On the SU(2) Wigner function dynamics

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Recibido el 14 de diciembre de 2001; aceptado el 10 de junio de 2002

We study the quantum dynamics of the $SU(2)$ quasiprobability distribution ("Wigner function") for the simple nonlinear Hamiltonian (finite analog of the Kerr medium, $H = S_z^2$). The quasiclassical approximation for the Wigner function and the corresponding evolution of mean values are considered and compared with the exact and classical solutions.

Keywords: Wigner function; quasiclassical approximation.

Se estudia la dinámica cuántica de la función de distribución de cuasiprobabilidad del grupo $SU(2)$ (función de Wigner) para un simple hamiltoniano no lineal (el análogo finito del medio de Kerr $H = S_z^2$). Se consideran la aproximación cuasiclásica para la función de Wigner y la evolución correspondiente de los valores medios. Se comparan las soluciones exacta, clásica y cuasiclásica.

Descriptores: Función de Wigner; aproximación cuasiclásica.

PACS: 03.65.Bz; 42.50.Fx

1. Introduction

The advantage of the phase-space description of a quantum system in terms of Wigner quasiprobability function consists in that reflecting all quantum features, the Wigner function keeps the closest correspondence with its classical counterpart [1, 2]. The dynamical equation for the Wigner function related to the Heisenberg-Weyl group has the form

$$\partial_t W(p,q,t) = [H(p,q), W(p,q,t)]_M \ , \tag{1}$$

where $H(p,q) = p^2/2m + V(q)$ is the system Hamiltonian and the Moyal bracket

$$[H(p,q), W(p,q,t)]_M = \frac{2}{\hbar}\sin\left[\frac{\hbar}{2}\left(\partial_q^{(H)}\partial_p^{(W)} - \partial_p^{(H)}\partial_q^{(W)}\right)\right] H(p,q)W(p,q,t)$$

reduces to the classical Poisson bracket in the limit $\hbar \to 0$ (here $\partial^{(H)}$ and $\partial^{(W)}$ act to the factors H and W correspondingly). For quadratic Hamiltonians Eq. (1) coincides with the equation for the classical distribution function, meanwhile in general case it can be expanded in powers of $\hbar$:

$$\partial_t W(p,q,t) = -\frac{p}{m}\partial_q W + \partial_q V \partial_p W - \frac{(\hbar/2)^2}{3!}\partial_q^3 V \partial_p^3 W + \dots \tag{2}$$

The third term in this case gives the quantum correction to the classical dynamics.

To provide the phase space description for spin systems Stratonovich [3] in 1956 introduced the quasiprobability distribution function on the sphere $(\theta,\phi) \in \mathcal{S}_2$ (see also Refs. 4, 5 and 6). This function is naturally related to the $SU(2)$ dynamical group and we will later call it the $SU(2)$ Wigner function. This function was proved to be very useful to visualize nonclassical properties of a collection of two-level atoms [7] and polarization optics [8]. Note that a rather general construction of a covariant Wigner function for exponential-type Lie groups was introduced in Refs. 9 and 10. In the particular case of the $SU(2)$ group it can be reduced to the Stratonovich definition [11].

The $SU(2)$ Wigner function is defined as follows:

$$W_\rho(\theta,\phi) = \mathrm{Tr}\left(\rho\hat{w}(\theta,\phi)\right), \tag{3}$$

where ρ is the system density matrix and $\hat{w}(\theta,\phi)$ is the Wigner operator

$$\hat{w}(\theta,\phi) = \frac{2\sqrt{\pi}}{\sqrt{2S+1}}\sum_{L=0}^{2S}\sum_{M=-L}^{L} Y_{L,M}^*(\theta,\phi)\,\hat{T}_{L,M}^{(S)}, \tag{4}$$

such that

$$\mathrm{Tr}\left[\hat{w}(\theta,\phi)\right] = 1, \quad \frac{2S+1}{4\pi}\int_{\mathcal{S}_2} d\Omega\hat{w}(\theta,\phi) = I \tag{5}$$

In Eq. (4) we use the spherical harmonics

$$Y_{L,M}(\theta,\phi) = (-1)^M Y_{L,-M}^*(\theta,\phi)$$

and the irreducible tensor operators $\hat{T}_{L,M}^{(S)}$ (Ref. 12, Eq. 2.4 (6)),

$$\hat{T}_{L,M}^{(S)} = \sqrt{\frac{2L+1}{2S+1}}\sum_{m,m'=-S}^{S} C_{S,m;L,M}^{S,m'}\,|S,m'\rangle\,\langle S,m|, \tag{6}$$

Here $C_{S,m;L,M}^{S,m'}$ are the Clebsch-Gordan coefficients which couple two representations of spin S and L ($0 \le L \le 2S$) to a total spin S. The function $W_\rho(\theta,\phi)$ is covariant under rotations and provides the overlap relation

$$\frac{2S+1}{4\pi}\int_{\mathcal{S}_2} d\Omega W_\rho(\theta,\phi)\,W_A(\theta,\phi) = \mathrm{Tr}\left(\rho A\right). \tag{7}$$

Here $d\Omega = \sin\theta \, d\theta \, d\phi$ is the invariant measure on the sphere and $W_A(\theta,\phi)$ is the Wigner symbol of the operator A,

$$W_A(\theta,\phi) = \mathrm{Tr}\left(\hat{A}\hat{w}(\theta,\phi)\right), \tag{8}$$

the density matrix can be reconstructed from the Wigner function (3) through the obvious relation

$$\rho = \frac{2S+1}{4\pi}\int_{\mathcal{S}_2} d\Omega \, \hat{w}(\theta,\phi) W_\rho(\theta,\phi). \tag{9}$$

In this paper we will consider the dynamics of the $SU(2)$ Wigner function for the simplest nontrivial example of the finite level analog of the Kerr Medium [10, 13, 14]. We will show that the leading order of the expansion in the inverse powers of $2S+1$ (the representation dimension) leads to the classical evolution of the initial distribution on the sphere. (As in the case of phase plane and the Heisenberg-Weyl dynamical group, one may distinguish between the classical evolution which obeys the classical Poisson bracket and the initial state which may be a quantum one.) This approximation describes well the initial stage of the dynamics (when one can neglect the self-interference). It allows us to calculate mean values of the spin operators and gives the results which are drastically better then the "naive" solution of the Heisenberg equations of motion with decoupled correlators. On the other hands, the quantum phenomena which follow from the self-interference (like Shcrödinger cats [10, 13]) are clearly beyond the scope of this semiclassical approximation.

We start in Sec. 2.1 with the case of linear dynamics. The Wigner function dynamics under Kerr Hamiltonian is discussed in Sec. 2.2. The evolution of mean values and the comparison with the classical dynamics from the decoupled correlators in the Heisenberg equations are considered in Sec. 3. The article ends up with Conclusions in Sec. 4. We give the proof of the dynamical equation for the Wigner function in Appendix A. The useful integral representation for the $SU(2)$ Wigner function is briefly described in Appendix B.

In the rest of paper we will consider only integer values of S, which corresponds to the $SO(3)$ group rather than $SU(2)$.

2. Wigner function dynamics

2.1. Linear dynamics

Let us consider the dynamics of the Wigner function $W_\rho(\theta,\phi)$ under the action of a Hamiltonian from the universal enveloping algebra of $su(2)$. First of all we note that due to the *covariance* of the Wigner function with respect to rotations, its evolution under a linear Hamiltonian

$$H = \omega_0 S_z + g_1 S_x + g_2 S_y \tag{10}$$

is equivalent to a rotation round some axis. In other words, the equation of motion for a linear Hamiltonian (10) can be reduced to a diagonal Hamiltonian

$$H \rightarrow H_d = U^\dagger H U = \omega S_z, \quad \omega = \sqrt{\omega_0^2 + g_1^2 + g_2^2}, \tag{11}$$

where U is the rotation from the $SU(2)$ group.

Substituting the density matrix in terms of Wigner function (9) to the equation of motion,

$$i\partial_t \rho = [H,\rho],$$

we obtain

$$i\int_{\mathcal{S}_2} d\Omega \, \hat{w}(\theta,\phi)\partial_t W_\rho(\theta,\phi) = \int_{\mathcal{S}_2} d\Omega \, \left[H,\hat{w}(\theta,\phi)\right] W_\rho(\theta,\phi). \tag{12}$$

Taking into account that

$$\left[S_z, \hat{T}^{(S)}_{L,M}\right] = M\hat{T}^{(S)}_{L,M},$$

we get

$$[S_z,\hat{w}(\theta,\phi)] = \frac{2\sqrt{\pi}}{\sqrt{2S+1}}\sum_{L=0}^{2S}\sum_{M=-L}^{L} M Y^*_{L,M}(\theta,\phi)\,\hat{T}^{(S)}_{L,M} = -i\partial_\phi \hat{w}(\theta,\phi). \tag{13}$$

Replacing the above equation into (12) and integrating by parts we obtain the following equation of motion for the Wigner function

$$\partial_t W_\rho(\theta,\phi) = -\omega\partial_\phi W_\rho(\theta,\phi).$$

Its solution is

$$W_\rho(\theta,\phi|t) = W_\rho(\theta,\phi-\omega t|t=0),$$

which, of course, corresponds to the above mentioned propriety that under the action of a linear Hamiltonian the initial Wigner function rotates with respect to some axis (the direction of this axis depends on the coefficients of the Hamiltonian in Eq. (10)) [4].

2.2. Kerr dynamics

Now let us consider the simplest non-linear Hamiltonian

$$H = \chi S_z^2, \tag{14}$$

which, in spite of its simplicity, leads to a number of interesting features, such as, for example, generation of squeezed atomic states [13] and atomic Shcrödinger cats [14]. Also, the Hamiltonian in Eq. (14) gives the simplest example when the quantum dynamics differs essentially from the corresponding classical one [10].

To find an approximate dynamical equation for the Wigner function under the action of the Hamiltonian (14) we use the expansion in the powers of small parameter

$$\varepsilon = \frac{1}{2S+1},$$

(the inverse dimension of representation). Neglecting terms of order ε^2 one can obtain the following equation (see Appendix A):

$$\partial_t W_\rho(\theta,\phi) = -\frac{\chi}{\varepsilon}\cos\theta\partial_\phi W_\rho(\theta,\phi) + \frac{\chi\varepsilon}{2}\left[\cos\theta\partial_\phi W_\rho(\theta,\phi) + \sin\theta\,\partial_\phi\partial_\theta W_\rho(\theta,\phi) + \cos\theta\partial_\phi\mathcal{L}^2 W_\rho(\theta,\phi)\right], \quad (15)$$

where $\mathcal{L}^2$ is a differential operator (Casimir operator on the sphere):

$$\mathcal{L}^2 = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]. \quad (16)$$

In the zero-order approximation we arrive at the following quasiclassical equation for the distribution function,

$$\partial_t W_\rho(\theta,\phi) = -\frac{\chi}{\varepsilon}\cos\theta\partial_\phi W_\rho(\theta,\phi), \quad (17)$$

with the solution,

$$W_\rho\left(\theta,\phi|t\right) = W_\rho\left(\theta,\phi - \frac{\chi t}{\varepsilon}\cos\theta \middle| t=0\right). \quad (18)$$

This quasiclassical Wigner function describes well the evolution for times $\chi t \leq 1$. Note that Eq. (17) corresponds to the classical evolution equation that involves the Poisson brackets on the sphere (expressed in terms of the angles θ,ϕ, see. *e.g.*, [15]) between the classical Hamiltonian function and the classical distribution function.

One observes that the Wigner function suffers the maximum deformation close to the poles of the sphere (in the opposite directions at the south and north poles), meanwhile the equator zone does not evolve at all. Of course, the poles themselves do not evolve because they corresponds to the eigenstates of the operator S_z, $|\pm S, S\rangle$. The Wigner function of an arbitrary eigenstate $|k,S\rangle$ of the operator S_z has a stationary form

$$W_k(\theta,\phi|t) = W_k(\theta,\phi|t=0) = \frac{2\sqrt{\pi}}{\sqrt{2S+1}}\sum_{L=0}^{2S}\sqrt{\frac{2L+1}{2S+1}}Y^*_{L,0}(\theta,\phi)C^{Sk}_{Sk,\,L0},$$

which does not depend on ϕ. In the limit of large representation dimensions, $S \gg 1$, and for the values of $|k| \sim S$, $W_k(\theta,\phi)$ can be approximated as follows (see Appendix B),

$$W_k(\theta,\phi) \simeq (-1)^{k+S} d^S_{kk}(2\theta)\left[1 + \frac{k}{S}\cos\theta\right],$$

where $d^S_{kk}(\vartheta) = \langle k,S|\exp\left(-i\vartheta S_y\right)|k,S\rangle$ is the d-function. In particular, the Wigner function corresponding to the state $|S,S\rangle$ takes the following simple form

$$W_{k=S}(\theta,\phi|t) = W_{k=S}(\theta,\phi|t=0) \simeq \cos^{2S}\theta\left[1+\cos\theta\right].$$

On the other hand, starting from the atomic coherent state initially located along the x-direction, $|\theta=\pi/2,\phi=0\rangle$,

$$S_x|\pi/2,0\rangle = (S/2)|\pi/2,0\rangle,$$

such that

$$|\pi/2,0\rangle = \frac{1}{2^S}\sum_{k=-S}^{S}\sqrt{\frac{(2S)!}{(S+k)!(S-k)!}}|k,S\rangle, \quad (19)$$

(this is an eigenstate of the S_x operator) we obtain from Eq. (18)

$$W_\rho(\theta,\phi|t) = \frac{2\sqrt{\pi}}{\sqrt{2S+1}}\sum_{L=0}^{2S}\sqrt{\frac{2L+1}{2S+1}} \times \sum_{M=-L}^{L} Y^*_{L,M}(\theta,\phi)\,e^{-i\tau M\cos\theta}\sum_{k,n=-S}^{S} C^{Sn}_{Sk,\,LM}\,\alpha_{kn},$$

where

$$\alpha_{kn} = \frac{(2S)!}{2^{2S}\sqrt{(S+k)!(S-k)!(S+n)!(S-n)!}},\ \tau = \frac{\chi t}{\varepsilon}.$$

In the limit $S \gg 1$, the Wigner function for the initial coherent state in Eq. (19) takes the following approximate form

$$W_\rho(\theta,\phi|t=0) \simeq \left(\sin\theta\cos\phi\right)^{2S-1}\left(1+\sin\theta\cos\phi\right),$$

leading to the evolution

$$W_\rho(\theta,\phi|t) \simeq \left(\sin\theta\cos\left(\phi - \frac{\chi t}{\varepsilon}\cos\theta\right)\right)^{2S-1} \times \left[1 + \sin\theta\cos\left(\phi - \frac{\chi t}{\varepsilon}\cos\theta\right)\right]. \quad (20)$$

It is worth noting that the above equation can be rewritten as follows:

$$W_\rho(\vec{n}\,(\theta,\phi)\,|t) \simeq f^{2S-1}(\vec{n},t)\left[1 + f(\vec{n},t)\right],$$

where

$$f(\vec{n},t) = n_x\cos\left(\frac{\chi t}{\varepsilon}n_z\right) + n_y\sin\left(\frac{\chi t}{\varepsilon}n_z\right).$$

Then, from the covariance of the Wigner function one can easily recover the evolution of an arbitrary initial coherent state [3, 4],

$$W_{g\cdot\rho}(\vec{n}\,(\theta,\phi)\,|t) = W_\rho(g^{-1}\cdot\vec{n}\,(\theta,\phi)\,|t).$$

Here $g\cdot\rho = T(g)\varrho T^{-1}(g),\ T(g)$ is the operator of finite rotation in the $2S+1$ dimensional representation of the $su(2)$ algebra, which transforms coherent states among themselves; $T(g)|\xi\rangle = e^{i\varphi}|\xi_g\rangle$, and $\vec{n} = (\cos\phi\sin\theta,\ \sin\phi\sin\theta, \cos\theta)$.

In the same manner, we can find evolution of the Wigner function for the initial superposition of atomic coherent states. For instance, for the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\theta = \frac{\pi}{2}, \phi = 0\rangle + |\theta = \frac{\pi}{2}, \phi = \pi\rangle\right),$$

(the Schrödinger cat state on the sphere [14]), the Wigner function takes the form

$$W_\rho\left(\theta, \phi|t\right) = f(\theta, \phi|t)^{2S} - \mathrm{Re}\left(f(\theta, \phi + \pi/2|t) + i\cos\theta\right)^{2S},$$

where $f(\theta, \phi|t) = \sin\theta\cos(\phi - \varepsilon^{-1}\chi t\cos\theta)$.

The rest of the terms in Eq. (15) (diffusion-like terms) describes quantum corrections to the quasiclassical motion in analogy to the quasiclassical expansion in Eq. (2). Precisely that terms are responsible for the formation of Schrödinger cats on the sphere.

It is worth noting that the term describing first quantum correction in Eq. (15) vanishes when $\varepsilon \to 0$ ($S \to \infty$). This property is specific for the Wigner-like quasidistribution functions (for the Heisenberg-Weyl group as well as for the $SU(2)$ group) and does not take place for other types of quasidistributions (see, for example, [16] for analysis of the Q-function evolution for the $SU(2)$ group). This is the main reason why the Wigner function is the most suitable tool for the analysis of quantum-classical correspondence.

3. Evolution of average values

Using the overlap relation in Eq. (7) and the "classical" Eq. (17) we can determine the evolution of average values of angular momentum operators:

$$\frac{d}{dt}\langle A\rangle = \frac{2S+1}{4\pi}\int_{\mathcal{S}_2} d\Omega W_A(\theta, \phi)\frac{d}{dt}W_\rho(\theta, \phi). \tag{21}$$

For this purpose we first obtain the Wigner symbols of S_j, $j = x, y, z$. Taking into account the following relations:

$$\begin{aligned} S_x &= A_S\left(T_{1-1} - T_{11}\right), \\ S_y &= iA_S\left(T_{1-1} + T_{11}\right), \\ S_z &= \sqrt{2}A_S T_{10}, \end{aligned} \tag{22}$$

where

$$A_S = \sqrt{\frac{S\left(S+1\right)\left(2S+1\right)}{6}},$$

one can easily obtain

$$\begin{aligned} Tr\left[S_x T_{LM}\right] &= A_S\left[\delta_{L1}\delta_{M-1} - \delta_{L1}\delta_{M1}\right], \\ Tr\left[S_y T_{LM}\right] &= -iA_S\left[\delta_{L1}\delta_{M-1} + \delta_{L1}\delta_{M1}\right], \\ Tr\left[S_z T_{LM}\right] &= \sqrt{2}A_S\delta_{L1}\delta_{M0}. \end{aligned}$$

Substituting the above expressions into Eq. (8) one gets

$$W_{S_j}(\theta, \phi) = \sqrt{S\left(S+1\right)}n_j, \tag{23}$$

where n_j are the components of the unitary vector

$$\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

In the same way we find

$$\begin{aligned} &W_{\frac{1}{2}\{S_j,S_k\}_+}(\theta, \phi) \\ &\quad = \frac{1}{2}\sqrt{\left(2S+3\right)\left(2S-1\right)S\left(S+1\right)}n_j n_k, \quad j \neq k, \end{aligned} \tag{24}$$

$$\begin{aligned} W_{S_j^2}(\theta, \phi) = \frac{1}{2}&\sqrt{\left(2S+3\right)\left(2S-1\right)S\left(S+1\right)} \\ &\times\left(n_j^2 - \frac{1}{3}\right) + \frac{S\left(S+1\right)}{3}. \end{aligned} \tag{25}$$

Replacing Eq. (23) into Eq. (21) and using the "quasiclassical" evolution Eq. (17) for the Wigner function we obtain after integration by parts over the angle ϕ (25):

$$\begin{aligned} \frac{d}{dt}\langle S_z\rangle =& 0, \\ \frac{d}{dt}\langle S_x\rangle =& -\frac{\chi}{2\varepsilon}\frac{2S+1}{4\pi}\sqrt{S\left(S+1\right)} \\ &\times\int_{\mathcal{S}_2} d\Omega\sin 2\theta\sin\phi W_\rho(\theta, \phi|t), \\ \frac{d}{dt}\langle S_y\rangle =& \frac{\chi}{2\varepsilon}\frac{2S+1}{4\pi}\sqrt{S\left(S+1\right)} \\ &\times\int_{\mathcal{S}_2} d\Omega\sin 2\theta\cos\phi W_\rho(\theta, \phi|t). \end{aligned}$$

Comparing with the relations in Eq. (24) one can obtain

$$\begin{aligned} \frac{d}{dt}\langle S_z\rangle &= 0, \\ \frac{d}{dt}\langle S_x\rangle &= -\chi\alpha_S\langle\{S_y, S_z\}_+\rangle, \\ \frac{d}{dt}\langle S_y\rangle &= \chi\alpha_S\langle\{S_x, S_z\}_+\rangle, \\ \alpha_S &= \frac{2S+1}{\sqrt{\left(2S+3\right)\left(2S-1\right)}}. \end{aligned} \tag{26}$$

The equations for the second order correlators can also be found. They involve the third order correlators. The solution of this infinite chain of equations can be obtained directly from Eq. (18), giving, for example, for the first order moments:

$$\begin{aligned}\langle S_z(t)\rangle =&\frac{2S+1}{4\pi}\sqrt{S(S+1)}\\&\times\int_{\mathcal{S}_2}d\Omega\cos\theta W_\rho(\theta,\phi-\frac{\chi t}{\varepsilon}\cos\theta|t=0))\\=&\frac{2S+1}{4\pi}\sqrt{S(S+1)}\\&\times\int_{\mathcal{S}_2}d\Omega\cos\theta W_\rho(\theta,\phi|t=0))=\langle S_z(t=0)\rangle,\\\langle S_x(t)\rangle =&\frac{2S+1}{4\pi}\sqrt{S(S+1)}\\&\times\int_{\mathcal{S}_2}d\Omega\sin\theta\cos\phi W_\rho(\theta,\phi-\frac{\chi t}{\varepsilon}\cos\theta|t)),\\\langle S_y(t)\rangle =&\frac{2S+1}{4\pi}\sqrt{S(S+1)}\\&\times\int_{\mathcal{S}_2}d\Omega\sin\theta\sin\phi W_\rho(\theta,\phi-\frac{\chi t}{\varepsilon}\cos\theta|t)).\end{aligned}$$

In particular, for the initial coherent state, Eq. (19), (the eigenstate of S_x operator) the above solution leads to the following expression for the average values:

$$\begin{aligned}&\langle S_x(t)\rangle=\sqrt{S(S+1)}\Gamma(S+3/2)\left(\frac{2}{z}\right)^{S+1/2}J_{S+1/2}(z),\\&\langle S_y(t)\rangle=0,\\&\qquad z=\chi t\,(2S+1)\,.\end{aligned}\tag{27}$$

We show in Fig. 1 the quasiclassical result (27) together with the exact quantum solution for the mean value of S_x,

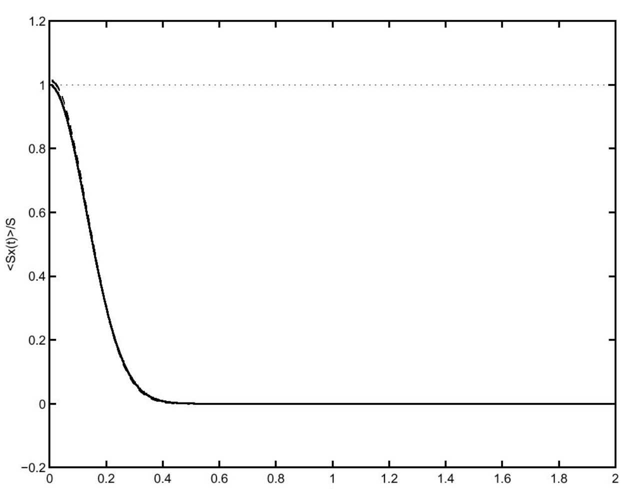

FIGURE 1. Evolution of the normalized average value $\langle S_x(t)\rangle/S$ from exact formula (28) (solid line); form quasiclassical approximation (27) (dashed line); from classical expression (30) (dotted line); $\chi=1$, $S=30$.

$$\langle S_x(t)\rangle=S\cos^{2S-1}\chi t\tag{28}$$

One may observe the good agreement between these two curves up to times $\chi t\sim 1$.

It is easy to see that Eqs. (26) coincide with the correspondent averaged quantum Heisenberg equations

$$\begin{aligned}\frac{d}{dt}\langle S_z\rangle&=0,\\\frac{d}{dt}\langle S_x\rangle&=-\chi\langle\{S_y,S_z\}_+\rangle,\\\frac{d}{dt}\langle S_y\rangle&=\chi\langle\{S_x,S_z\}_+\rangle,\end{aligned}\tag{29}$$

except the factor α_S, which in the limit $S\gg 1$ tends to unity $\alpha_S=1+O(S^{-2})$. It is interesting to note that the quasiclassical evolution equation for the Wigner function does not lead to the classical equations of motion for the average values of the angular momentum operators (in the sense that we do not arrive at equations with decoupled correlators). This means that even in the limit of large dimension of representation S, the evolution under the Kerr Hamiltonian conserves some of its quantum features (though, the evolution equation for the Wigner function has a simple solution shown in Eq. (18)).

The difference between quasiclassical, Eq. (18), and classical solutions is apparent at the level of average values. Indeed, after averaging the system of Eqs. (29) over some state and decoupling the correlators one obtains the classical system

$$\begin{aligned}\frac{d}{dt}\langle S_z\rangle&=0,\\\frac{d}{dt}\langle S_x\rangle&=-2\chi\langle S_y\rangle\langle S_z\rangle,\\\frac{d}{dt}\langle S_y\rangle&=2\chi\langle S_x\rangle\langle S_z\rangle.\end{aligned}$$

(This system can also be obtained form the classical Hamiltonian $H=\chi S_z^2$ with the Poisson brackets $\{S_i,S_j\}=S_k,\ i,j,k=x,y,z$.) The classical equations have a solution,

$$\begin{aligned}\langle S_x(t)\rangle_{cl}=&\langle S_x(0)\rangle\cos 2\langle S_z(0)\rangle\chi t\\&-\langle S_y(0)\rangle\sin 2\langle S_z(0)\rangle\chi t,\\\langle S_y(t)\rangle_{cl}=&\langle S_x(0)\rangle\sin 2\langle S_z(0)\rangle\chi t\\&+\langle S_y(0)\rangle\cos 2\langle S_z(0)\rangle\chi t.\end{aligned}\tag{30}$$

These solutions are usually called parametric approximation. Note, that if the correlators are decoupled not at the level of the Heisenberg Eqs. (29) but in the nonlinear equation of motion for the operator $S_x(t)$, it leads just to a small change in the frequency of oscillations in Eq. (30).) One can see that the "classical" solution diverges from the exact one, Eq. (28); even for very small times. Indeed, the Taylor expansion of the exact solution of Eq. (28) gives:

$$\langle S_x(t)\rangle\approx S-S(S-1/2)(\chi t)^2+O\left(t^4\right),$$

while the classical solution, Eq. (30), has quite different behavior for the initial coherent state, Eq. (19), just maintains constant

$$\langle S_x(t)\rangle_{cl} = S.$$

On the other hand, the classical solution (30) can be rewritten in terms of spherical angles, $S_z = S\cos\theta$, $S_x = S\sin\theta\cos\phi,\ S_y = S\sin\theta\sin\phi$,

$$\theta = \text{const}, \quad \phi(t) = \phi(0) + 2S\chi t\cos\theta$$

and we see, that the quasiclassical evolution of the Wigner function, Eq. (18), corresponds to the motion of every point of the initial (quantum) distribution on the sphere along the classical trajectory (compare with the Heisenberg-Weyl case, [17-19]).

4. Conclusions

In summary, we have considered the evolution of the $SU(2)$ Wigner quasiprobability function on the sphere under the action of the simplest nonlinear Hamiltonian - the "finite Kerr medium". We obtained the exact equation of motion for the Wigner function and solved it for the case of large representation dimensions. In this "quasiclassical" limit different parts of the initial distribution rotate with different velocities (depending of the angle θ) which leads to a deformation of the initial distribution (without self-interference). This "quasiclassical" Wigner function leads to the results which are essentially different from what follows from the classical equations of motion for mean values (parametric approximation). The quasiclassical Wigner function describes well the system dynamics up to times $\chi t \sim 1$ (while the parametric approximation fails for these times).

Appendix A: Derivation of the dynamical equation for the Wigner function

Here we will derive the dynamical equation for the Wigner function for the case of the finite Kerr medium. Firstly, we note that due to Eq. (13)

$$\begin{aligned}\left[S_z^2, \hat{w}(\theta,\phi)\right] &= \{S_z, [S_z, \hat{w}(\theta,\phi)]\}_+ \\ &= i\partial_\phi \{S_z, \hat{w}(\theta,\phi)\}_+. \end{aligned} \tag{31}$$

(Here, $\{\ldots,\ldots\}_+$ stands for anticommutator.) Taking into account that

$$S_z = A_S \hat{T}_{1,0}^{(S)}, \quad A_S = \sqrt{\frac{S(S+1)(2S+1)}{3}}$$

and using the definition (4) we represent $\{S_z, \hat{w}(\theta,\phi)\}_+$ in the following form

$$\begin{aligned}\{S_z, \hat{w}(\theta,\phi)\}_+ &= A_S \frac{2\sqrt{\pi}}{\sqrt{2S+1}} \\ &\times \sum_{L=0}^{2S}\sum_{M=-L}^{L} Y^*{}_{L,M}(\theta,\phi)\left\{\hat{T}_{1,0}^{(S)}, \hat{T}_{L,M}^{(S)}\right\}_+. \end{aligned} \tag{32}$$

The anticommutator of two irreducible tensor operators can be presented as a linear form in irreducible tensor operators,

$$\begin{aligned}\left\{\hat{T}_{1,0}^{(S)}, \hat{T}_{L,M}^{(S)}\right\}_+ &= \sqrt{3\,(2L+1)}\sum_{L'}[(-1)^{L'} - (-1)^{L}] \\ &\times C_{LM10}^{L'M}\left\{\begin{array}{ccc} L & 1 & L' \\ S & S & S\end{array}\right\}\hat{T}_{L'M}^{(S)},\end{aligned}$$

where $\left\{\begin{array}{ccc} L & 1 & L' \\ S & S & S\end{array}\right\}$ is a $6j$-symbols. The values of the Clebsch-Gordan coefficients are

$$C_{LM10}^{L+1M} = \left[\frac{(L+M+1)\,(L-M+1)}{(2L+1)\,(L+1)}\right]^{1/2},$$

$$C_{LM10}^{L-1M} = -\left[\frac{(L+M)\,(L-M)}{(2L+1)\,(L+1)}\right]^{1/2},$$

and we get

$$\begin{aligned}\left\{\hat{T}_{1,0}^{(S)}, \hat{T}_{L,M}^{(S)}\right\}_+ &= 2\sqrt{3}(-1)^{L+1}\left[\sqrt{\frac{(L+1)^2 - M^2}{L+1}}\right. \\ &\times \left\{\begin{array}{ccc} L & 1 & L+1 \\ S & S & S\end{array}\right\}\hat{T}_{L+1M}^{(S)} \\ &\left. -\sqrt{\frac{L^2-M^2}{L}}\left\{\begin{array}{ccc} L & 1 & L-1 \\ S & S & S\end{array}\right\}\hat{T}_{L-1M}^{(S)}\right].\end{aligned}$$

Substituting the above equation into Eq. (32) and taking into account the values of $6j$-symbols

$$\begin{aligned}\left\{\begin{array}{ccc} L & 1 & L+1 \\ S & S & S\end{array}\right\} &= \frac{(-1)^{L+1}}{2} \\ &\times \left[\frac{(2S+L+2)\,(2S-L)\,(L+1)}{(2L+1)\,(2L+3)\,S\,(S+1)\,(2S+1)}\right]^{1/2},\end{aligned}$$

$$\begin{aligned}\left\{\begin{array}{ccc} L & 1 & L-1 \\ S & S & S\end{array}\right\} &= \frac{(-1)^{L}}{2} \\ &\times \left[\frac{(2S+L+1)\,(2S-L+1)\,L}{(2L+1)\,(2L-1)\,S\,(S+1)\,(2S+1)}\right]^{1/2},\end{aligned}$$

we obtain after some algebra

$$\{S_z, \hat{w}(\theta,\phi)\}_+ = \frac{2\sqrt{\pi}}{\sqrt{2S+1}} \sum_{L=0}^{2S} \sum_{M=-L}^{L} \hat{T}_{L,M}^{(S)} \times \left[\sqrt{\frac{(L^2-M^2)\left((2S+1)^2-L^2\right)}{(2L+1)(2L-1)}} Y_{L-1,M}^* + \sqrt{\frac{((L+1)^2-M^2)\left((2S+1)^2-(L+1)^2\right)}{(2L+3)(2L+1)}} Y_{L+1,M}^* \right].$$

Using the recurrence relations for the spherical harmonics

$$Y_{L+1,M}(\theta,\phi)\sqrt{((L+1)^2-M^2)} = [\sin\theta\,\partial_\theta + (L+1)\cos\theta]\sqrt{\frac{2L+3}{2L+1}} Y_{L,M}(\theta,\phi),$$

$$Y_{L-1,M}(\theta,\phi)\sqrt{(L^2-M^2)} = [-\sin\theta\,\partial_\theta + L\cos\theta]\sqrt{\frac{2L-1}{2L+1}} Y_{L,M}(\theta,\phi),$$

we get

$$\{S_z, \hat{w}(\theta,\phi)\}_+ = \frac{2\sqrt{\pi}}{\sqrt{2S+1}} \sum_{L=0}^{2S} \sum_{M=-L}^{L} \frac{1}{2L+1} \hat{T}_{L,M}^{(S)} \times \{f_S(L+1)[\sin\theta\,\partial_\theta + (L+1)\cos\theta] - f_S(L)[\sin\theta\,\partial_\theta - L\cos\theta]\} Y_{L,M}^*(\theta,\phi), \quad (33)$$

where

$$f_S(L) = \sqrt{(2S+L+1)(2S-L+1)} = \frac{1}{\varepsilon}\sqrt{1-\varepsilon^2 L^2}, \qquad \varepsilon = \frac{1}{2S+1}.$$

After some algebra, Eq. (33) can be rewritten as follows

$$\{S_z, \hat{w}(\theta,\phi)\}_+ = \left(\frac{\cos\theta}{2\varepsilon}\hat{F}(\varepsilon) + \varepsilon\left[\frac{\cos\theta}{2} + \sin\theta\partial_\theta\right]\hat{F}^{-1}(\varepsilon)\right)\hat{w}(\theta,\phi), \quad (34)$$

where the operator function $\hat{F}(\varepsilon)$ depends on the operator of the total angular momentum $\mathcal{L}^2$,

$$\mathcal{L}^2 Y_{L,M}(\theta,\phi) = L(L+1)Y_{L,M}(\theta,\phi),$$

and

$$\hat{F}(\varepsilon) = \left[2-\varepsilon^2\left(2\mathcal{L}^2+1\right)+2\sqrt{1-\varepsilon^2\left(2\mathcal{L}^2+1\right)+\varepsilon^4\mathcal{L}^4}\right]^{1/2}.$$

In the limit of large representation dimensions, $S\to\infty$, we can expand square roots in the above equation in powers of a small parameter ε,

$$\hat{F}(\varepsilon) \simeq 2 - \varepsilon^2\frac{(2\mathcal{L}^2+1)}{2},$$

which gives

$$\{S_z, \hat{w}(\theta,\phi)\}_+ \simeq \left[\frac{1}{\varepsilon}\cos\theta - \frac{\varepsilon}{2}\left(\sin\theta\,\partial_\theta + \cos\theta(\mathcal{L}^2+1)\right) + O(\varepsilon^3)\right]\hat{w}(\theta,\phi).$$

Substituting the above equation into Eq. (31) and then into Eq. (12), and integrating by parts (using the fact that $\mathcal{L}^2$ is a self-adjoin operator on the sphere) we obtain the equation of motion in Eq. (15) for the Wigner function.

Appendix B: Approximate Wigner functions for some special states

In this Appendix we obtain approximate expressions for the Wigner functions for the angular momentum coherent state and S_z operator eigenstates. For this purpose we use the integral representation for the Wigner-Stratonovich operator, Eq. (4) [20],

$$\hat{w}(\theta,\phi) = \int_0^{2\pi} d\omega e^{-i\omega\,\vec{n}\cdot\vec{S}} f(\omega), \quad (35)$$

where the weight function $f(\omega)$ is defined as follows

$$f(\omega) = \frac{1}{2\pi}\sum_{L=0}^{2S} i^L \frac{2L+1}{2S+1}\chi_L^S(\omega). \quad (36)$$

Making use the integral representation (35) one can find simple expressions for $SU(2)$ Wigner function for different states of angular momentum in the limit of large dimension of representation $S \gg 1$.

a) *Eigenstate of the operator* S_z,

$$S_z|k,S\rangle = k|k,S\rangle.$$

The density matrix has the form

$$\rho = |k,S\rangle\langle k,S|.$$

Then from Eqs. (8) and (35) we get

$$W_k(\theta,\phi) = \int_0^{2\pi} d\omega\, d_{kk}^S(\beta)\, e^{-i(\alpha+\gamma)} f(\omega), \quad (37)$$

where

$$d_{kk}^S(\beta) = \langle k,S|e^{-i\beta S_y}|k,S\rangle$$

is the Wigner SU(2) representation function and the Euler angles α,β,γ are related to the set of parameters (ω,θ,ϕ) according to

$$\sin\frac{\beta}{2} = \sin\theta\sin\frac{\omega}{2}, \quad \tan\frac{\alpha+\gamma}{2} = \cos\theta\tan\frac{\omega}{2}.$$

As expected, the Wigner function (37) does not depend on the angle ϕ.

b) $SU(2)$ *coherent state* $|\xi\rangle$.

For simplicity we consider a coherent state located on the equator $|\xi = 1\rangle$. The density matrix has the form

$$\rho = \sum_{k,n=-S}^{S} c_k c_n^* |k, S\rangle\langle n, S|,$$

$$c_k = \frac{1}{2^S}\sqrt{\frac{(2S)!}{(S-k)!(S+k)!}}.$$

The Wigner function takes the form

$$W_{CS}(\theta,\phi) = \int_0^{2\pi} d\omega I(\theta,\phi,\omega) f(\omega), \qquad (38)$$

where

$$I(\theta,\phi,\omega) = \sum_{k,n=-S}^{S} c_k c_n^* \langle n, S|e^{-i\beta S_y}|k, S\rangle e^{-i\alpha n - i\gamma k}$$
$$= \left(\cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2} + i\sin\frac{\beta}{2}\sin\frac{\alpha-\gamma}{2}\right)^{2S},$$

and (α,β,γ) are the Euler angles which can be expressed in terms of polar angles according to

$$\cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2} = \cos\frac{\omega}{2}, \quad \tan\frac{\alpha+\gamma}{2} = \cos\theta\tan\frac{\omega}{2},$$
$$\frac{\alpha-\gamma}{2} = \phi - \frac{\pi}{2},$$

giving

$$I(\theta,\phi,\omega) = \left(\cos\frac{\omega}{2} - i\sin\frac{\omega}{2}\sin\theta\cos\phi\right)^{2S}.$$

The representations in Eqs. (37) and (38) are much simpler that corresponding expressions given in Ref. 7.

One can show that the function $f(\omega)$ in the limit case of large representation dimensions, $S \gg 1$ takes the following asymptotic form [20] (see also Ref. 21),

$$f(\omega) \to (-1)^S\left[\delta(\omega-\pi) - \frac{i}{S}\frac{\partial}{\partial\omega}\delta(\omega-\pi)\right],$$
$$S \to \infty, \quad (39)$$

where the limit is understood in a weak sense. This allows us to find approximate expressions for the Wigner functions $W_D(\theta,\phi)$ and $W_{CS}(\theta,\phi)$ giving

$$W_k(\theta,\phi) = (-1)^{S+k} d^S_{kk}(2\theta)\left[1 + \frac{k}{S}\cos\theta\right],$$

$$W_{CS}(\theta,\phi) = (\sin\theta\cos\phi)^{2S-1}\left[1 + \sin\theta\cos\phi\right].$$

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