Shift of saddle-node bifurcation points in modulated Hénon map

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We study the influence of a harmonic parametric modulation on the positions of critical points in the low-dissipative Hénon map with coexisting period-1 and period-3 attractors. The shift of the saddle-node bifurcation and crisis points depends strongly on the modulation frequency and amplitude. Resonance phenomena play a significant role in the displacement of the attractor boundaries as well as in attractor annihilation due to boundary crisis.

Keywords: Nonlinear dynamics; parametrical modulation; coexisting attractors; crisis.

Se estudia la influencia de la modulación paramétrica armónica sobre la posición de los puntos críticos en el mapa de Hénon de baja disipación con atractores coexistentes de periodo-uno y periodo-tres. El corrimiento en la bifurcación de saddle-node y de los puntos de crisis depende fuertemente de la frecuencia y amplitud de modulación. El fenómeno de resonancia juega un papel significativo en el desplazamiento de las fronteras del atractor así como en la aniquilación de éste debido a crisis de frontera.

Descriptores: Dinámica no lineal; modulación paramétrica; atractores coexistentes; crisis.

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1. Introduction

Many nonlinear systems exhibit two or more dynamical equilibrium states for a given set of parameters; some of that states are regular (periodic) and other are chaotic. This phenomenon known as "generalized multistability" [1] is attracted much attention because it is general and appears in variety systems, such as electronic circuits [2], lasers [3], geophysical models [4], and mechanical systems [5], in addition to some standard models like Hénon map [6] and Duffing oscillator [7]. Multistability has been also found in many biological systems [8, 9], including neurons [10], the human proprioceptive system [11], and visual perception [12]. The organization of the basins of attraction in such systems is governed primarily by the ordering of homoclinic and heteroclinic connections of regular saddles [13]. When a control parameter is varied, the system changes the basin of attraction normally through a *saddle-node bifurcation* (SNB). The SNB is fundamental in the study of nonlinear systems since this is one of the most basic processes by which a pair of periodic orbits are created; one of them is always unstable (the saddle), while the other periodic orbit is always stable (the node) [14].

In this paper we offer a method to control bifurcation in a bistable system. In particular, we investigate how boundaries between coexisting attractors change their positions in the parametrically modulated Hénon map. This control can be realized by shifting or removing the critical points. Assume that a system operates at some points in the phase space. As the system parameters slowly vary, it can undergo some bifurcations. Due to some reasons these bifurcations may be undesirable for the system performance and in this case it is convenient to state the control problem in terms of control of bifurcations. As known from the theory of bifurcation [15], bifurcations may be classified into continuous and discontinuous (catastrophic) bifurcations depending on whether the system states vary continuously or discontinuously as the bifurcation parameter gradually varies through its critical value. Although both of them can affect the system behavior the discontinuous bifurcation is much dangerous because it can result in unbounded bifurcated solutions as well as the situation whereby the bifurcated solutions exist only on some fixed time interval. The problem of bifurcation control can be posed as: (i) shifting or removing the bifurcation points in the parameter space or (ii) changing the nature of bifurcation [16]. In this work we deal with the former type of the control.

The shift of the SNB point has been already observed in a CO_2 laser when a control parameter is linearly increased or decreased in the vicinity of the SNB point [17, 18] and also when a parameter is periodically modulated [19]. The goal of this work is to illustrate the applicability of our approach to the control of discrete-time system with two coexisting attractors. Here we illustrate the method by the example of the Hénon map. In particular, we study how the SNB and crisis points can be moved when a periodic modulation is applied to the control parameter. The Hénon map is a popular example of two-dimensional quadratic mapping which produces a discrete-time system with chaotic behavior. Recently control and synchronization algorithms for the Hénon map have been utilized for secure communications [20] and for control of pathological rhythms in some models of cardiac activity [21]. The study of the Hénon map is very important because it is one of the simplest dynamical systems allowed coexistence of attractors, and therefore the results obtained with the Hénon map can be generalized to other more complex systems.

The paper is organized as follows. In Sec. 2 we describe our approach to control the position of the critical points in the parametrically modulated low dissipative Hénon map in which period-1 and period-3 attractors coexist. In Sec. 3, we present the results of calculation and discuss the physical mechanism of dynamical behavior of the system under the parametrical modulation. Finally, the main conclusions are given in Sec. 4.

2. Method

The Hénon map is described by the following difference equations [22]:

$$x_{n+1} = 1 - \mu x_n^2 + y_n, \tag{1}$$

$$y_{n+1} = -Jx_n, \tag{2}$$

where x_n and y_n are the scalar state variables which can be measured as time series, μ is the parameter to which the control can be applied. The Jacobian J ($0 \leq J \leq 1$) is related to dissipation. The dynamics of the Hénon map is well studied (see, for instance, Ref. 23) and its fixed points are given by

$$(x_1, y_1) = \left(\frac{-J - 1 + \sqrt{(J+1)^2 + 4\mu}}{2\mu}, -Jx_1 \right), \tag{3}$$

$$(x_2, y_2) = \left(\frac{-J - 1 - \sqrt{(J+1)^2 + 4\mu}}{2\mu}, -Jx_2 \right) \tag{4}$$

and the corresponding eigenvalues are

$$\lambda_{1,2} = -\mu x \pm \sqrt{(\mu x)^2 - J}. \tag{5}$$

In our simulations we consider low dissipative case when $J = 0.9$.

In Fig. 1 we show the bifurcation diagram where the period-1 and period-3 attractors coexists in the parameter range of $0.92 < \mu < 1.18$. This diagram is calculated by taking different initial conditions that allows us to plot two stable attractors in the same diagram (shown by the open and closed dots). The period-3 branch ultimately destroyed by boundary crisis, presumably by the collision with the regular period-3 saddle. The crisis occurs at $\mu \approx 1.18$ when we increase the control parameter while the system stays on the period-3 branch. This point indicates the upper boundary of the coexistence of the attractors. We call this bifurcation a *forward* bifurcation (FB). When μ decreases, the SNB appears at $\mu \approx 0.92$. This point indicates the position of the lower boundary of the period - 3 attractor. We call this bifurcation a *backward* bifurcation (BB). The application of a periodic modulation to the control parameter μ can change radically the position of these critical points. Now consider how these bifurcation points are shifted when the parametrical modulation is added.

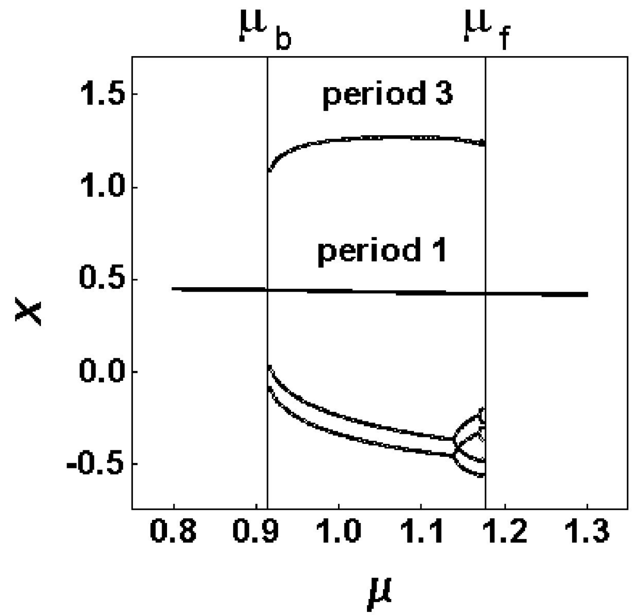

FIGURE 1. Bifurcation diagram of the Hénon map with coexisting period-1 and period-3 attractors without modulation ($\mu_c = 0$). The vertical lines indicate the positions of the backward and forward critical points, μ_b and μ_f.

The method uses the following control algorithm:

$$\mu = \mu_0 \left\{ 1 \pm 0.5\mu_c \left[1 - \cos\left(2\pi f_c n\right) \right] \right\}, \tag{6}$$

where μ_c and f_c are the amplitude and frequency of the control, μ_0 is the initial control parameter without modulation (*i.e.*, when $\mu_c = 0$ or $f_c = 0$), and n is a number of the iteration or time. We use the sign (+) for searching the position of the FB point and the sign (-) for the BB point.

The application of the parametrical modulation Eq. (6) results in dynamical deformation of the basins of attraction of coexisting attractors [25] that reveals itself as a shift in the position of the critical points so that the period-3 branch in the bifurcation diagram in Fig. 1 can be prolonged or restricted. The new position of the FB is found when we increase the control parameter μ_0 while the system stays on the period-3 branch until this attractor is destroyed by the boundary crisis. The maximum value of μ at which the period 3 exists, $\mu_{\max} = \mu_0(1 + \mu_c)$, yields the position of the FB point $\mu_f = \mu_{\max}$. The shifted position of the BB point can be found by decreasing μ_0 until the period 3 is destroyed in the SNB. The minimal value of μ at which this happens, $\mu_{\min} = \mu_0(1 - \mu_c)$, indicates the new position of the BB point $\mu_b = \mu_{\min}$.

3. Results and discussions

The control modulation changes radically the range of coexistence of the attractors. The position of the critical points depends on the control frequency and amplitude. Figures 2 and 3 show the three-dimensional surface of the shifted positions of the BB and FB points in the parameter space of

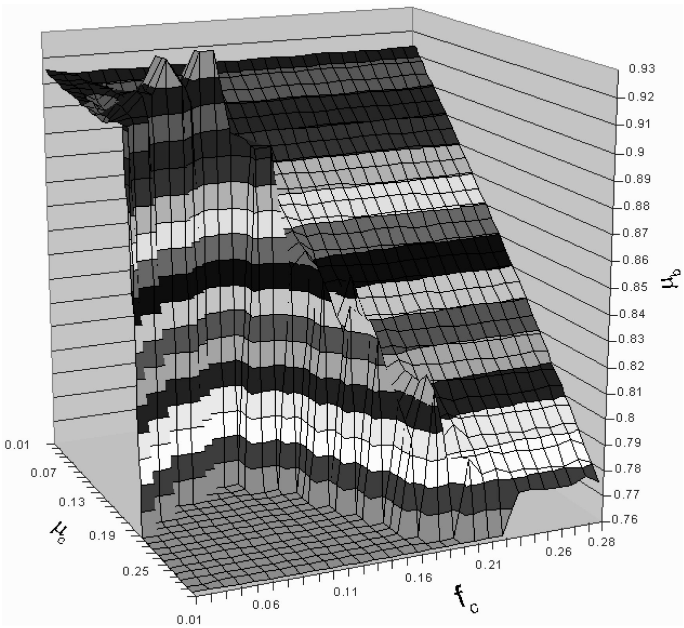

FIGURE 2. Shifted position of the saddle-node bifurcation point versus modulation frequency f_c and amplitude μ_c.

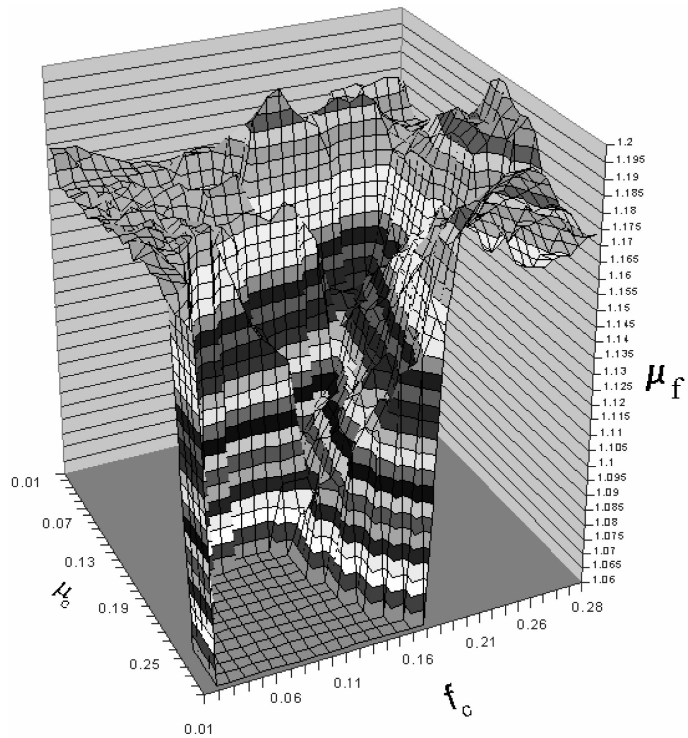

FIGURE 3. Shifted position of the crisis point versus f_c and μ_c.

f_c and μ_c. One can see that for certain modulation parameters the system can remain on the period-3 branch even if the parameter μ during its variation becomes out of the stability range for the uncontrolled system, *i.e.* when $\mu_b < 0.92$ and $\mu_f > 1.18$. Thus, the parameter modulation allows one to spread the attractor boundaries by shifting the critical points to the new positions.

As seen from Figs. 2 and 3 the dynamics of the FB is more complex than the dynamics of the BB, because when μ_0 is increased not only crisis but also supercritical period-doubling bifurcations are involved in the dynamics. The latter bifurcations are also shifted when the modulation is applied and their positions depend on the modulation parameters [28].

Let us analyze first the dynamics of the BB shown in Fig. 2. One can distinguish three regions in the figure: (i) At very low frequencies ($f_c \lesssim 0.06$) and amplitudes ($\mu_c \lesssim 0.1$) the position of the BB point is almost independent on the modulation parameters (quasi stationary modulation); (ii) at high modulation frequencies ($f_c \gtrsim 0.1$) the BB point is displaced linearly with μ_c; and (iii) at low frequencies ($f_c \lesssim 0.2$) and high amplitudes ($\mu_c \gtrsim 0.15$) the period-3 attractor is destroyed. The plane regions on the bottom of Figs. 2 and 3 represent the area where the period 3 does not exist and hence the system becomes monostable. At the boundary of this area (the sudden fall of the surfaces to the plane area) crisis of the attractor is observed. Two peaks in Fig. 2 at small μ_c correspond to the contraction of the period-3 attractor when f_c is close to the frequency of relaxation oscillations ($f_r \approx 0.05$) and its second harmonic for the average value of modulated parameter μ. For larger μ_c the attractor undergoes crisis and disappears. The disappearance of one of the coexisting attractors in a system with parameter modulation known as *attractor annihilation* have been recently demonstrated in the Hénon map [24], in coupled Duffing oscillators [26], and in a loss-modulated CO_2 laser [27]. As we shall show below, this phenomenon results from boundary crisis of the period - 3 attractor due to a resonant interaction of the modulation frequency with the relaxation oscillation frequency of the attractor.

In order to understand the complex behavior of the critical points, we study the relaxation oscillations in the Hénon map in the range of bistability. In Fig. 4 we show the relaxation oscillations in one branch of the bifurcation diagram of the period 3 without the control modulation ($\mu_c = 0$). The frequency of relaxation oscillations depends on the parameter μ_0 as seen from Fig. 5 where we plot the period of the relaxation oscillations in the units of the number of iterations n. The frequency of the relaxation oscillation $f_r = 1/(3n)$. One can see that the period n decreases exponentially when μ_0 approaches the boundary of the period-3 branch.

Now consider how the relation between the modulation frequency and the frequency of the relaxation oscillations influences on the position of the FB point. In Fig. 6 we plot the parameter μ_0 at which crisis of the period-3 attractor appears versus the modulation frequency f_c at fixed $\mu_c = 0.15$ (crosses). In fact, this plot is the section of the 3D graphs of Fig. 3, but instead of μ_f we use μ_0 as an ordinate. The crosses in Fig. 6 indicate the boundary between the bistable (period 1 + period 3) and monostable (only period 1) states. As for the BB point, one can distinguish three ranges in the dynamics of the FB point: (i) At low f_c ($f_c < 0.04$) the position of this

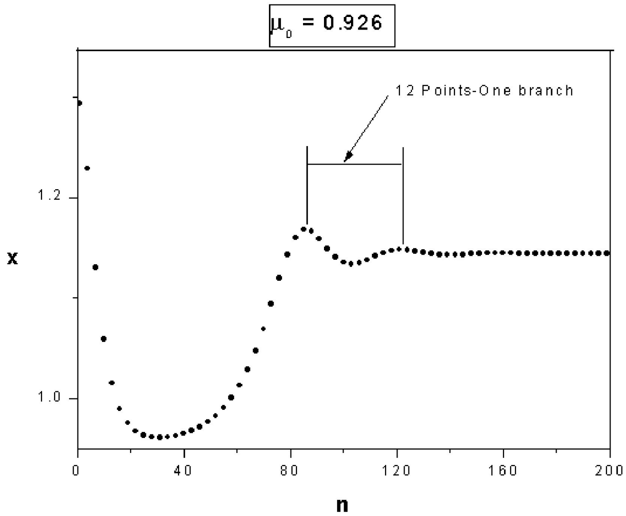

FIGURE 4. Relaxation oscillations in transients in one branch of the period 3 at $\mu_0 = 0.926$. Period $n = 12$ corresponds to the third subharmonic of the relaxation oscillation frequency.

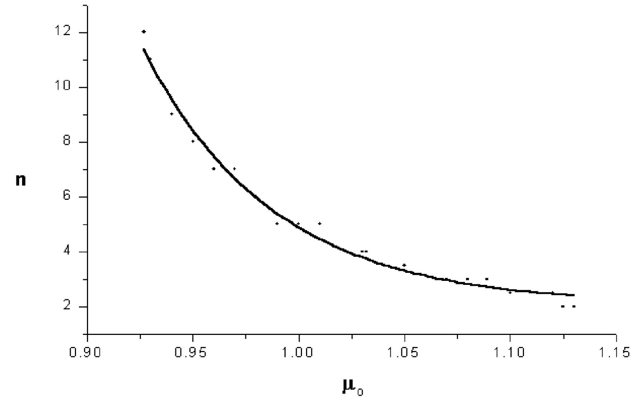

FIGURE 5. Period of relaxation oscillations in one branch of the period 3 versus μ_0.

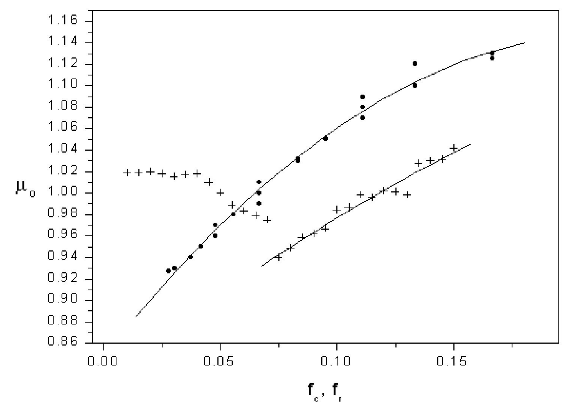

FIGURE 6. Bifurcation parameter μ_0 at which crisis of period-3 attractor appears versus f_c at $\mu_c = 0.15$ (crosses) and f_r (dots). Above the crosses the only period-1 attractor exists while below the crosses two attractors coexist.

point is almost independent on f_c; (ii) at high f_c ($f_c > 0.075$) the crisis point is moved to the right side of the bifurcation diagram shown in Fig. 1 with increasing f_c, so that the period-3 branch is expanded for $f_c > 0.13$; and (iii) the intermediate range ($0.04 < f_c < 0.07$) where the FB point is moved to the left side of the bifurcation diagram, so that the period-3 branch is retracted. At $f_c = 0.075$ the crisis point jumps to the new position due to the sudden contraction of the period-3 attractor. In the same figure we plot the frequency of relaxation oscillations, f_r, as an abscissa for corresponding μ_0. We find that the jump occurs when f_c coincides with f_r for the average value of the modulated parameter μ. Thus, we may conclude that the optimal condition for crisis is fulfilled when $f_c \simeq f_r$.

4. Conclusions

In this work we have studied the influence of a harmonic parametrical modulation on positions of the critical points in the Hénon map with coexisting period-1 and period-3 attractors. We have shown that these positions depend strongly on the frequency and amplitude of modulation. Depending on the modulation parameters the period-3 branch can be either expanded or retracted. The crucial factor in deterministic dynamics of the bifurcation points is the relationship between the modulation frequency and the frequency of relaxation oscillations. One of the coexisting attractors can undergo crisis and disappear when the modulation frequency is close to the frequency of relaxation oscillations. We underline the importance of our approach to the problem of control of oscillations. The traditional approach to the control is to stabilize the existing, perhaps unstable, solutions of the system. Here the problem can be also stated as to prevent the system from the birth of undesirable solution at the bifurcation points. It should be noted that the approach is rather generic; the same methodology has been employed in Refs. 24, 26, 28–31 to control a period-doubling bifurcation. We believe that our method may be applied to other nonlinear systems with coexisting attractors.

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