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Parametric Negations of Probability Distributions and Fuzzy Distribution Sets

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Abstract. Negation of probability distributions (PD) was initially introduced by Yager as a transformation of probability distributions representing linguistic terms like High Price, into probability distributions representing linguistic terms like Not High Price. Further, different negations of PD and formal definitions of negations of probability distributions have been proposed, and several classes of such negations have been studied. Here we give a new look at negators dependent and not dependent on probability distributions. We consider different parametric representations of linear negators and analyze relationships between the parameters of these representations. We introduce a new parametric negation of probability distributions based on the involutive negation of PD. Recently it was proposed to consider probability distributions as fuzzy distribution sets, which paved the way for the extension of many concepts and operations of fuzzy sets on probability distributions. From such a point of view, Yager's negation of probability distributions is an extension of the standard negation of fuzzy logic, also known as Zadeh's negation. In this paper, using this approach, we extend the parametric Yager's and Sugeno's negation of fuzzy logic on probability distributions and study their properties. Considered parametric negations of probability distributions can be used in the models of probabilistic reasoning.

Keywords. Probability distribution, fuzzy logic, negation, Yager's negation, Zadeh's negation, Sugeno's negation.

1 Introduction

In [1], R. Yager considered the problem of how to define a probability distribution Not High Price if it is given a probability distribution with linguistic interpretation High Price. To address this problem, he proposed a transformation of probability distributions (PD) called a negation. Later, many papers introduced and studied different types of negations of probability distributions [2-18].

The paper [8] studied the general properties of negations of probability distributions, introduced the class of PD-independent linear negators, and showed that linear negators can be represented as a convex combination of Yager's and uniform negators. In [11], linear negations were represented as a parametric generalization of Yager's negation. In [9], it was shown that linear negations are contracting negations, such that the multiple negations of probability distributions have as a limit the uniform distribution.

In [13], it was shown that repeated application of independent negations leads to a progressive loss of information for the general class of φ-entropies. In [9] it was introduced an involutive negation of probability distributions. Involutive operations play an important role in Boolean [19], De Morgan and Kleene algebras [20], fuzzy sets theory, and fuzzy logic [21-23], in the definition and construction of association measures and correlation functions [24], etc.

The involutive negation of probability distributions [9] was used in the definition of the correlation between frequency distributions, bar charts, and categorical variables[25] and in the analysis of relationships between them [26]. Here we introduce parametric negation of probability distributions based on involutive negation.

Recently, it was proposed to consider probability distributions as fuzzy sets, called fuzzy distribution sets [27]. Such interpretation of PD paves the way for the extension of fuzzy concepts and operations on probability distributions.

From this point of view, Yager's negation of probability distributions is an extension of the standard negation of fuzzy logic, also known as Zadeh's negation [21, 28]. In this paper, we extend parametric families of Yager's and Sugeno's negations from fuzzy logic [23, 29, 30] on probability distributions.

Section 2 gives definitions of probability negators and negations of probability distribution. In Sections 3 and 4, we consider known pd-independent and ps-dependent parametric probability negators.

In Section 5, we introduce a new parametric family of probability negators based on the involutive negation of probability distributions. In Sections 6 and 7, we propose probability negators as extensions of Yager's and Sugeno's parametric negations of fuzzy logic. The last section contains the Conclusion.

2 Negation of Probability Distributions

Consider some basic concepts related to negations of probability distributions [8, 9].

A **probability distribution** (*PD*) of the length *n* is an n-tuple $P = (p_1, \dots, p_n)$ satisfying the following properties for all $i = 1, \dots, n$:

$$0 \le p_i \le 1, \qquad \sum_{i=1}^{n} p_i = 1.$$

The elements of such n-tuple can be considered as probabilities of mutually exclusive outcomes $X = \{x_1, \dots, x_n\}$, $n \ge 2$, of some experiments. For simplicity, we fix the ordering of elements of the set X according to the indexes of these elements, such that p_i is the probability of x_i. In this paper, we are not interested in the elements of X but only consider concepts related to the operations of negation of probability distributions.

Below is an example of the simplest probability distribution called the **uniform distribution** [8]:

$$P_U = \left(\frac{1}{n}, \dots, \frac{1}{n}\right).$$

This probability distribution plays an important role in the analysis of negations of PD.

Denote $\mathcal{P}_n$ the set of all probability distributions of the length n. A **negation of probability distributions** is a function $neg: \mathcal{P}_n \to \mathcal{P}_n$ such that for any probability distribution $P = (p_1, \dots, p_n)$ in $\mathcal{P}_n$ the probability distribution $neg(P) = Q = (q_1, \dots, q_n)$, satisfies for all $i = 1, \dots, n$, the following properties:

$$\text{if } p_i \le p_j, \text{ then } q_i \ge q_j. \qquad (1)$$

From (1), it follows:

$$\text{if } p_i = p_j \text{ then } q_i = q_j.$$

Let $P = (p_1, \dots, p_n)$ and $Q = (q_1, \dots, q_n)$ are two probability distributions from $\mathcal{P}_n$ satisfying (3), then we will say that Q ***is a negation of*** P. Note, that from (1), it does not follow that P is a negation of Q. For given P there can exist several different negations of P.

If for a given $P = (p_1, \dots, p_n)$ there exists a function $N: [0,1] \to [0,1]$ transforming P in its negation $Q = (q_1, \dots, q_n)$ by: $q_i = N(p_i)$, $i = 1, \dots, n$, such that $Q = (N(p_1), \dots, N(p_n))$, then we will say that N is a **probability negator for** P. For a negator N, the following properties are fulfilled:

$$0 \le N(p_i) \le 1,$$

$$\sum_{i=1}^{n} N(p_i) = 1,$$

$$\text{if } p_i \le p_j \text{ then } N(p_i) \ge N(p_j).$$

If the function N does not depend on P and it is a negator for all PD in $\mathcal{P}_n$ then N is called a **pd-independent negator**. If the function N is a negator for all probability distributions P in $\mathcal{P}_n$ but depends on probability distributions then N is called a **pd-dependent negator**.

In the following sections, we consider examples of pd-independent and pd-dependent negators.

3 PD-Independent Parametric Negators

PD-independent negator $N: [0,1] \to [0,1]$ depends only on the value p, but does not depend on PD where this value can appear. Hence, if N is PD-independent negator, then for any PD $P = (p_1, \dots, p_n)$ and $Q = (q_1, \dots, q_n)$ in $\mathcal{P}_n$, it holds:

$$\text{if } p_i = q_j, \text{ then } N(p_i) = N(q_j). \quad (2)$$

Consider the parametric family of linear, pd-independent, negators and corresponding negations of probability distributions $P = (p_1, \dots, p_n)$.

Uniform negator $N_U(p)$ and uniform negation $neg_U(P)$ [8]:

$$N_U(p) = \frac{1}{n}, \text{ for any } p \text{ in } [0,1],$$

$$neg_U(P) = (N_U(p_1), \dots, N_U(p_n)) == \left(\frac{1}{n}, \dots, \frac{1}{n}\right) = P_U.$$

Yager's negator $N_Y(p)$ and Yager's negation $neg_Y(P)$ [1]:

$$N_Y(p) = \frac{1-p}{n-1},$$

$$neg_Y(P) = (N_Y(p_1), \dots, N_Y(p_n)) = \left(\frac{1-p_1}{n-1}, \dots, \frac{1-p_n}{n-1}\right).$$

The following parametric extension of Yager's negator and Yager's negation have been introduced in [11]:

$$N_{\delta Y}(p) = \frac{1-\delta p}{n-\delta}, \quad 0 \le \delta \le 1, \quad (3)$$

$$neg_{\delta Y}(P) = \left(\frac{1-\delta p_1}{n-\delta}, \dots, \frac{1-\delta p_n}{n-\delta}\right),$$

When $\delta = 0$, the $neg_{\delta Y}$ negation coincides with uniform negation neg_U, and when $\delta = 1$, the $neg_{\delta Y}$ negation coincides with Yager's negation.

Generally, $N_{\delta Y}$ negator coincides [11] with the class of linear negators N_L [8] that are convex combinations of uniform negator and Yager's negator, and have several equivalent parametric representations [8]:

$$N_L(p) = \alpha N_U(p) + (1-\alpha) N_Y(p) = \alpha \frac{1}{n} + (1-\alpha)\frac{1-p}{n-1}, \quad 0 \le \alpha \le 1, \quad (4)$$

$$N_L(p) = N_L(1) + (1 - nN_L(1))\frac{1-p}{n-1}, \quad (5)$$

where $N_L(1) \in [0, \frac{1}{n}]$, and:

$$N_L(p) = N_L(0) + (1 - nN_L(0))p, \quad (6)$$

where $N_L(0) \in [\frac{1}{n}, \frac{1}{n-1}]$.

There exist the following relationships between parameters of different representations of linear negators (3)-(6).

From $N(1)$, one can obtain:

$$N(0) = \frac{1-N(1)}{n-1},$$

$$\delta = \frac{1-nN(1)}{1-N(1)},$$

$$\alpha = nN(1).$$

From $N(0)$, one can obtain:

$$N(1) = 1-(n-1)N(0),$$

$$\delta = n - \frac{1}{N(0)},$$

$$\alpha = n - n(n-1)N(0).$$

From δ, one can obtain:

$$N(1) = \frac{1-\delta}{n-\delta},$$

$$N(0) = \frac{1}{n-\delta},$$

$$\alpha = n\frac{1-\delta}{n-\delta}.$$

From α, one can obtain:

$$N(1) = \frac{\alpha}{n},$$

$$N(0) = \frac{n-\alpha}{n(n-1)},$$

$$\delta = \frac{n(1-\alpha)}{n-\alpha}.$$

4 Parametric PD-Dependent Negators

Generally, for PD-dependent negator, the values $N(p_i)$ for probability values p_i in a PD $P = (p_1, \dots, p_n)$ depend on other values p_j in P, and (2) does not fulfill.

The negator of Zhang et al. [5] is a parametric pd-dependent negator with parameter $k \geq 1$:

$$N_Z(p_i) = \frac{1-p_i^k}{n-\sum_{j=1}^{n} p_j^k}, \quad k \geq 1. \tag{7}$$

When $k = 1$ it coincides with Yager's negator N_Y. When k approaches infinity this negator approaches uniform negator N_U.

5 Involutive Negator Based Parametric Negators

Here we introduce a parametric extension of involutive negation of probability distributions. A negation neg of probability distributions is ***involutive*** if for any PD P in $\mathcal{P}_n$ it is fulfilled:

$$neg(neg(P)) = P.$$

Let $P = (p_1, \dots p_n)$ be a probability distribution. Denote $\max(P) = \max\{p_1, \dots, p_n\}$, $\min(P) = \min\{p_1, \dots, p_n\}$ and $MP = \max(P) + \min(P)$. The involutive negator of probability distributions was introduced by Batyrshin in [9] as follows:

$$N_B(p_i) = \frac{max(P)+min(P)-p_i}{n(max(P)+min(P))-1} = \frac{MP-p_i}{nMP-1}, \tag{8}$$

It is clear that this negator is pd-dependent. Denote:

$$neg_B(P) = Q = (q_1, \dots, q_n) = (N_B(p_1), \dots, N_B(p_n)).$$

Calculating this negation, in (8) we need to use $max(P)$ and $min(P)$. But calculating:

$$neg_B(neg_B(P)) = neg_B(Q) = (N_B(q_1), \dots, N_B(q_1)),$$

In (8) instead of P and p_i, we need to use Q and q_i hence instead of $max(P)$ and $min(P)$ we need to use $max(Q)$ and $min(Q)$.

Here we introduce the parametric extension of the involutive negator (8), depending on the parameter $0 \leq \gamma \leq 1$:

$$N_{\gamma B}(p_i) = \frac{max(P) + min(P) - \gamma p_i}{n(max(P) + min(P)) - \gamma} = \frac{MP - \gamma p_i}{nMP - \gamma}, \tag{9}$$

When $\gamma = 1$ it coincides with involutive negator N_B. When $\gamma = 0$, this negator coincides with uniform negator N_U. The nearer the value γ to 1, the more similar this negator to involutive negator.

6 Generation of Negators

Theorem 1 [8,27]. Let $f(p)$ be a non-negative non-increasing real-valued function satisfying for all probability distributions $P = (p_1, \dots, p_n)$ the property: $\sum_{j=1}^{n} f(p_j) > 0$, then the function N defined for all $i = 1, \dots, n$ by:

$$N(p_i) = \frac{f(p_i)}{\sum_{j=1}^{n} f(p_j)} \tag{10}$$

Is a negator of probability distributions.

The function f in (10) will be called here a **generator** of the negator N.

In the next section, we consider how to generate negators of probability distributions using negations of fuzzy logic.

7 Fuzzy Logic Based Parametric Probability Negators

In [27], it was proposed to consider probability distribution functions as membership functions of fuzzy sets subject to the sum of membership values equal to 1.

These fuzzy sets are called fuzzy distribution sets. Such interpretation of probability distributions gives the possibility to extend on them almost all concepts of fuzzy sets. In fuzzy logic, the truth or membership value, like the probability, takes value in the interval [0,1].

A negation in fuzzy logic is defined as a decreasing function $N: [0,1] \to [0,1]$ taking values $N(0) = 1$, and $N(1) = 0$. Hence, such a negation function can be used as a generator of probability negator and negation of probability distributions in (10). Consider some popular negations of fuzzy logic [21, 23, 28-30].

Standard or **Zadeh's negation** of truth values $x \in [0,1]$ in fuzzy logic is defined as follows [21]:

$$N(x) = 1 - x.$$

Using in (10) Zadeh's negation as negation of probability values $f(p) = 1 - p$, we obtain Yager's negator for probability distributions:

$$N_Y(p_i) = \frac{1-p_i}{\sum_{j=1}^{n}(1-p_j)} = \frac{1-p_i}{n-\sum_{j=1}^{n} p_j} = \frac{1-p_i}{n-1}.$$

Sugeno's negation of truth values $x \in [0,1]$ in fuzzy logic is defined as follows [23,29]:

$$N(x) = \frac{1-x}{1+\omega x}, \text{ where } \omega > -1.$$

Using $f(p) = \frac{1-p}{1+\omega p}$ we obtain from (10) the following probability negator generated by Sugeno negation of fuzzy logic:

$$N_{\omega S}(p_i) = \frac{\frac{1-p_i}{1+\omega p_i}}{\sum_{j=1}^{n}\frac{1-p_j}{1+\omega p_j}} = \frac{1-p_i}{(1+\omega p_i)\sum_{j=1}^{n}\frac{1-p_j}{1+\omega p_j}}, \quad (11)$$

where $\omega > -1$.

Yager's negation of truth values $x \in [0,1]$ in fuzzy logic is defined as follows [30]:

$$N(x) = \sqrt[\tau]{1 - x^{\tau}}, \text{ where } \tau > 0.$$

Using (10) we obtain the following probability negator for probability distributions:

$$N_{\tau Y}(p_i) = \frac{\sqrt[\tau]{1-p_i^{\tau}}}{\sum_{j=1}^{n}\sqrt[\tau]{1-p_j^{\tau}}}, \text{ where } \tau > 0. \quad (12)$$

8 Conclusion

The paper considered different parametric probability negators that can be used for construction of parametric families of negations of probability distributions. We considered several equivalent parametric representations of pd-independent linear negators introduced before, and formulated relationships between parameters of these negators.

We introduced a new parametric family of probability negators based on involutive negator. Finally, considering a probability distribution as a fuzzy distribution set, we extended Sugeno's and Yager's parametric negations used in fuzzy logic on probability negators. Considered parametric probability negators can be used for construction of negations of probability distributions in the models of probabilistic reasoning.

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