Appendix

Appendix A. Figures.

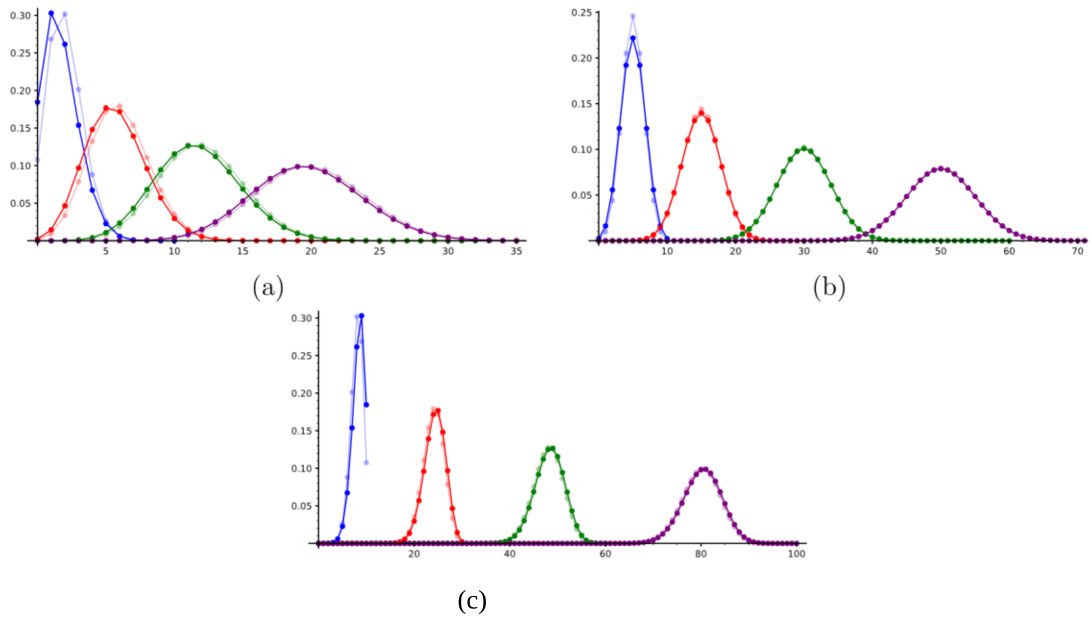


Figure I. *In this figure are plotted some simulations of* B_n *(dark line) and its approximating binomial distribution (light line). In all the plots, it was considered:* $\varepsilon = 1/n^2$ *and* $n = 10$ *in blue;* $n = 30$ *in red;* $n = 60$ *in green and* $n = 100$ *in purple. Plot* (a) *is made for* $q = 0.2$*; plot* (b) *is made for* $q = 0.5$*, and plot* (c) *for* $q = 0.8$.

Figura I. *En esta figura se grafican algunas simulaciones de* B_n *(línea oscura) y su aproximación a una distribución binomial (línea clara). En todas las gráficas se consideró:* $\varepsilon = 1/n^2$ *y* $n = 10$ *en azul;* $n = 30$ *en rojo;* $n = 60$ *en verde y* $n = 100$ *in morado. La gráfica* (a) *está hecha con* $q = 0.2$*; la* (b) *con* $q = 0.5$*, y la* (c) *con* $q = 0.8$.

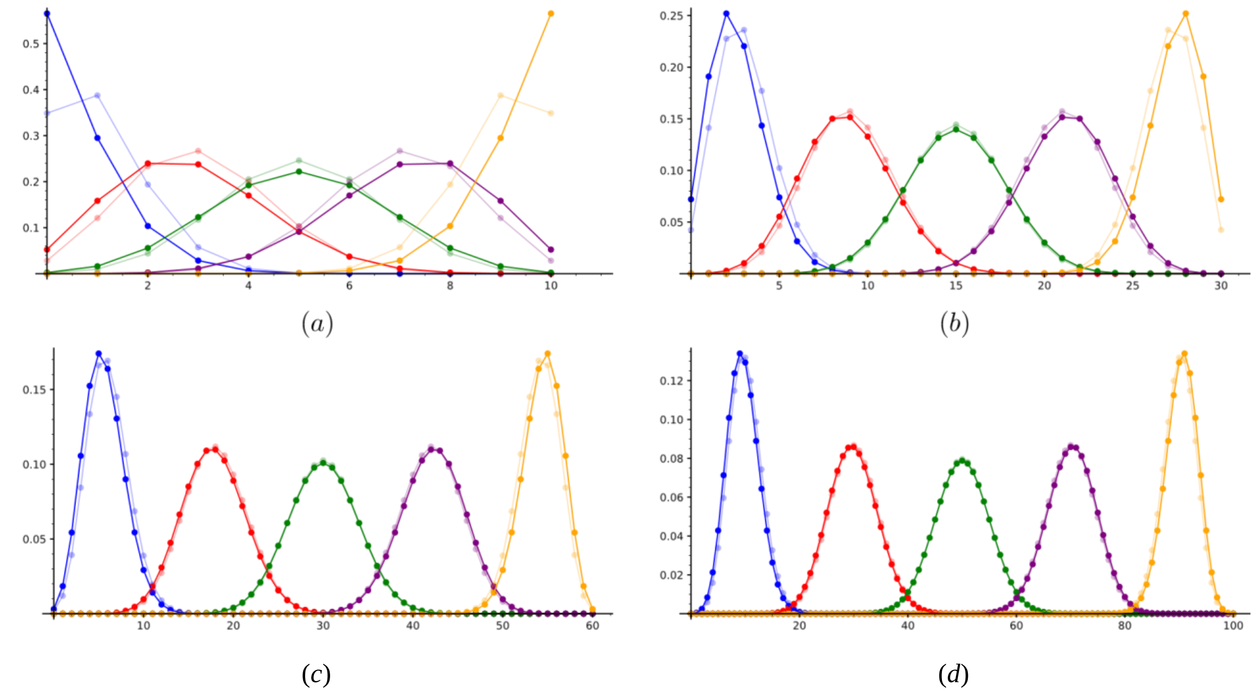
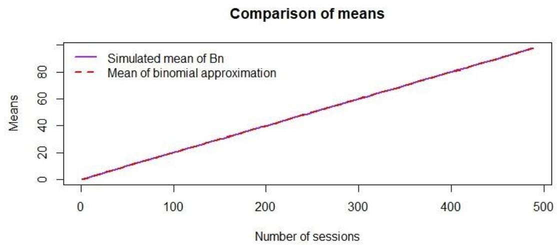
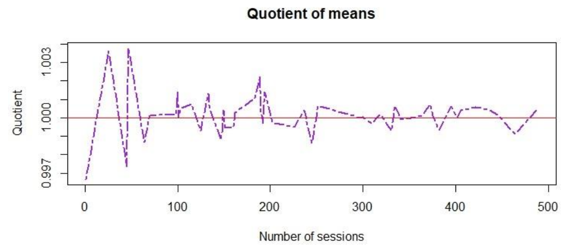


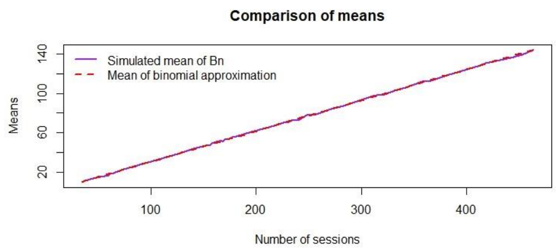
Figure II. *In this figure are plotted some simulations of* B_n *(dark line) and its approximating binomial distribution (light line). In all the plots, it was considered:* $\varepsilon = 1/n^2$ *and* $q = 0.1$ *in blue;* $q = 0.3$ *in red;* $q = 0.5$ *in green;* $q = 0.7$ *in purple and* $q = 0.9$ *in yellow. Plot* (a) *is made for* $n = 10$*; plot* (b) *is made for* $n = 30$*; plot* (c) *is made for* $n = 60$*, and plot* (d) *for* $n = 100$

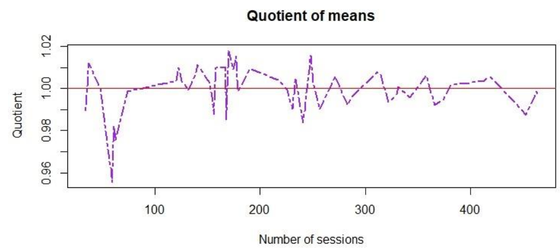
Figura II. *En esta figura se grafican algunas simulaciones de* B_n *(línea oscura) y su aproximación a una distribución binomial (línea clara). En todas las gráficas se consideró:* $\varepsilon = 1/n^2$ *y* $q = 0.1$ *en azul;* $q = 0.3$ *en rojo;* $q = 0.5$ *en verde;* $q = 0.7$ *in morado y* $q = 0.9$ *en amarillo. La gráfica* (a) *está hecha con* $n = 10$*; la* (b) *con* $n = 30$*; la* (c) *con* $n = 60$*, y la* (d) *con* $n = 100$.





(a) (b)

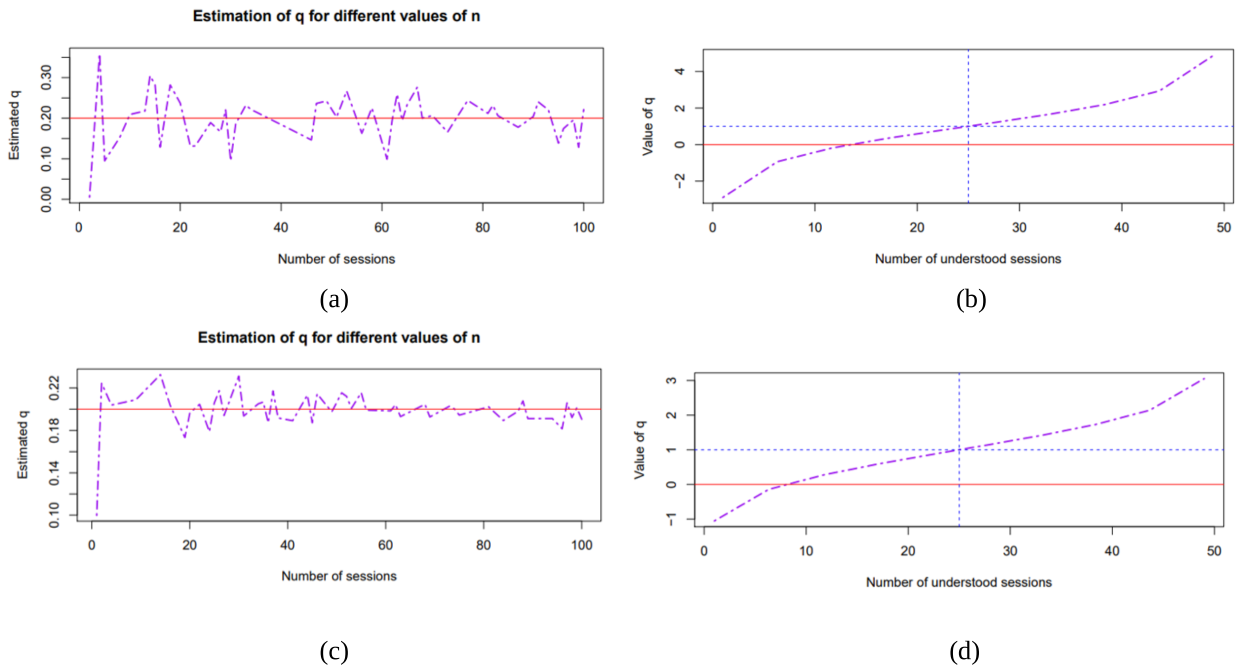




(c) (d)

Figure III. *It was simulated the mean of* B_n *for* n *up to 500 and it was compared with the mean of the binomial approximation for* $q = 0.2$. *Similarly, the quotients were plotted* $\frac{\boldsymbol{E}[B_n]}{\boldsymbol{n}\bar{F}(1-q)}$ *for the cases when* F *is the uniform distribution* (a)-(b) *and the logistic distribution* (c)-(d).

Figura III. *Se simuló la media de* B_n *para n hasta 600 y se comparó con la media de la aproximación binomial para* $q = 0.2$. *De igual manera se graficaron los cocientes* $\frac{\boldsymbol{E}[B_n]}{\boldsymbol{n}\bar{F}(1-q)}$ *para los casos en que* F *es la distribución uniforme* (a)-(b) *y la distribución logística* (c)-(d).



(a) (b)

(c) (d)

Figure IV. *In this plot it is illustrated the estimation of q using Theorem 5 for* (a)-(b) *logistic model and* (c)-(d) *standard normal model. Plots* (b) *and* (d) *show the behavior of q varying the values* $E\,[B_n]$ *when* $n\ =\ 50$.
Figura IV. *En esta gráfica se ilustra la estimación de q usando el Teorema 5 para* (a)-(b) *una distribución logística y* (c)-(d) *una distribución normal estándar. Las gráficas* (b) y (d) *muestran el comportamiento de q mientras varían los valores de* $E\,[B_n]$ *cuando* $n\ =\ 50$.

Appendix B. Proof of the main theorems

Proof of Theorem 1. The result holds for $m = 1$ by the definition of p_1. We proceed by induction, assuming that for an integer $k \geq 1$:

$$p_k = \overline{F}_1(1-q)\prod_{j=1}^{k-1}\left[\overline{F}_j(1-q-\varepsilon)-\overline{F}_j(1-q+\varepsilon)\right] + \sum_{i=1}^{k-1}\overline{F}_i(1-q+\varepsilon)\prod_{j=i+1}^{k-1}\left[\overline{F}_j(1-q-\varepsilon)-\overline{F}_j(1-q+\varepsilon)\right].$$

By equation (2):

$$p_{k+1} = \overline{F}_k(1-q-\varepsilon)p_k + \overline{F}_k(1-q+\varepsilon)(1-p_k) = p_k\left[\overline{F}_k(1-q-\varepsilon)-\overline{F}_k(1-q+\varepsilon)\right] + \overline{F}_k(1-q+\varepsilon).$$

Hence, we obtain from the induction hypothesis:

$$p_{k+1} = \left[\overline{F}_k(1-q-\varepsilon) - \overline{F}_k(1-q+\varepsilon)\right]\overline{F}_1(1-q)\prod_{j=1}^{k-1}\left[\overline{F}_j(1-q-\varepsilon) - \overline{F}_j(1-q+\varepsilon)\right]$$
$$+\left[\overline{F}_k(1-q-\varepsilon) - \overline{F}_k(1-q+\varepsilon)\right]\sum_{i=1}^{k-1}\overline{F}_i(1-q+\varepsilon)\prod_{j=i+1}^{k-1}\left[\overline{F}_j(1-q-\varepsilon) - \overline{F}_j(1-q+\varepsilon)\right]$$
$$+\overline{F}_k(1-q+\varepsilon)$$
$$= \overline{F}_1(1-q)\prod_{j=1}^{k}\left[\overline{F}_j(1-q-\varepsilon) - \overline{F}_j(1-q+\varepsilon)\right]$$
$$+\sum_{i=1}^{k}\overline{F}_i(1-q+\varepsilon)\prod_{j=i+1}^{k}\left[\overline{F}_j(1-q-\varepsilon) - \overline{F}_j(1-q+\varepsilon)\right].$$

The result now follows ■

Proof of Theorem 2.

1. If the student has not understood the first $n-1$ sessions from a total of n, it follows from the construction of the model that the student's parameter for understanding the nth-session becomes $1-q+(n-2)\varepsilon$. Hence:

$$\mathbb{P}[B_n = 0] = \mathbb{P}[B_{n-1} = 0, Y(n) = 0] = \mathbb{P}[Y(n) = 0 | B_{n-1} = 0]\mathbb{P}[B_{n-1} = 0] = F(1-q+(n-1)\varepsilon)\mathbb{P}[B_{n-1} = 0].$$

2. Let A_n denote the number of sessions that the student has not understood from a total of n. Then the event $B_n = n$ is the same as $\{A_n = 0\}$ and hence $\mathbb{P}[B_n = n] = \mathbb{P}[A_n = 0]$ Now the result in 1 yields:

$$\mathbb{P}[B_n = n] = \mathbb{P}[A_{n-1} = 0]\left(1 - F(1-q+(n-1)\varepsilon)\right) = \mathbb{P}[B_{n-1} = n-1]\,\overline{F}\,(1-q+(n-1)\varepsilon).$$

3. Let $U_k(n) := \{(x_1,\ldots,x_n) \in \{0,1\}^n : x_1 + \cdots + x_n = k\}$, then:

$$\mathbb{P}[B_n = k] \tag{6}$$
$$= \mathbb{P}[B_{n-1} = k, Y(n) = 0] + \mathbb{P}[B_{n-1} = k-1, Y(n) = 1]$$
$$= \sum_{(x_1,\ldots,x_{n-1})\in U_k(n-1)} \mathbb{P}[Y(1) = x_1, \ldots, Y(n-1) = x_{n-1}, Y(n) = 0]$$
$$+ \sum_{(x_1,\ldots,x_{n-1})\in U_k(n-1)} \mathbb{P}[Y(1) = x_1, \ldots, Y(n-1) = x_{n-1}, Y(n) = 1]$$
$$= \sum_{(x_1,\ldots,x_{n-1})\in U_k(n-1)} \mathbb{P}[Y(n) = 0 \mid Y(1) = x_1, \ldots, Y(n-1) = x_{n-1}]\mathbb{P}[Y(1) = x_1, \ldots, Y(n-1) = x_{n-1}]$$
$$+ \sum_{(x_1,\ldots,x_{n-1})\in U_k(n-1)} \mathbb{P}[Y(n) = 1 \mid Y(1) = x_1, \ldots, Y(n-1) = x_{n-1}]\mathbb{P}[Y(1) = x_1, \ldots, Y(n-1) = x_{n-1}]. \tag{7}$$

Note that, given the configuration $Y(1) = x_1, \dots, Y(n-1) = x_{n-1}$ in which the student understood exactly k sessions, we have added k times ε to the quality parameter q. Moreover, the maximum number of times we may add or subtract ε in a total of n sessions equals $n-1$, since we do not add or subtract anything in session one. Hence if the student understood k of n sessions, they did not understand $n-k$ and we have subtracted $n-1-k$ times ε. This means that understanding session n depends on the parameter:

$$q + k\varepsilon - (n-1-k)\varepsilon = q - (n-1-2k)\varepsilon.$$

It follows that the student does not understand session n with probability $F(1-q-(n-1-2k)\varepsilon)$. Since this holds for any given configuration $Y\,(1) = x_1, \dots, Y\,(n-1) = x_{n-1}$, in which the student has understood exactly k sessions, we obtain:

$$\sum_{(x_1,\dots,x_{n-1})\in U_k(n-1)} \mathbb{P}[\,Y(n) = 0 \mid Y(1) = x_1, \dots, Y(n-1) = x_{n-1}\,]\mathbb{P}[Y(1) = x_1, \dots, Y(n-1) = x_{n-1}]$$

$$= F(1-q-(n-1-2k)\varepsilon) \sum_{(x_1,\dots,x_{n-1})\in U_k(n-1)} \mathbb{P}[Y(1) = x_1, \dots, Y(n-1) = x_{n-1}]$$

$$= F(1-q-(n-1-2k)\varepsilon)\mathbb{P}[B_{n-1} = k]. \tag{8}$$

Analogously:

$$\sum_{(x_1,\dots,x_{n-1})\in U_k(n-1)} \mathbb{P}[\,Y(n) = 1 \mid Y(1) = x_1, \dots, Y(n-1) = x_{n-1}\,]\mathbb{P}[Y(1) = x_1, \dots, Y\,(n-1) = x_{n-1}]$$

$$= \overline{F}\big(1-q-\big(n-1-2(k-1)\big)\varepsilon\big)\mathbb{P}[\,B_{n-1} = k-1]. \tag{9}$$

The result follows by substituting equations (8) and (9) in equation (7).

Proof of Theorem 3. First, we prove the case when $k \notin \{0, n\}$. Following the arguments in the proof of Theorem 2, we have:

$$\mathbb{P}[B_n = k] = F(1-q-(n-1-2k)\varepsilon) \sum_{(x_1,\dots,x_{n-1})\in U_k(n-1)} \mathbb{P}[Y(1) = x_1, \dots, Y(n-1) = x_{n-1}]$$

$$+\overline{F}\big(1-q-\big(n-1-2(k-1)\big)\varepsilon\big) \sum_{(x_1,\dots,x_{n-1})\in U_{k-1}(n-1)} \mathbb{P}[Y(1) = x_1, \dots, Y(n-1) = x_{n-1}], \tag{10}$$

Where:

$$U_k(n) = \{(x_1, \dots, x_n) \in \{0,1\}^n : x_1 + \dots + x_n = k\}.$$

From the hypothesis $n^2\varepsilon \to 0$, it follows that $\varepsilon \to 0$. Hence, by the assumption of continuity of F, the following convergences as $n \to \infty$ hold:

$$F(1-q-(n-1-2k)\varepsilon) \to F(1-q) \textit{ and } \overline{F}\big(1-q-\big(n-1-2(k-1)\big)\varepsilon\big) \to \overline{F}(1-q).$$

Note from the matrix representation given in (4) that $\mathbb{P}[Y(1) = x_1, \dots, Y(n-1) = x_{n-1}]$ can be expressed as:

$$p_{n-1,k} := \prod_{j=1}^{n-1} \overline{F}^{v_1(j)}(1-q-u_1(j)\varepsilon+u_0(j)\varepsilon)F^{1-v_1(j)}[1-q-u_1(j)\varepsilon+u_0(j)\varepsilon],$$

Where:

$$u_1(j) = \sum_{a=1,x_a=1}^{j} (-1)^{x_a}, \quad u_0(j) = \sum_{a=1,x_a=0}^{j} (-1)^{x_a},$$

And $v_1(j) = 1$ if student understood session j. Now we have the following bounds for $p_{n-1,k}$:

$$\overline{F}^k(1-q+u_0(j)\varepsilon)F^{n-1-k}(1-q-u_1(j)\varepsilon) \leq p_{n-1,k} \leq \overline{F}^k(1-q-u_1(j)\varepsilon)F^{n-1-k}(1-q+u_0(j)\varepsilon). \quad (11)$$

Since the terms with the tail $\overline{F}$ converge to $\overline{F}(1-q)$ and their exponents do not depend on n, we only need to prove that:

$$\frac{\mathrm{F}^{\mathrm{n-1-k}}(1-\mathrm{q}-\mathrm{u}_1(\mathrm{j})\varepsilon)}{\mathrm{F}^{\mathrm{n-1-k}}(1-\mathrm{q})} \to 1, \quad \mathrm{n} \to \infty,$$

Or equivalently:

$$(n-1-k)\log\left(\frac{F(1-q-u_1(j)\varepsilon)}{F(1-q)}\right) \to 0, \quad n \to \infty.$$

Using the hypothesis $n^2\varepsilon \to 0$ we may write $\varepsilon = cn^{-2-\eta}$ for some $c, \eta > 0$. Applying L'Hôpital's rule we obtain:

$$\lim_{n\to\infty} n\log\left(\frac{F(1-q-u_1(j)\varepsilon)}{F(1-q)}\right) = cu(j)(2+\eta)\lim_{n\to\infty}\frac{n^{-3-\eta}f(1-q-u_1(j)cn^{-2-\eta})}{(-n^{-2})F(1-q-u_1(j)cn^{-2-\eta})} = 0.$$

From the limit above and (11) it follows that:

$$\frac{p_{n-1,k}}{\overline{F}^k(1-q)F^{n-1-k}(1-q)} \to 1, n \to \infty. \quad (12)$$

Now let us consider the term:

$$\sum_{(x_1,\ldots,x_{n-1})\in U_k(n-1)} \mathbb{P}[Y(1) = x_1, \ldots, Y(n-1) = x_{n-1}].$$

Since $|U_k(n-1)| = \binom{n-1}{k}$, using the result in equation (12), for an arbitrary $\beta > 0$ and sufficiently large n we have:

$$1-\beta \leq \sum_{(x_1,\ldots,x_{n-1})\in U_k(n-1)} \frac{\mathbb{P}[Y(1) = x_1, \ldots, Y(n-1) = x_{n-1}]}{\binom{n-1}{k}\overline{F}^k(1-q)F^{n-1-k}(1-q)} \leq 1+\beta.$$

The result follows by letting $n \to \infty$ and $\beta \to 0$. The result for the second term in equation (10) is obtained analogously. Now we proceed in a similar way to prove that $\mathbb{P}[B_n = 0]$ is asymptotically equivalent to $p_n(0)$. It might be easily checked that $\mathbb{P}[B_n = 0] = F(1-q)\prod_{j=1}^{n-1} F(1-q+j\varepsilon)$, hence:

$$1 = \left(\frac{F(1-q)}{F(1-q)}\right)^n \leq \frac{F(1-q)\prod_{j=1}^{n-1} F(1-q+j\varepsilon)}{F^n(1-q)} \leq \left(\frac{F(1-q+n\varepsilon)}{F(1-q)}\right)^{n-1}.$$

Using the representation $\varepsilon = cn^{-\eta-2}$ and L'Hôpital's rule again, we obtain:

$$\lim_{n\to\infty}(n-1)\log\left(\frac{F(1-q+n\varepsilon)}{F(1-q)}\right) = -c(\eta+2)\lim_{n\to\infty}(n-1)^2\, n^{-\eta-2}F(1-q+cn^{-\eta-1})f(1-q+cn^{-\eta-1}) = 0.$$

The proof for $\mathbb{P}[B_n = n]$ is analogous. ■

Proof of Theorem 4. We let $a_n = \sum_{j=1}^{n} j\mathbb{P}[B_n = j]$ and $b_n = n\overline{F}(1-q)$ and calculate the limit:

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}. \qquad (13)$$

According to Stolz-Césaro Theorem, if the limit above exists, then $\frac{a_n}{b_n}$ converges and its limit coincides with that in (13).

First, we calculate the difference $a_{n+1} - a_n$. It follows from the relations in Theorem 2 that:

$$\begin{aligned} a_{n+1} - a_n &= \sum_{j=1}^{n} j\mathbb{P}[B_n = j]F(1-q+(n-1-2j)\varepsilon) + \sum_{j=1}^{n} j\mathbb{P}[B_n = j-1]\overline{F}\big(1-q+\big(n-1-2(j-1)\big)\varepsilon\big) \\ &\quad + (n+1)\mathbb{P}[B_{n+1} = n+1] - \sum_{j=1}^{n} j\mathbb{P}[B_n = j]. \end{aligned}$$

By adding and subtracting 1 in the second term above and changing the index $j-1$ to j, we obtain:

$$
\begin{aligned}
a_{n+1} - a_n \quad &= \sum_{j=1}^{n} j\mathbb{P}[B_n = j]F(1 - q + (n - 1 - 2j)\varepsilon) \\
&\qquad + \sum_{j=1}^{n-1} j\mathbb{P}[B_n = j]\overline{F}(1 - q + (n - 1 - 2j)\varepsilon) \\
&\qquad + \sum_{j=1}^{n} P[B_n = j - 1]\overline{F}\big(1 - q + \big(n - 1 - 2(j - 1)\big)\varepsilon\big) \\
&\qquad + (n + 1)P[B_{n+1} = n + 1] - \sum_{j=1}^{n} jP[B_n = j] \\
&= -nP[B_n = n]\overline{F}(1 - q + (n - 1 - 2n)\varepsilon) \\
&\qquad + \sum_{j=0}^{n-1} P[B_n = j]\overline{F}(1 - q + (n - 1 - 2j)\varepsilon) \\
&\qquad + (n + 1)P[B_n = n]\overline{F}(1 - q - n\varepsilon) \\
&= -(n + 1)P[B_n = n]\overline{F}(1 - q - (n + 1)\varepsilon) \\
&\qquad + \sum_{j=0}^{n} P[B_n = j]\overline{F}(1 - q + (n - 1 - 2j)\varepsilon) \\
&\qquad + (n + 1)P[B_n = n]\overline{F}(1 - q - n\varepsilon) \\
&= (n + 1)P[B_n = n]\left(\overline{F}(1 - q - n\varepsilon) - \overline{F}(1 - q - (n + 1)\varepsilon)\right) \\
&\qquad + \sum_{j=0}^{n} P[B_n = j]\overline{F}(1 - q + (n - 1 - 2j)\varepsilon). \quad (14)
\end{aligned}
$$

By relation 2 in Theorem 2, $\mathbb{P}[B_n = n]$ tends to zero. Furthermore, using the hypothesis on ε and the density of F we obtain from L'Hôpital's rule that:

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{\overline{F}(1 - q - n\varepsilon) - \overline{F}(1 - q - (n + 1)\varepsilon)}{(n + 1)^{-1}} \\
&= -\lim 1_{n\to\infty} (n + 1)^2\varepsilon[f(1 - q - n\varepsilon) - f(1 - q - (n + 1)\varepsilon)] \\
&= 0.
\end{aligned}
$$

It follows that the first term in the right-hand side of (14) tends to zero. On the other hand, for $0 \leq j \leq n$ we have:

$$\overline{F}(1 - q + (n - 1)\varepsilon) \leq \overline{F}(1 - q + (n - 1 - 2j)\varepsilon) \leq \overline{F}(1 - q - (n + 1)\varepsilon).$$

Hence, using that $\sum_{j=0}^{n} \mathbb{P}[B_n = j] = 1$ we obtain:

$$\overline{F}(1 - q + (n - 1)\varepsilon) \leq \sum_{j=0}^{n} P[B_n = j]\overline{F}(1 - q + (n - 1 - 2j)\varepsilon) \leq \overline{F}(1 - q - (n + 1)\varepsilon),$$

Therefore:

$$\lim_{n\to\infty}(a_{n+1}-a_n)=\lim_{n\to\infty}\sum_{j=0}^{n}P[B_n=j]\overline{F}(1-q+(n-1-2j)\varepsilon)=\overline{F}(1-q).$$

The limit above and the equality $b_{n+1}-b_n=\overline{F}(1-q)$ yield:

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=1.$$

The result now follows by Stolz-Césaro's Theorem. ■

Proof of Theorem 5. First, we show that $\frac{E[B_n]}{n}\to\overline{F}(1-q)$. By Theorem 4 above, for an arbitrary $\varepsilon>0$ and sufficiently large n we have:

$$\left|\frac{E[B_n]}{n\overline{F}(1-q)}-1\right|<\varepsilon\Rightarrow\overline{F}(1-q)\left|\frac{E[B_n]}{n\overline{F}(1-q)}-1\right|<\overline{F}(1-q)\Rightarrow\left|\frac{E[B_n]}{n\overline{F}(1-q)}-\overline{F}(1-q)\right|<\varepsilon,$$

Where in the last implication we have used that $\overline{F}(1-q)<1$. Now under the assumption that $\overline{F}^{-1}$ exists and it is continuous we obtain:

$$1-\lim_{n\to\infty}\overline{F}^{-1}\left(\frac{E[B_n]}{n}\right)=1-\overline{F}^{-1}\left(\overline{F}(1-q)\right)=q.$$

■

Appendix C. The case of the uniform distribution

In this section we present some important quantities in the particular case when the students initial distribution for understanding is uniform in $[0,1]$. Throughout this section we assume that ε is such that $[1-q-(n-1)\varepsilon,1-q+(n-1)\varepsilon]\subseteq[0,1]$, which we name as Hypothesis 1.

Proposition C.1. Let F be the uniform distribution over $(0,1)$ and assume Hypothesis 1 holds. Then for $i\le n$ we have:

$$\overline{F}_i(1-q-m\varepsilon)=(1+2\varepsilon)^{i-1}\left\{q-\tfrac{1}{2}\right\}+\tfrac{1}{2}+m\varepsilon.$$

Proof. Recall that $F_1=F$. The result holds for $i=1$, since:

$$\overline{F}_1(1-q-m\varepsilon)=q+m\varepsilon.$$

We proceed by induction, assuming that for an integer $j\ge1$ we have:

$$\overline{F}_j(1-q-m\varepsilon)=(1+2\varepsilon)^{j-1}\left\{q-\frac{1}{2}\right\}+\frac{1}{2}+m\varepsilon.$$

By the Law of Total Probability and induction hypothesis:

$$\begin{aligned}\overline{F}_{j+1}(1-q-m\varepsilon) &= \overline{F}_j(1-q-(m+1)\varepsilon)\overline{F}_j(1-q) + \overline{F}_j(1-q-(m-1)\varepsilon)\left(1-\overline{F}_j(1-q)\right)\\ &= \left[(1+2\varepsilon)^{j-1}\left\{q-\frac{1}{2}\right\}+\frac{1}{2}+(m+1)\varepsilon\right]\left[(1+2\varepsilon)^{j-1}\left\{q-\frac{1}{2}\right\}+\frac{1}{2}\right]\\ &+\left[(1+2\varepsilon)^{j-1}\left\{q-\frac{1}{2}\right\}+\frac{1}{2}+(m-1)\varepsilon\right]\left[\frac{1}{2}-(1+2\varepsilon)^{j-1}\left\{q-\frac{1}{2}\right\}\right]\\ &= \frac{1}{2}(1+2\varepsilon)^{j-1}\left\{q-\frac{1}{2}\right\}+\frac{1}{2}\left[\frac{1}{2}+(m+1)\varepsilon\right]+\left[(1+2\varepsilon)^{j-1}\left\{q-\frac{1}{2}\right\}\right]\left[\frac{1}{2}+(m+1)\varepsilon\right]\\ &+\frac{1}{2}(1+2\varepsilon)^{j-1}\left\{q-\frac{1}{2}\right\}+\frac{1}{2}\left[\frac{1}{2}+(m-1)\varepsilon\right]-\left[(1+2\varepsilon)^{j-1}\left\{q-\frac{1}{2}\right\}\right]\left[\frac{1}{2}+(m-1)\varepsilon\right]\\ &= (1+2\varepsilon)^{j-1}\left\{q-\frac{1}{2}\right\}+\frac{1}{2}+m\varepsilon+2\varepsilon\left[(1+2\varepsilon)^{j-1}\left\{q-\frac{1}{2}\right\}\right] = (1+2\varepsilon)^{j}\left\{q-\frac{1}{2}\right\}+\frac{1}{2}+m\varepsilon.\end{aligned}$$

Hence the result follows. ■

Theorem C.1. Let F be the uniform distribution over $(0,1)$ and assume Hypothesis 1 holds. Then for $m \le n$ we have:

$$p_m = \frac{1}{2} + (1+2\varepsilon)^{m-1}\left(q-\frac{1}{2}\right).$$

Proof. For $1 \le j \le m-1$, it follows from Proposition C.1 that:

$$\overline{F}_j(1-q-\varepsilon) - \overline{F}_j(1-q+\varepsilon) = 2\varepsilon.$$

Since $\overline{F}(1-q) = q$, by Theorem 1 we have:

$$\begin{aligned}p_m &= \overline{F}_1(1-q)\prod_{j=1}^{m-1}\left[\overline{F}_j(1-q-\varepsilon)-\overline{F}_j(1-q+\varepsilon)\right] + \sum_{i=1}^{m-1}\overline{F}_i(1-q+\varepsilon)\prod_{j=i+1}^{m-1}\left[\overline{F}_j(1-q-\varepsilon)-\overline{F}_j(1-q+\varepsilon)\right]\\ &= q[2\varepsilon]^{m-1} + \left\{\frac{1}{2}-\varepsilon\right\}\sum_{i=1}^{m-1}(2\varepsilon)^{m-1}(2\varepsilon)^{-i} + \left\{q-\frac{1}{2}\right\}\sum_{i=1}^{m-1}(1+2\varepsilon)^{i-1}[2\varepsilon]^{m-1-i}\\ &= [2\varepsilon]^{m-1}\left[q+\left\{\frac{1}{2}-\varepsilon\right\}\sum_{i=1}^{m-1}(2\varepsilon)^{-i}+\frac{q-\frac{1}{2}}{1+2\varepsilon}\sum_{i=1}^{m-1}\left(\frac{1+2\varepsilon}{2\varepsilon}\right)^{i}\right]. \qquad (15)\end{aligned}$$

Using the identities:

$$\sum_{i=1}^{n-1}\left(\frac{1+2\varepsilon}{2\varepsilon}\right)^{i} = [1+2\varepsilon]\left[\left(\frac{1+2\varepsilon}{2\varepsilon}\right)^{n-1}-1\right] \text{ and } \sum_{i=1}^{n-1}(2\varepsilon)^{-i} = \left[\frac{1}{1-2\varepsilon}\right]\left[\left(\frac{1}{2\varepsilon}\right)^{n-1}-1\right],$$

The right hand of equation (15) becomes:

$$\begin{aligned}&[2\varepsilon]^{m-1}\left[q+\left(\frac{1}{2}-\varepsilon\right)\left[\frac{1}{1-2\varepsilon}\right]\left[\left(\frac{1}{2\varepsilon}\right)^{m-1}-1\right]+\frac{q-\frac{1}{2}}{1+2\varepsilon}[1+2\varepsilon]\left[\left(\frac{1+2\varepsilon}{2\varepsilon}\right)^{m-1}-1\right]\right]\\ &= [2\varepsilon]^{m-1}\left[\frac{1}{2}\left(\frac{1}{2\varepsilon}\right)^{m-1}+q\left(\frac{1+2\varepsilon}{2\varepsilon}\right)^{m-1}-\frac{1}{2}\left(\frac{1+2\varepsilon}{2\varepsilon}\right)^{m-1}\right]\\ &= \left[\frac{1}{2}+q(1+2\varepsilon)^{m-1}-\frac{1}{2}(1+2\varepsilon)^{m-1}\right].\end{aligned}$$

Hence, we obtain:

$$p_m = \frac{1}{2} + (1 + 2\varepsilon)^{m-1}\left(q - \frac{1}{2}\right),$$

And the result follows. ■