A Fast Method for Plotting Binary Solids Composed of a Large Number of Voxels

Un Método Rápido para Graficar Sólidos Binarios Compuestos de una Gran Cantidad de Voxels

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Abstract

We present a fast and efficient method for plotting rigid solids composed of a large number of voxels. This method is based on the concept of the contact surface area. *The contact surface area corresponds to the sum of the contact surface areas of the face-connected voxels of solids. A relation between the area of the surface enclosing the volume and the contact surface area is presented. We analyze the minimum and maximum contact surface areas. Finally, we present a result using a binary solid of the real world.*

Keywords: Binary Solids, Voxel-Based Objects, Plotting.

Resumen

Se presenta un método rápido y eficiente para graficar objetos sólidos y rígdos compuestos por un gran número de voxels. Este método se basa en el concepto del área de superficie de contacto. *El área de superficie de contacto corresponde a la suma de las áreas de las superficies de contacto de los voxels vecinos de los sólidos. La relación entre la superficie envolvente y la de contacto del volumen es definida. Se analizan las áreas de las superficies de contacto mínimas máximas. Finalmente, se presenta un resultado del método usando datos reales de un sólido binario.*

Palabras clave: Sólidos Binaŕios, Objetos Voxelizados y Graficación.

1 Introduction

The study of rigid solids is an important part in computer vision. In the content of this work, we present the concept of *contact surface area* for rigid solids composed of voxels (Bribiesca, 1998). Thus, solids are represented as 3D arrays of voxels which are marked as filled with matter. The representation of solids by means of spatial occupancy arrays can require much storage if resolution is high, since space requirements increase as the cube of linear resolution (Ballard and Brown, 1982). Nevertheless, at present with the declining cost of computer memory and storage devices, explicit spatial occupancy arrays are often used. An advantage of using this kind of representation is that slices through objects may be easily produced.

Several authors have been using different kinds of representations for solids: rigid solids represented by their boundaries or enclosing surfaces are shown in Requicha (1980) and Besl and Jain (1985); constructive solid geometry schemes are presented and analized in Voelcker and Requicha (1977) and Boyse (1979); generalized cylinders as 3D volumetric primitives are shown in Soroka (1979), Soroka and Bajcsy (1976), and Brooks (1983); and superquadrics in Pentland (1986).

Solid objects are normally represented by their enclosing surfaces, commonly defined as graphics primitives. In this manner, the interior of a solid object is not explicity represented. On the other hand, solid object voxelization has not been sufficiently studied (Fang and Chen, 2000). Voxelization is mainly used as a preprocessing step in the current volume graphics process. The voxelization process needs to be done on-the-fly after each change of the model for volume rendering and other volume-related applications. In order to support dynamic scenes and interactive applications, it is necessary that fast voxelization algorithms be used (Manohar, 1999). This work deals with a fast method for plotting voxel-based objects.

This paper is organized as follows. In Section 2 we give a set of concepts and definitions and present the contact surface area and its relation to the enclosing surface area. In Section 3 we give the method here proposed for plotting rigid solids composed of a large number of voxels. Section 4 presents some results using digital elevation models as rigid solids and finally, in Section 5 we give some conclusions.

2 Concepts and Definitions

In order to introduce our concept of contact surface area we use volumetric representation for rigid solids by means of spatial occupancy arrays. Thus, the solids are represented as 3D arrays of voxels which are marked as filled with matter. Furthermore, shape is referred as shape-of-object, and object is considered as a geometric solid composed of voxels.

In the content of this work, *area* is a numerical value expressing 2D extent in a plane, or sometimes it is used to mean the interior region itself (Karush, 1989). Another consideration is the assumption that an *entity* has been isolated from the real world. This is called the *rigid solid*, and is defined as a result of previous processing. In this work, the length of all the edges of voxels is considered equal to one.

In this section, we define the contact surfaces for rigid solids composed of voxels. Also, we define the relation between the contact surface area and the area of the surface enclosing the volume. This relation between the areas of the surfaces can be used in different polyhedrons, which cover up space. In this case, we demonstrate the above mentioned using hexahedrons. In order to introduce our plotting method, a number of geometrical concepts are defined below:

The *area A of the enclosing surface* of a rigid solid composed of a finite number n of voxels, corresponds to the sum of the areas of the external plane polygons of the voxels which form the visible faces of the solid.

The *contact surface area A_c* of a rigid solid composed of a finite number n of voxels, corresponds to the sum of the areas of the contact surfaces which are common to two voxels.

The *relation between the areas of the enclosing surface and the contact surface.* For any binary solid composed of n voxels, the relation between the areas is defined by:

$$2A_c + A = aFn, \qquad (1)$$

where A_c is the contact surface area, A is the area of the enclosing surface, a is the area of the face of the voxel ($a = 1$ is assumed in the forward examples corresponding to figures 1 to 3), and F is the number of plane polygons or faces of the voxel (Bribiesca, 2000).

Geometrically, it means that the sum of two times the contact surface area plus the enclosing surface area is equal to the total sum of the polygon areas of all the voxels of the solid. Notice that the above-mentioned relation is preserved for solids having holes and inner holes.

By Eq. (1), the contact surface area is:

$$A_c = \frac{aFn - A}{2}. \qquad (2)$$

The *contact surface area is maximized by a digital cube.* In the digital domain, when we are using regular hexahedrons the contact surface area is maximized to the form of the used polyhedron. Thus, if the solids are described using voxels ($F = 6$) the contact surface area is maximized by a digital cube. Fig. 1 shows this: in the figures 1(a) and (b) we show a digital cube and a digital sphere composed of 19,683 voxels each one. The cube root of 19,683 is a positive integer: $\sqrt[3]{19683} = 27$, i.e. 19,683 is a perfect cube. The contact surface area for this digital cube is equal to 56,862 and for this digital sphere is equal to 56,453. Notice that the maximum contact surface area belongs to the digital cube. Here we do not prove this, we only illustrate it.

The *minimum contact surface area $A_{c_{min}}$* for a solid composed of n voxels is given by:

$$A_{c_{min}} = a(n - 1); \qquad (3)$$

this minimum value is attained if each voxel has only one contact face, for example when the voxels are aligned or arranged as illustrated in Fig. 2(o).

The *maximum contact surface area $A_{c_{max}}$* for a solid composed of n voxels, where $n = m^3$ is a perfect cube, is easily obtained from Eq. (2):

$$A_{c_{max}} = 3a(m^3 - m^2). \qquad (4)$$

If n is not a perfect cube, say $m_1^3 < n < m_2^3$ for some successive integers m_1 and m_1, then:

$$3a(m_1^3 - m_1^2) < A_{c_{max}} < 3a(m_2^3 - m_2^2), \qquad (5)$$

and a rough estimate of $A_{c_{max}}$ is given by

$$A_{c_{max}} \approx 3a(n - \sqrt[3]{n})^2). \qquad (6)$$

Fig. 2 shows examples of solids composed of 27 voxels each one. Notice that the maximum contact surface area corresponds to the solid in Fig. 2(a), its value is 54 and may be obtained by Eq. (4). Figures 2(b), (c), (d),...,(o) show different examples of solids which have descending contact surface areas in steps of 2. Thus, for each solid in Fig. 2 we have: (a) $A_c = 54$ and $A = 54$; (b) $A_c = 52$ and $A = 58$; (c) $A_c = 50$ and $A = 62$; ... ; (o) $A_c = 26$ and $A = 110$. Notice that the values of

the contact surface areas of the solids in Fig. 2 are decreasing in steps of 2 whereas the values of the enclosing surface areas are increasing in steps of 4. The minimum contact surface area belongs to the solid presented in the Fig. 2(o), its contact surface area is 26, which may be obtained using the Eq. (3). The contact surface area decreases linearly.

3 The Proposed Method for Plotting Solids

The enclosing surface area of a solid corresponds to the sum of the areas of visible faces of the solid. On the other hand, the contact surfaces correspond to the hidden faces of the solid. Therefore, when a solid is plotted the contact surfaces must be eliminated from the plotting, this decreases the computation greatly. Thus, the number of hidden faces which must be eliminated of any plotting is equal to $2A_c$. Subsequently, the visible faces of the solid are represented in a standard vector file format, this produces a wireframe plotting in which all lines are represented, including those that would be hidden by faces. To eliminate these hidden lines from the plotting, the HIDE command is used (which is programmed in most 3D CAD software). The final effect is that objects are plotted as opaque solids.

A simple algorithm for eliminating the hidden faces of a solid S of n voxels is as follows. Assume the voxels are unit cubes. Let P be the smallest rectangular parallelepiped containing S and let the lengths of its sides be denoted α, β, and γ. A three-dimensional binary matrix M of $\alpha\beta\gamma$ elements is used to represent P and S. The elements of M correspond in the obvious way to the voxels of P, and an element of M is 1 if and only the corresponding voxel is in S. Each element of M is visited once following a natural order and a bounded number of operations (simple boolean comparisons and writings to an ouput file). Thus, the time (and the space) complexity of the algorithm is $O(\alpha\beta\gamma)$. The worst-case complexity is $O(n^3)$, it is achieved for solids like the one shown in Fig. 2(o) when $\alpha = \beta = \gamma \approx n/3$. The best-case complexity is $O(n)$, obviously this is the case when $\alpha\beta\gamma = n$ (i.e. when $S = P$, as for the cube shown in Fig. 2(a)). In applications as the one to be presented in the next section, it is common that each S fills more than a constant fraction of its corresponding P, thus $\alpha\beta\gamma \leq kn$ holds for the considered family of solids and the complexity is $O(n)$.

4 Results

In this section, we present an example of digital elevation models of "El Valle de México". Those models are digital representations of the Earth's surface. Generally speaking, a digital elevation model is generated as a uniform rectangular grid organized in profiles. In this case, the digital elevation models are represented as rigid solids composed of a large number of voxels (Karabassi et al., 1999). The digitalization of these models is based on 1:250,000 scale contours.

Fig. 3 shows the digital elevation model of the volcano "Iztaccíhuatl". This volcano is to the east of "El Valle de México" and what is shown is the western side of the mountain. In Fig. 3(a) the volcano is represented by a 3D mesh of 200×200. The elevation data values of the model presented in this study were increased to enhance their characteristics. Fig. 3(b) illustrates the volcano as a rigid solid composed of $960,903$ voxels, where $A = 121,544$ and $A_c = 2,821,937$ ((6) gives $A_{c_{max}} \approx 2,853,496$). Therefore, the number of hidden faces which were eliminated is equal to $5,643,874$. When the volcano has this representation it is possible to use morphological operators to erode it, simplify binary solid data, and preserve essential shape characteristics (Bribiesca, 1998). Of course, when we are using morphological operators all the amount of information of the binary solid is considered. Notice that in Fig. 3 (c) the volcano Iztaccíhuatl represented as a binary solid was cut across to present its interior without inner faces, which were eliminated using the proposed method.

5 Conclusions

Using the concept of contact surfaces a fast and efficient method for plotting rigid solids composed of a large number of polyhedrons, is defined. Suggestions for further work: *prove* the concept of contact surfaces for rigid solids using different polyhedrons (which cover up space, for instance non-regular octahedrons), in order to obtain a better representation and plotting; when we are working in the digital domain using rigid solids composed of polyhedrons, *demonstrate* that the contact surface area is maximized to the form of the used polyhedron.

Acknowledgements

<div align="center">

(a)

n=19683

$A_c = 56862$

(b)

n=19683

$A_c = 56453$

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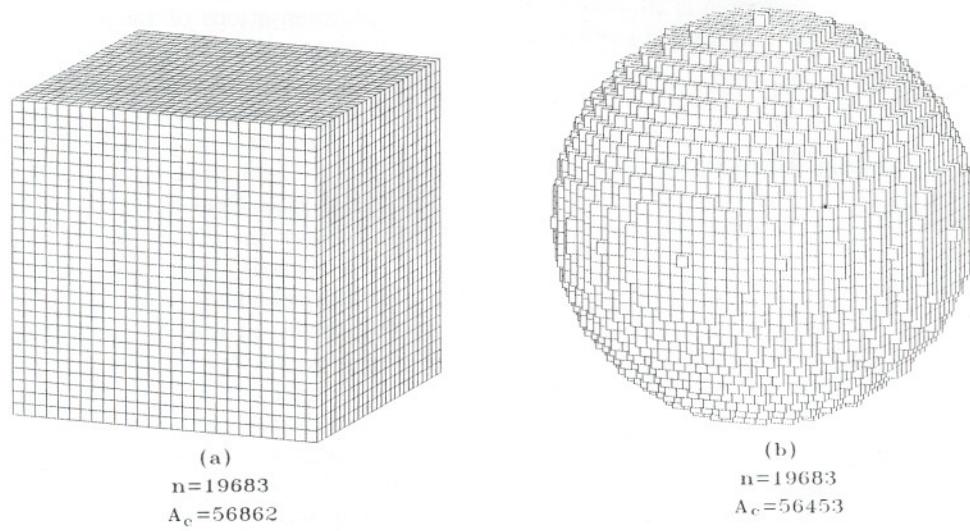
Figure 1. The contact surface area is maximized by a digital cube: (a) a digital cube composed of 19,683 voxels, its contact surface area is equal to 56,862; (b) a digital sphere composed of 19,683 voxels, its contact surface area is equal to 56,453.


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(a) $A_C = 54$ (b) $A_C = 52$ (c) $A_C = 50$ (d) $A_C = 48$ (e) $A_C = 46$

(f) $A_C = 44$ (g) $A_C = 42$ (h) $A_C = 40$ (i) $A_C = 38$ (j) $A_C = 36$

(k) $A_C = 34$ (l) $A_C = 32$ (m) $A_C = 30$ (n) $A_C = 28$ (o) $A_C = 26$

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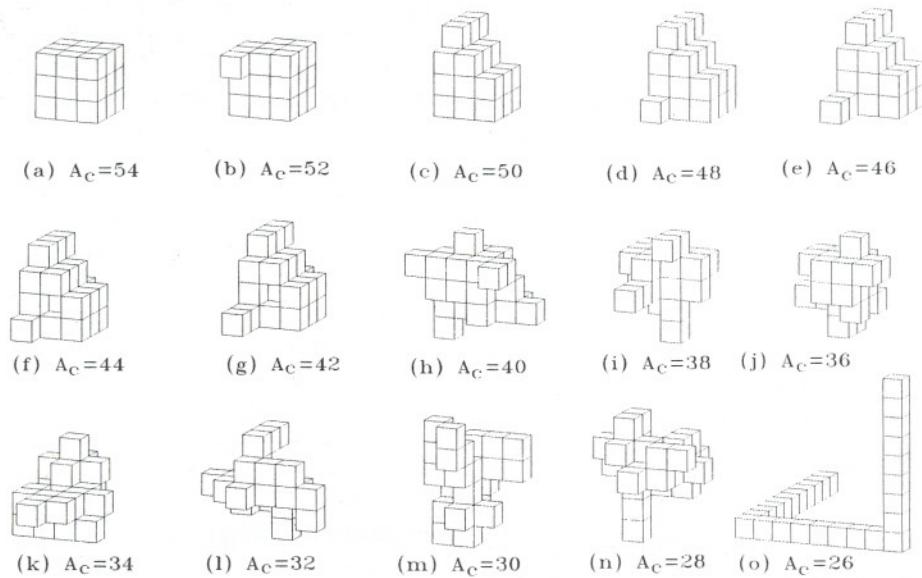
Figure 2. Contact surface areas in descending order for different solids composed of 27 voxels aech one: (a) the maximum contact surface area; (b); (c); (d);...; (o) this example of solid has the minimum contact surface area.

(a)

(b)

(c)
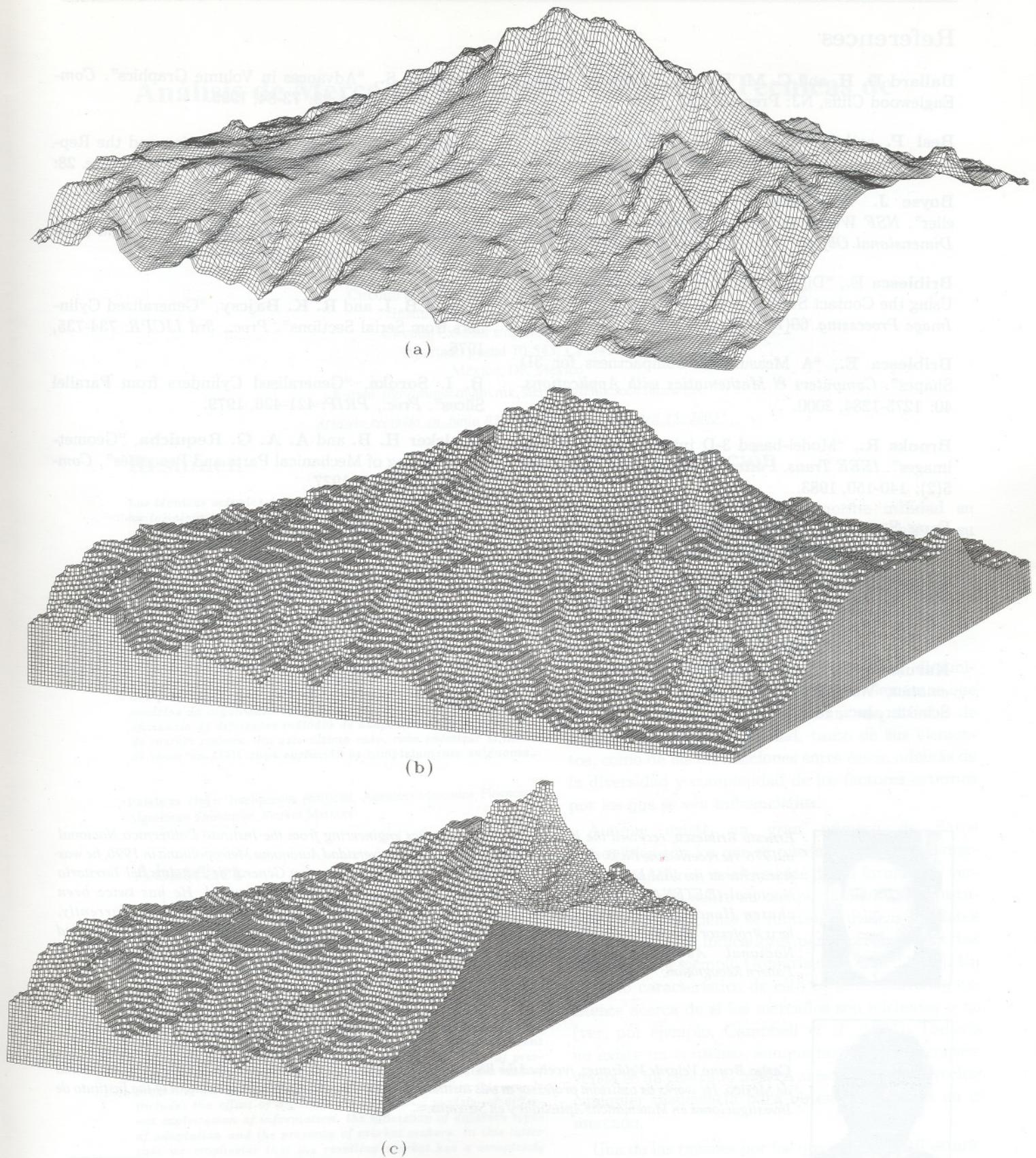
Figure 3: The digital elevation model of the volcano «Iztaccíhuatl»: (a) represented by a 3D mesh of 200 x 200 elements; (b) composed of 960,903 voxels; (c) the volcano without inner faces.

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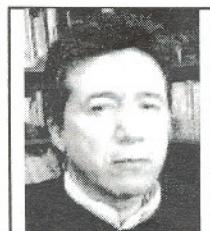
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