the upper bound for the value of the efficiency; in fact the larger the ratio V_3/V_1, the bigger the efficiency becomes. The comparison between the upper bound calculated with the approximations (15) and the values obtained numerically in Refs. 10

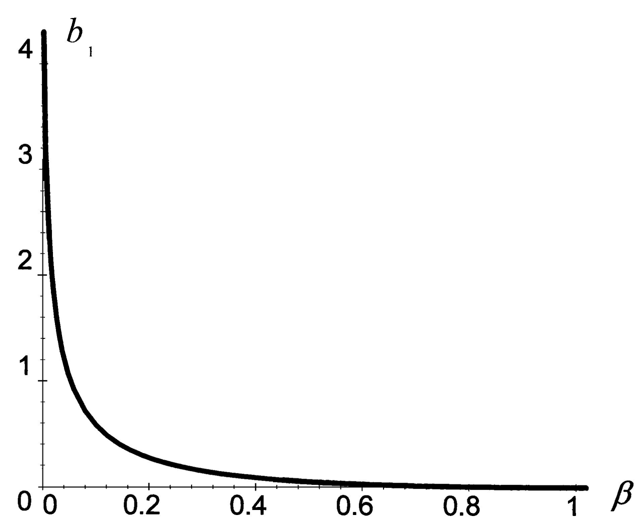

FIGURE 3. First order coefficient $b_1 = b_1(\beta)$, in an interval $0 < \beta < 1$ of the power series (20).

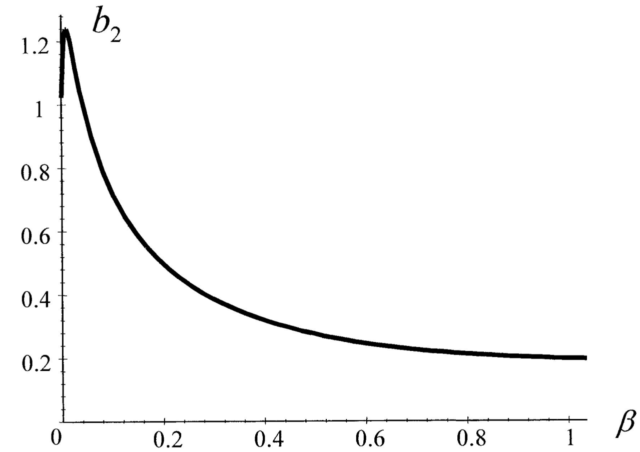

FIGURE 4. Second order coefficient $b_2 = b_2(\beta)$, in an interval $0 < \beta < 1$ of the power series (20).

and 11 are shown in colums 4, 5 and 6 of Table I, where one can appreciate that it was appropiate to take these approximations. In fact, the semi-sum property shown in Eq. (3) and Eq. (19) can be appreciated in colum 6. It is important to point out that in Refs. 10 and 11 this property was only numerically found; in contrast, in the present work this property has been derived anlytically by means of Eq. (19). As we can see, the efficiency $\eta_{PDP} = 1 - z_{PDP}$ is a more relistic model than the reported in Refs. 10 and 11.

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