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A Stabilizable Control Laws For a Rotational Pendulum: A Trajectory Planning Approach

Leyes de control Estabilizadoras para un péndulo rotacional: Una Planificación de Trayectorias

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Abstract

We propose two simple controls for the regulation of an under actuated rotational pendulum. Both controllers are based on the Lyapunov approach; the first is a simple passive control which makes the closed-loop solution converges asymptotically to an equilibrium manifold. The second approach is a combination of the Lyapunov and the off-line trajectory planning approaches to move the pendulum from an equilibrium point to another equilibrium point, both point belonging to an equilibrium manifold. The last task is accomplished in an approximated fashion. The results are illustrated by means of digital computer simulations.
Keywords: Lyapunov-based control, Trajectory Planning and Under Actuated Systems.

Resumen

Se proponen dos controles simples para la regulación de un péndulo rotacional sub-actuado. Ambos controles están basados en el enfoque de Lyapunov, el primero es un control pasivo simple que hace que la solución de lazo cerrado converja asintóticamente a una variedad (manifold) de equilibrio. El segundo enfoque es una combinación de los enfoques de Lyapunov y el de planeación de trayectoria fuera de línea para mover el péndulo de un punto de equilibrio a otro punto de equilibrio, ambos pertenecientes a una variedad (manifold) de equilibrio. La última tarea se logra de forma aproximada. Los resultados se ilustran mediante simulaciones hechas en una computadora digital.
Palabras clave: Control basado en el enfoque de Lyapunov, Planeación de Trayectoria y Sistemas Sub-actuados.

1. Introduction

Control of under actuated mechanical systems has received considerable attention in the last two decades (de Jager and Nijmeijer, 2000). The main feature of this kind of system is that the number of independent control inputs is less than the number of degrees of freedom to be controlled. These control problems are of both practical and theoretical interest (we recommend to seeing Jager and Nijemeijer, 2000, Fantoni and Lozano, 2002, Olfati, 1999, Bloch et. al., 2000, 2002, and Furuta et. al., 1992). The main challenge of many of these problems, is that these devices are non-feedback linearizable by means of a dynamic state feedback (Jakubczyk and Respondek, 1997 , Spong, 1997, Isidori, 1995), and are not locally controllable around their equilibrium points (Shiriaev, 2000, Brockett, 1983, and Reyhanoglu, 1999). These drawbacks make it difficult to carry out some controlling tasks. For example, to force an under actuated system to follow a desired trajectory is not easy and in many cases, it is not possible to completely solve the problem. However, in various practical cases, that task can be partially solved by taking some suitable approximations, such as discarding some non-linearities (Hauser et. al., 1992 and Sira, 2000). It is worth mentioning that many traditional methods for nonlinear control design like backstepping (Isidori, 1995 and Krstic et. al., 1995), forwarding (Mazenk and Praly, 1996, Sepulchre et. al., 1997, Teel, 1996 and Olfati, 1999), high gain/low design (Sepulchre et. al., 1997, and Khalil, 1996), and sliding mode control (Utkin, 1992, and Christopher and Spurgeons, 1998) are not directly applicable to controlling these mechanical systems.

In this article, we present a Lyapunov approach for the design of two asymptotic stabilizing feedback controllers, for the regulation of a very simple under actuated mechanical system; the first controller makes the system converge asymptotically to an equilibrium manifold and the second controller is a combination of the Lyapunov and the off-line trajectory planning

approaches, for carrying out the equilibrium to equilibrium regulation, in an approximated fashion. This particular mechanical device is an *ad-hoc* example for the application of Lyapunov's theory, due to the fact that it is not locally controllable around its equilibrium points. The methodology has been used to design stabilizing control laws in other under actuated systems, such as the Tora system (Escobar *et. al.*, 1999, and Bupp *et. al.*, 1995), in the regulation of a mass-spring system (Sira and Llanes, 2000) and in the stabilization of the inverted pendulum around its homoclinic orbit (Lozano *et. al.*, 2000, and Fantoni and Lozano, 2002).

The remainder of this work is organized as follows. Section 2 covers a brief description of the dynamic model and presents the most important properties of the system. Section 3 presents the two control laws. In both closed-loop controllers, Lyapunov stability theorem and LaSalle's invariance principle theorem were used to analyze the asymptotic stability of the desired equilibrium point. In both cases, we present computer simulation results depicting the performance of the system to the particular feedback controller option. Section 4 contains the concluding remarks.

2. Equation of Motion

Consider the under actuated system described in Fig. 1. It consists of a rigid frame $\overline{OO'A}$ joined to the axis of an electrical motor (which turns around $\overline{OO'A}$). At point A, a free rotating planar pendulum is attached. The plane of the pendulum is parallel to the rotating plane $\overline{OO'A}$. For simplicity, the pendulum is a very light rod of length l and mass m. To describe the motion, let us introduce a reference system with origin at O, such that the axis of the motor is perpendicular to the plane OXY. Let θ be the angle which the projection of $O'A$ makes with the OX axis and φ the angle between $\overline{AB}$ and the minus direction of the OZ axes.

The coordinates of the pendulum's center of mass are given by

$$\begin{aligned} x &= (a + l\sin\varphi)\cos\theta \\ y &= (a + l\sin\varphi)\sin\theta \\ z &= l(\cos\varphi - 1) \end{aligned} \tag{1}$$

where a is the length of segment $\overline{OA'}$.

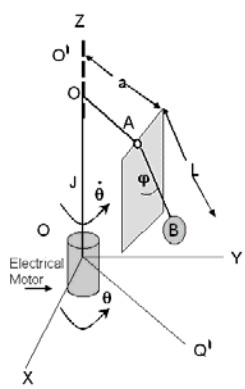

Fig. 1 Rotational pendulum

2.1 Dynamic Model

The total kinetic energy and the potential energy U of this system are given by:

$$K = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right); U = mgz\ , \tag{2}$$

where J is the inertial momentum of the frame and the motor's rotor.

Differentiating *x*, *y* and *z* from (1) and substituting into relations (2) we obtain, after some algebraic manipulation, that kinetic energy and potential energy are given by,

$$K(q,\dot{q}) = \frac{1}{2}\left(J + m(a + l\sin\varphi)^2\right)\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\varphi}^2;$$
$$U(q) = -mgl(\cos\varphi - 1).$$

where $q \cong (q_1, q_2) = (\theta, \varphi)$.

The Lagrangian function of the system is evidently given by

$$L(q,\dot{q}) = K(q,\dot{q}) - U(q).$$

Following the traditional Euler-Lagrange procedure, we get the following set of differential equations

$$\left(J + m(a + l\sin\varphi)^2\right)\ddot{\theta} + 2ml(a + l\sin\varphi)\cos\varphi\dot{\theta}\dot{\varphi} = 0, \qquad (3)$$
$$ml^2\ddot{\varphi} - ml(a + l\sin\varphi)\cos\varphi\dot{\theta}^2 + mgl\sin\varphi = 0.$$

Clearly, the system dynamics may be described by

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u_f \qquad (4)$$

where $M(q)$ is the inertia matrix

$$M(q) = \begin{bmatrix} \left(J + m(a + l\sin\varphi)^2\right) & 0 \\ 0 & ml^2 \end{bmatrix}$$

$C(q,\dot{q})$ is the Coriolis matrix

$$C(q,\dot{q}) = \begin{bmatrix} (a + l\sin\varphi)\dot{\varphi} & (a + l\sin\varphi)\dot{\theta} \\ -(a + l\sin\varphi)\dot{\theta} & 0 \end{bmatrix}$$

$G(q)$ is the gravity vector

$$G(q) = \begin{bmatrix} 0 \\ gml\sin\varphi \end{bmatrix};$$

and finally u_f is an external generalized force given by

$$u_f = \begin{bmatrix} \tau_\theta \\ 0 \end{bmatrix}.$$

where τ_θ is the torque of the motor.

It can be easily seen that equations given in (4) define an under actuated system, because it has only one input τ_θ and two degrees of freedom θ and φ. Now, if $\tau_\theta = 0$ and $\varphi \in [0, 2\pi)$, the system (4) has two equilibrium points: the unstable equilibrium defined by $(\varphi = \pi, \theta = 0, \dot{\varphi} = 0)$, and the stable given by $(\varphi = 0, \theta = 0, \dot{\varphi} = 0)$.

Remark 1. ***Linearization of system (4) around the stable equilibrium point produces***

$$\delta\ddot{\theta}\left(J + m(a + l\varphi)^2\right) = \delta\tau \ , \ \delta\ddot{\varphi} + g\delta\varphi = 0,$$

and evidently, we can claim that the system is not locally controllable around its stable equilibrium point.

On the other hand, if the input action is a constant different from zero, it establishes another equilibrium manifold, which is characterized by the following relation:

$$\dot{\bar{\theta}} = \sqrt{\frac{g\sin\bar{\varphi}}{(a + l\sin\bar{\varphi})\cos\bar{\varphi}}} \ . \qquad (5)$$

Indeed, it is obtained substituting $\Big|_{\dot{\varphi} = \ddot{\varphi} = 0}$ into the second equation of (3). Now, we finish this section presenting the following mechanical properties, that system (4) satisfies:

P.1) $M(q)$ is definite positive. **P.2**) The matrix $H = M - 2C$ is skew-matrix

$$H = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix}.$$

with $x = lm\cos\varphi(a + l\sin\varphi)\theta$.

Hence $z^T Hz = 0$ for any $z \in \Re$. **P.3**) Vector $G(q)$ satisfies the following relation

$$G(q) = \frac{\partial P(q)}{\partial q} \text{ where } P(q) = mgl(1 - \cos\varphi)$$

P.4) The operator $\tau_\theta \to \theta$ is passive (see Fantoni and Lozano, 2002). For this, one verifies, after use of properties **P2** and **P3,** that the time derivative of total energy defined as

$$E(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q), \text{ is given by } \dot{E} = \dot{\theta}\tau_\theta$$

3. Stabilization

The control objective is to move the planar pendulum from a initial angular equilibrium position to another angular equilibrium position, by the action of the motor's angular velocity, assuming that the pendulums's angular position φ lies in $E\varphi$, where $E_\varphi = \{\varphi : -\pi/2 < \varphi < \pi/2\}$[1]. That is, from a set of initial conditions $(q(0), \dot{q}(0))$, such that $q_2(0) \in E\varphi$, we desire to change the present pendulum's angular position to another constant angular position, which is characterized by the equilibrium manifold defined previously in (5). For solving this problem a simple linear feedback passivity controller and an off-line trajectory planning controller are proposed.

3.1 A simple linear feedback control law

Consider the following controller

$$\tau_\theta = -k\left(\dot{\theta} - \dot{\bar{\theta}}\right); \ k > 0 \qquad (6)$$

Evidently, the above controller produces the following closed-loop system:

[1] Relation (12) gives sufficient conditions to assure that $\varphi \in E_\varphi$.

$$\left(J + m(a + l\sin\varphi)^2\right)\ddot{\theta} + 2ml(a + l\sin\varphi)\cos\varphi\dot{\theta}\dot{\varphi} = -k\left(\dot{\theta} - \dot{\bar{\theta}}\right)$$
$$ml^2\ddot{\varphi} - ml(a + l\sin\varphi)\cos\varphi\dot{\theta}^2 + mgl\sin\varphi = 0 \qquad (7)$$

To prove that (7) is asymptotically stable[2], we introduce the following Lyapunov function,

$$V_1(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + mgl(\cos\bar{\varphi} - \cos\varphi).$$

The time derivative of V_1 along the trajectories of the closed loop system equations (7) is given, after using the system equations and the passivity properties, by next relation

$$\dot{V}_1(q,\dot{q}) = -k\left(\dot{\theta} - \dot{\bar{\theta}}\right)^2 \le 0. \qquad (8)$$

Since V_1 is negative semi-definite, we can only conclude that the signals $\left\{\left(\dot{\theta} - \bar{\theta}\right), \varphi, \dot{\varphi}\right\}$ are bounded. We need to employ La Salle's invariance theorem to guarantee that the closed-loop system is asymptotically stable (Khalil, 1996).

Now, let us define the invariant set:

$$S = \left\{(q,\dot{q}) : \dot{V}(q,\dot{q}) = 0\right\} = \left\{-k\left(\dot{\theta} - \dot{\bar{\theta}}\right)^2 : \dot{\theta} = \dot{\bar{\theta}}\right\} \qquad (9)$$

On the set S the first equation of (7) becomes,

$$(a + l\sin\varphi)\cos\varphi\dot{\varphi}\dot{\bar{\theta}} = 0 \text{; with } \dot{\bar{\theta}} \ne 0. \qquad (10)$$

Integrating the last relation, from t to $t+\delta$ with $\delta > 0$, we obtain:

$$(a + l\sin\varphi(t))^2 = (a + l\sin\varphi(t+\delta))^2 \qquad (11)$$

Clearly, $\varphi(t)$ is a constant on the set S and also we have $\ddot{\varphi} = \dot{\varphi} = 0$. Hence, from the second equation of (7), we guarantee that $\varphi = \bar{\varphi}$

Notice that if the initial conditions satisfy

$$V_1(q(0),\dot{q}(0)) \le mgl, \qquad (12)$$

Then we guarantee that $\varphi \in E\varphi$. This is due to the fact that V_1 is a non-increasing function (see 8). The last relation defines the region of attraction of the closed loop system.

We summarize the previous stability analysis, as follows:

Proposition 1. ***Consider the closed loop system (7), under the assumption that the initial conditions satisfy relation (12). Then the equilibrium point ($\varphi = \bar{\varphi}, \dot{\theta} = \bar{\theta}, \dot{\varphi} = 0$) is asymptotically stable.***

3.1.1 Simulation Results

Fig. 2 shows the closed-loop responses of the linear controller. We use system (3) with the parameters *m=0.01* [Kg], *l=0.25* [m], *a=0.05* [m] and *L=0.0025* [Nw-m/s^2]. The control parameters were selected as k=0.5 and the desired pendulum's angular position was chosen $\bar{\varphi} = 1.1$ [rad]. All the initial conditions were set to zero, except the initial angular position which was set as $q_2(0) = -1$

[2] We only make regulation in the variables $\left\{\left(\dot{\theta} - \dot{\bar{\theta}}\right), \phi - \bar{\phi}, \dot{\phi}\right\}$

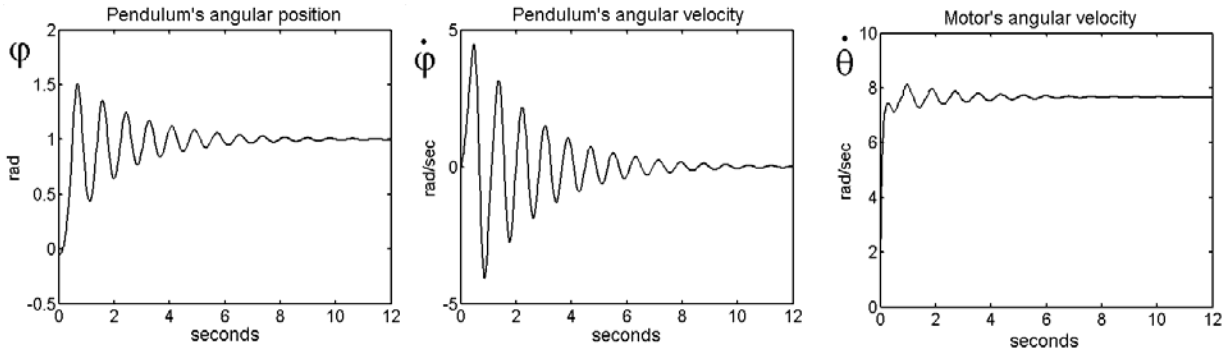

Fig. 2. Close-Loop response of feedback controller

3.2 The transfer manoeuvre task

Suppose that we want to carry out, in a finite time interval $\lfloor t_0, t_f \rfloor$, a transfer manoeuvre from an equilibrium to another equilibrium, by means of the motor's angular velocity . In other words, we desire to change the pendulum from a given equilibrium position $(\theta(t_0), \varphi(t_0))$ to another equilibrium position $(\theta(t_f), \varphi(t_f))$ based on the motor's rotational velocity. The manoeuvre task will be carried out following a nominal specified trajectory.

Let us suppose that the pendulum's angular position given by,

$$\varphi_*(t) = \begin{cases} \varphi_0 & t < t_0 \\ \psi(t, t_0, t_f) & t_0 \le t \le t_f \\ \varphi_f & t > t_f \end{cases}, \qquad (13)$$

where $\psi(t, t_0, t_f)$ is a smooth polynomial spline, interpolating between *0* and *1*,

$$\psi(t, t_0, t_f) = \delta^5(t)\left[\alpha_1 - \alpha_2\delta(t) + \ldots + \alpha_6\delta^6(t)\right]$$

where $\delta(t) = \left(\dfrac{t - t_0}{t_f - t_0}\right)$, $\alpha_1 = 252$, $\alpha_2 = 1050$, $\alpha_3 = 1800$, $\alpha_4 = 1575$, $\alpha_6 = 700$ and $\alpha_6 = 126$.

From the second equation of (3) we obtain the following relation for the motor's angular velocity, given by

$$\dot{\theta}_* = \sqrt{\frac{l\ddot{\varphi} + g\sin\varphi_*}{(a + l\sin\varphi_*)\cos\varphi_*}} \qquad (14)$$

and evidently, the velocity references vector is defined by:

$$\dot{q}_*^T = [\theta_*, \dot{\varphi}_*],$$

where variable $\dot{\varphi}_*$ is computed taking the time derivative of equation (14).

Since system (3) is not feedback linearizable and not locally controllable around the origin (see **Remark 1**), it is not possible to exactly solve the mentioned manoeuvre task. Despite the fact that there is no precise solution, the problem can be partially solve by taking some suitable approximations, as we indicate in the following assumption:

A1. We disregard the terms: $ml^2\dot{\varphi}_*(t)$, $ml^2\dot{\varphi}(t)$ and $ml^2\ddot{\varphi}(t) \cong 0$. That is, we change the pendulum's angular position by means of very slow movements of the pendulum.

The previous assumption can be validated by a numerical simulation. For example, taking into account the above experiment, we estimate that $ml^2\dot{\varphi}(t) \cong 10^{-3}$. Consequently, we can use assumption **A1**.

3.2.1 An off-line trajectory planning controller

We proceed to design a passivity feedback controller to perform the transfer manoeuvre task, in an approximate fashion. Let us introduce the following Lyapunov function[3] :

$$V_2(q,\dot{q}) = \frac{1}{2}(\dot{q}-\dot{q}_*)^T M(q)(\dot{q}-\dot{q}_*) + mgl(\cos\varphi_* - \cos\varphi).$$

Differentiating V_2 along the trajectories of (4), produces, after using the passivity properties and the assumption **A1**, the following approximation,

$$\dot{V}_2(q,\dot{q}) = (\dot{\theta}-\dot{\theta}_*)(\tau_\theta - m_{11}(q)\ddot{\theta}_*), \tag{15}$$

with $m_{11}(q) = J + m(a + l\sin\varphi)^2$ [4]

From the above relation we may propose the feedback controller as,

$$\tau_\theta = -k(\dot{\theta}-\dot{\theta}_*) + m_{11}(q)\ddot{\theta}_*. \tag{16}$$

Thus, substituting the input (16) into (15), we guarantee

$$\dot{V}_2 = (q,\dot{q}) = -k(\dot{\theta}-\dot{\theta}_*)^2.$$

To finish the asymptotic stability proof of the closed loop system, defined by equations (4) and (16), we must apply LaSalles' invariance theorem and follow the same arguments that were used in the previous proposition (see equations (9) and (11)).

Finally, the region of attraction of the closed loop system is defined by,

$$V_2(q(0),\dot{q}(0)) \le mgl. \tag{17}$$

The above relation follows directly, due the fact that V_2 is also a non-increasing function.

The above discussion can be summarized in the following proposition.

Proposition 2 ***Consider the system (4) and the controller (16) with positive constant k, under the assumption that initial conditions satisfy relation (17). Then, the closed loop system solution converges asymptotically to the nominal values*** $\varphi = \varphi_*$, $\theta = \theta_*$ ***and*** $\dot{\varphi} = \dot{\varphi}_*$.

Therefore, the feedback controller (16) can partially solve the transfer task from one pendulum's equilibrium point to another of its equilibrium points.

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[3] Recalling that we solely consider the case when $\varphi \in E_\varphi$, and we do not make regulation in θ.

[4] Ander assumption **A.1**, it is easy to check that we can approximate to zero the following terms: $mgl\dot{\varphi}_*(\sin\varphi_* - \sin\varphi)$ and all the centripetal forces.

3.2.2 Simulation Results

Fig. 3 shows the controlled responses of the system variables. The proposed controller is shown to make the system follow the specified nominal angular position of the pendulum φ_*, despite initial state deviation $\varphi(0) = -0.1$ [rad]. The time parameters were set as $t_0 = 0$ sec and $t_f = 10$ sec. All the initial conditions and the physical parameter were the same as in the previous simulations.

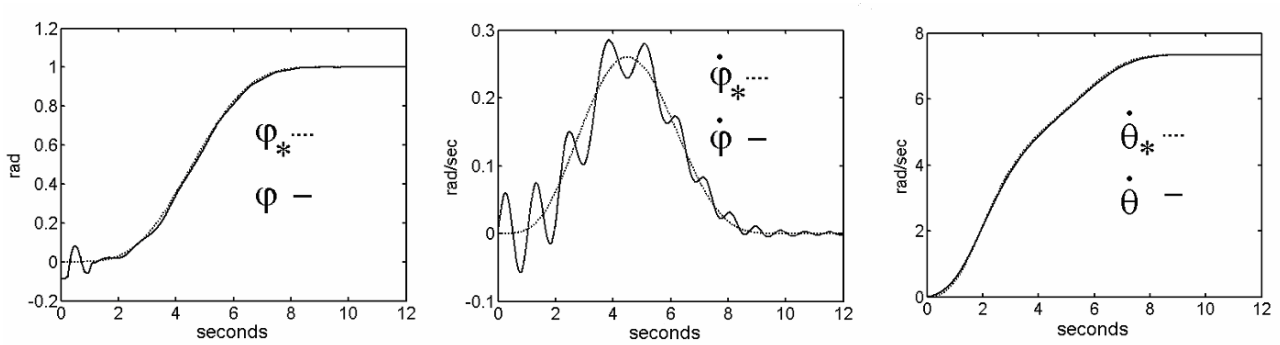

Fig. 3. Closed-Loop performance of feedback controller
Based on trajectory planning approach

4. Conclusions

In this work, we have presented two feedback control schemes for the stabilization of the rotational pendulum. Both schemes are based on the Lyapunov approach. The first is a simple passive linear control, and the second is a combination of the Lyapunov and the off-line trajectory planning approaches. The first control solely solves the regulation problem; the second, transfers (in an approximated way) the pendulum from an equilibrium point to another equilibrium point. These equilibrium points are completely characterized by the equilibrium manifold defined in (5). This task is carried out neglecting some suitable terms, because the pendulum angular position changes very slowly. The closed-loop performance of both controllers has been shown to be quite satisfactory as assessed from the numerical experiments. Lyapunov's method in conjunction with La Salle's invariance theorem has been applied in both cases for the asymptotic stability analysis.

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