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The Subalgebra Lattice of A Finite Diagonal–Free Two–Dimensional Cylindric Algebra

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Abstract. Diagonal–free two–dimensional cylindric algebras (**Df**$_2-$algebras for short) are Boolean algebras enriched with two existential quantifiers which commute. **Df**$_2-$algebras were introduced by A. Tarski, L. Chin and F. Thompson with the purpose of providing an algebraic device for the study of the first–order predicate calculus with two variables. This work is devoted to problems related to finite **Df**$_2-$algebras. More precisely, we study and describe the family of subalgebras of a given finite **Df**$_2-$algebra. Then, identifying the algebras of this family which are isomorphic, we provide a full description of the lattice of all non–isomorphic subalgebras of a given finite **Df**$_2-$algebra.

Keywords. Finite Boolean algebras, diagonal–free two–dimensional cylindric algebras, lattice of subalgebras.

1 Introduction

Cylindric algebras were introduced by A. Tarski in the 1940s with the intention of providing an algebraic counterpart to the first–order predicate calculus. As a general reference we mention the fundamental work by Henkin, Monk and Tarski [7].

In particular, the class of diagonal–free two–dimensional cylindric algebras constitute an algebraic counterpart to the first–order predicate calculus without identity and considering just two variable symbols in the language.

Formally a diagonal–free two–dimensional cylindric algebra is a Boolean algebra enriched with two existential quantifiers which commute.

This class of algebras will be denoted **Df**$_2$, in agreement with the notation introduced in [7]. Besides, the class **Df**$_2$ constitute a variety (that is, it can be described by means of a finite number of equations) and has been widely studied.

However, little research has pursued to investigate those problems inherent to finite algebras. On the other hand, a monadic Boolean algebra is any pair $(A, \exists)$ formed by a Boolean algebra A enriched with an existential quantifier $\exists$ defined on A (see [6]) and, within the context of cylindric algebras, these algebras are diagonal-free one-dimensional cylindric algebras or **Df**$_1-$algebras.

As we said, the variety **Df**$_2$ has been widely investigated by different authors. Among other known results, it can be mentioned that D. Monk studied the lattice $\Lambda(\mathbf{Df}_2)$ of all subvarieties of **Df**$_2$ and proved that it has $\aleph_0$ elements (subvarieties).

This author also showed that every element of $\Lambda(\mathbf{Df}_2)$ has a finite base and a decidable equational theory. Later, N. Bezhanishvili, in [1], proved that every proper subvariety of **Df**$_2$ is locally finite although **Df**$_2$ is not.

On the other hand, some problems inherent to finite algebras have also been studied.

For instance, in [3], the author exhibited a connection between **Df**$_2-$algebras and pairs formed by a monadic Boolean algebra and a certain subalgebra of it; and as a consequence, it was obtained a formula to calculate the number of monadic subalgebras of a given finite monadic Boolean algebra.

Also, in [4], formulas for computing the number of **Df**$_2-$algebra structures that can be defined over a finite Boolean algebra as well as the fine spectrum of **Df**$_2$ were obtained.

Finally, the lattice $\Lambda(\mathbf{Df}_2)$ was studied and a full description of the poset of all its joint–irreducible elements was given.

Besides, in [5] A. V. Figallo and C. M. Gomes studied the variety of $\mathbf{T_k m}-$algebras, this is, monadic Boolean algebras endowed with a monadic automorphism of period k and established, in the finite case, the relationship between this variety and the variety $\mathbf{Df}_2$.

It is worth mentioning that the study of the lattice of all subalgebras of an abstract algebra has interested many authors.

For instance, G. Birkhoff and O. Frink, [2], characterized the subalgebra lattices of universal algebras as algebraic lattices.

On the other hand, in [8], the author proved that every algebraic lattice is isomorphic to the subalgebra lattice of a square of some universal algebra.

The purpose of this paper is to study some properties related to the subalgebras of a finite diagonal–free two–dimensional cylindric algebra.

In section 2, we recall some well-known facts about **Df**$_2-$algebras, we emphasize, in particular, those which refer to finite algebras and which were stated in [3]; [4] and [7].

The main results of this work are in section 3. There, we define an order over the family of certain partitions of the set of atoms of a finite **Df**$_2-$algebra.

As a consequence of this and other results stated in section 2, we obtain a full description of the lattice of all subalgebras of a finite **Df**$_2-$algebra.

2 Preliminaries

In this section, we shall review some notions and results concerning finite **Df**$_2-$algebras will be used to obtain the main result of this work. We refer the interested reader to the references [3, 4].

Recall that a Boolean algebra is a structure $\mathbb{A} = (A, \vee, \wedge, \neg, 0, 1)$ such that $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice with first element 0, last element 1 and where $\neg a$ is the Boolean complement of a, for every $a \in A$.

A **Df**$_2-$algebra is a triple $(\mathbb{A}, \exists_1, \exists_2)$, where $\mathbb{A}$ is a Boolean algebra and $\exists_1$, $\exists_2$ are quantifiers defined on $\mathbb{A}$ that commute, that is $\exists_1$ and $\exists_2$ are unary operators on A, $\exists_i : A \to A$ ($i = 1, 2$), that verify the following conditions:

$$\exists_i 0 = 0, \tag{1}$$
$$x \leqslant \exists_i x, \tag{2}$$
$$\exists_i(x \wedge \exists_i y) = \exists_i x \wedge \exists_i y, \tag{3}$$
$$\exists_i \exists_j x = \exists_j \exists_i x. \tag{4}$$

For $1 \leqslant i, j \leqslant 2$ and $i \neq j$. The first three are the defining conditions of existential quantifier. In what follows we will denote the Boolean algebra with n atoms by $\mathbb{B}_n$ and by $\Pi(\mathbb{B}_n)$ the set of its atoms.

It is well known that there is an onto and one-to-one correspondence between the family of all quantifiers that can be defined over $\mathbb{B}_n$, and the family of all Boolean subalgebras of $\mathbb{B}_n$. Indeed, if S is a subalgebra of $\mathbb{B}_n$, then the map $\exists : \mathbb{B}_n \to \mathbb{B}_n$ defined by:

$$\exists(x) = \bigwedge\{s \in S : x \leq s\}. \tag{5}$$

Is a quantifier which will be called the quantifier associated to S. Moreover, all quantifiers on $\mathbb{B}_n$ can be obtained in this way.

On the other hand, every subalgebra S of $\mathbb{B}_n$ induces a partition $\mathcal{P}_S$ of the set $\Pi(\mathbb{B}_n)$ of its atoms which will be called partition induced by S and it is obtained, by considering the set $\Pi(S)$ of the atoms of S, in the following way:

$$C \in \mathcal{P}_S, \tag{6}$$

iff

$$\text{there is } s \in \Pi(S) \text{ such that } s = \bigvee_{a \in C} a. \tag{7}$$

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Conversely, every partition $\mathcal{P}$ of $\Pi(\mathbb{B}_n)$ induces a subalgebra $S_{\mathcal{P}}$ of $\mathbb{B}_n$ as follows: for every $C \in \mathcal{P}$, we consider the element $a_C = \bigvee_{a\in C} a$.

Then, $S_{\mathcal{P}}$ is the Boolean subalgebra generated by the set $\{a_C : C \in \mathcal{P}\}$. From the above, we can conclude that there is an onto and one-to-one correspondence between the family of all quantifiers that can be defined over $\mathbb{B}_n$, and the family of all partitions of $\Pi(\mathbb{B}_n)$.

Let $\exists$ be an arbitrary quantifier defined on $\mathbb{B}_n$ and let $\mathcal{P}$ be the partition of $\Pi(\mathbb{B}_n)$ associated to $\exists$. Then, we denote by $\mathcal{P}(x)$ the set:

$$\{C \in \mathcal{P} : \bigvee_{a\in C} a \leqslant x\}. \tag{8}$$

For each $x \in \exists\mathbb{B}_n$. The following definition plays an important role when dealing with finite $\mathbf{Df}_2-$algebras and was introduced in [3].

Definition 1. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two partitions of $\Pi(\mathbb{B}_n)$. For each $C \in \mathcal{P}_i$, we will call m_j–saturated of C, and we will denote it by $m_j(C)$, the least (in the sense of inclusion) subset of $\mathcal{P}_j$ which verifies $C \subseteq \bigcup_{F\in m_j(C)} F$, for $1 \leqslant i,j \leqslant 2$ and $i \neq j$.

Then, we can determine the m_j–saturated of any $C \in \mathcal{P}_i$, with $i \neq j$, $1 \leqslant i,j \leqslant 2$, as it is indicated in the next lemma.

Lemma 1. If $C \in \mathcal{P}_i$ and $b = \bigvee_{a\in C} a$, then $m_j(C) = \mathcal{P}_j(\exists_j b)$, with $1 \leqslant i,j \leqslant 2$ and $i \neq j$.

Another characterization of $m_j(C)$, for any $C \in \mathcal{P}_i$, is given next.

Lemma 2. If $C \in \mathcal{P}_i$, then $m_j(C) = \{D \in \mathcal{P}_j : C \cap D \neq \emptyset\}$, $1 \leqslant i,j \leqslant 2$ and $i \neq j$.

Remark 1. If $C \in \mathcal{P}_i$ and $b = \bigvee_{a\in C} a$, then $\exists_j b$ can be calculated in the following way:

$$\exists_j b = \bigvee_{\substack{a\in D\\ D\in m_j(C)}} a. \tag{9}$$

For $1 \leqslant i,j \leqslant 2$ and $i \neq j$.

Next, we define a binary relation between two partitions of $\Pi(\mathbb{B}_n)$.

Definition 2. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two partitions of $\Pi(\mathbb{B}_n)$. We will say that $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$ and we will write $\mathcal{P}_2 \succ \mathcal{P}_1$, if for each $C \in \mathcal{P}_1$ there exists $\mathcal{U} \subseteq \mathcal{P}_1$ such that:

$$\bigcup_{G\in m_2(C)} G = \bigcup_{F\in\mathcal{U}} F. \tag{10}$$

Remark 2. It is not difficult to check that the subset $\mathcal{U}$, mentioned in Definition 2, is unique. Therefore, from now on, for each $C \in \mathcal{P}_1$, we will denote with $\mathcal{U}_C$ the only subset of $\mathcal{P}_1$ such that:

$$\bigcup_{G\in m_2(C)} G = \bigcup_{F\in\mathcal{U}_C} F. \tag{11}$$

A characterization of $\mathcal{U}_C$, for every $C \in \mathcal{P}_i$, is stated in the following lemma.

Lemma 3. If $C \in \mathcal{P}_i$ and $b = \bigvee_{a\in C} a$, then $\mathcal{U}_C = \mathcal{P}_i(\exists_j b)$, with $1 \leqslant i,j \leqslant 2$ and $i \neq j$.

In what follows, we will write $\mathcal{P}_2 \approx \mathcal{P}_1$ to indicate that each of the partitions is a refinement of the other. The following three results are the most important in this section and, as we shall see later, they will be very useful. A detailed proof of them can be found in [3].

Theorem 1. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two partitions of $\Pi(\mathbb{B}_n)$ and $\exists_1$, $\exists_2$ their associated quantifiers. Then the following conditions are equivalent:

1. $\exists_1$ and $\exists_2$ commute,
2. $\mathcal{P}_1 \approx \mathcal{P}_2$.

Lemma 4. Let $(\mathbb{B}_n, \exists)$ be a finite monadic Boolean algebra, S a Boolean subalgebra of $\mathbb{B}_n$, and let $\mathcal{P}_2$ and $\mathcal{P}_1$ be the partitions of $\Pi(\mathbb{B}_n)$ associated to the quantifier $\exists$ and the subalgebra S, respectively. Then the following conditions are equivalent:

1. S is a monadic subalgebra of $(\mathbb{B}_n, \exists)$,
2. $\mathcal{P}_2 \succ \mathcal{P}_1$.

Lemma 5. Let $\mathcal{P}_1, \mathcal{P}_2$ be two partitions of $\Pi(\mathbb{B}_n)$. If $\mathcal{P}_2 \succ \mathcal{P}_1$, then $\mathcal{P}_1 \succ \mathcal{P}_2$.

3 Df$_2-$Subalgebras of a Finite Df$_2-$Algebra

In this section, we shall present a correspondence between the family of all subalgebras of a given **Df**$_2-$algebra $\mathbb{A} = (\mathbb{B}_n, \exists_1, \exists_2)$ and a certain family of partitions of the set of its atoms $\Pi(\mathbb{B}_n)$.

This will allow us to establish a characterization of the lattice of all subalgebras of $\mathbb{A}$. A characterization of the subalgebras of a finite **Df**$_2-$algebra is the following:

Lemma 6. Let $\mathbb{A} = (\mathbb{B}_n, \exists_1, \exists_2)$ be a finite **Df**$_2-$algebra, $\mathcal{P}_i$ the partition of $\Pi(\mathbb{B}_n)$ associated to $\exists_i$, $i = 1, 2$, and S a Boolean subalgebra of $\mathbb{A}$. Then the following conditions are equivalent:

1. S is a **Df**$_2-$subalgebra of $(\mathbb{B}_n, \exists_1, \exists_2)$,
2. $\mathcal{P}_S \approx \mathcal{P}_i$ for $i = 1, 2$, with $\mathcal{P}_S$ the partition of $\Pi(\mathbb{B}_n)$ associated to S.

Proof. It is consequence of Lemma 4. □

If $\mathbb{A} = (\mathbb{B}_n, \exists_1, \exists_2)$ is a given finite **Df**$_2-$algebra, we denote the set of all **Df**$_2-$subalgebras of $\mathbb{A}$ by $\mathscr{S}(\mathbb{A})$ and the set of all partitions $\mathcal{P}$ of $\Pi((\mathbb{B}_n)$ such that $\mathcal{P} \approx \mathcal{P}_i$ for $i = 1, 2$, by $\mathscr{P}(\mathbb{A})$, where $\mathcal{P}_i$ is the partition of $\Pi((\mathbb{B}_n)$ associated to $\exists_i$. Then, from the previous lemma, the following corollary is inferred:

Corollary 3.1. $\mathscr{S}(\mathbb{A})$ and $\mathscr{P}(\mathbb{A})$ have the same cardinality.

Now we will endow $\mathscr{P}(\mathbb{A})$ with an order relation $\preceq$ defined as follows:

$$\mathcal{P} \preceq \mathcal{P}' \quad (12)$$

iff

$$\text{For all } C \in \mathcal{P}', \text{ there is } Q \subseteq \mathcal{P} \text{ such that } C = \bigcup_{D \in Q} D. \quad (13)$$

Then we have:

Lemma 7. Let $\mathbb{A} = (\mathbb{B}_n, \exists_1, \exists_2)$ be a finite **Df**$_2-$algebra. Then, the ordered sets $(\mathscr{S}(\mathbb{A}), \subseteq)$ and $(\mathscr{P}(\mathbb{A}), \preceq)$ are antiisomorphic.

Proof. Let $\alpha : \mathscr{S}(\mathbb{A}) \to \mathscr{P}(\mathbb{A})$ be the application defined by:

$$\alpha(S) = \mathcal{P}_S \text{ for each } S \in \mathscr{S}(\mathbb{A}), \quad (14)$$

where $\mathcal{P}_S$ is the partition of $\Pi(\mathbb{B}_n)$ associated to S. It is not difficult to check that α is one-to-one and onto. Now, let $S_1, S_2 \in \mathscr{S}(\mathbb{A})$ such that (1) $S_1 \subseteq S_2$. For each $C \in \alpha(S_1)$, let:

$$d = \bigvee_{a \in C} a. \quad (15)$$

Then, $d \in \Pi(S_1)$. From (1) $d \in S_2$ and so, we may assert that $d = \bigvee_{\substack{b \in \Pi(S_2) \\ b \leqslant d}} b$. Therefore,

$$C = \bigcup_{D \in \mathcal{P}_{S_2}(d)} D. \quad (16)$$

And so, $\alpha(S_2) \preceq \alpha(S_1)$. On the other hand, suppose that (2) $\alpha(S_2) \preceq \alpha(S_1)$, and let $d \in \Pi(S_1)$. Then:

$$d = \bigvee_{a \in C} a. \quad (17)$$

For some $C \in \mathcal{P}_{S_1}$. Then, from (2), we have that $C = \bigcup_{D \in Q} D$, with $Q \subseteq \mathcal{P}_{S_2}$. Let us assume that $Q = \{D_1, D_2, \ldots, D_r\}$ and let $b_i = \bigvee_{a \in D_i} a$ where $1 \leqslant i \leqslant r$. Then, $b_i \in \Pi(S_2)$ and $d = \bigvee_{i=1}^{r} b_i$, that is to say, $d \in S_2$. In this way, $S_1 \subseteq S_2$. □

Our next objective is to determine necessary and sufficient conditions for two elements of $\mathscr{S}(\mathbb{A})$ to be isomorphic. For this purpose, let S_1 and S_2 be two elements of $\mathscr{S}(\mathbb{A})$.

For each $C \in \mathcal{P}_{S_1}$ ($D \in \mathcal{P}_{S_2}$) we will denote the saturated of C (D) in the partition $\mathcal{P}_i$ by $m_i^{S_1}(C)$ ($m_i^{S_2}(D)$). Besides, we will denote by $\mathcal{U}_C^{i\,S_1}$ ($\mathcal{U}_D^{i\,S_2}$) the least subset of $\mathcal{P}_{S_1}$ ($\mathcal{P}_{S_2}$), such that:

$$\bigcup_{H \in m_i^{S_1}(C)} H = \bigcup_{G \in \mathcal{U}_C^{i\,S_1}} G \quad \left(\bigcup_{I \in m_i^{S_2}(D)} I = \bigcup_{F \in \mathcal{U}_D^{i\,S_2}} F \right). \quad (18)$$

Lemma 8. Let $\mathbb{A} = (\mathbb{B}_n, \exists_1, \exists_2)$ be a finite **Df**$_2-$algebra, S_1 and S_2 **Df**$_2-$subalgebras of $\mathbb{A}$. Then, the following conditions are equivalent.

1. S_1 and S_2 are isomorphic,
2. there is a bijection $f : \mathcal{P}_{S_1} \to \mathcal{P}_{S_2}$ such that:

$$\bigcup_{G \in f(\mathcal{U}_C^{i\,S_1})} G = \bigcup_{H \in m_i^{S_2}(f(C))} H \qquad (19)$$

For each $C \in \mathcal{P}_{S_1}$ and $i = 1, 2$.

Proof. (i) $\Rightarrow$ (ii). Let S_1 and S_2 be isomorphic **Df**$_2-$subalgebras of $\mathbb{A}$, and let $\phi : S_1 \to S_2$ be the corresponding **Df**$_2-$isomorphism. Let us define $f : \mathcal{P}_{S_1} \to \mathcal{P}_{S_2}$ by:

$$f(C) = D \in \mathcal{P}_{S_2}, \qquad (20)$$

iff

$$\phi(\bigvee_{a \in C} a) = \bigvee_{b \in D} b \text{ for every } C \in \mathcal{P}_{S_1}. \qquad (21)$$

Then, it is clear that f is well defined. Besides, since $\phi|_{\Pi(S_1)}$ is a one-to-one and onto correspondence between $\Pi(S_1)$ and $\Pi(S_2)$, we can assert that f is one-to-one and onto. Let us prove that, for each $C \in \mathcal{P}_{S_1}$, it holds:

$$\bigcup_{G \in f(\mathcal{U}_C^{i\,S_1})} G = \bigcup_{H \in m_i^{S_2}(f(C))} H. \qquad (22)$$

For $i = 1, 2$. Suppose that $s_1 = \bigvee_{a \in C} a$, then:

$$\phi(s_1) = \phi(\bigvee_{a \in C} a) = s_2 = \bigvee_{b \in f(C)} b. \qquad (23)$$

With $s_1 \in \Pi(S_1)$ and $s_2 \in \Pi(S_2)$. It can be verified without any difficulty that:

$$\begin{aligned} ccl(\exists_i s_1) &= \phi(\bigvee_{\substack{a \in H \\ H \in m_i^{S_1}(C)}} a) \\ &= \phi(\bigvee_{\substack{a \in G \\ G \in \mathcal{U}_C^{i\,S_1}}} a) \\ &= \phi(\bigvee_{G \in \mathcal{U}_C^{i\,S_1}} \bigvee_{a \in G} a). \end{aligned} \qquad (24)$$

And, since $\bigvee_{a \in G} a \in S_1$ for every $G \in \mathcal{U}_C^{i S_1}$, we get:

$$\phi(\exists_i s_1) = \bigvee_{G \in \mathcal{U}_C^{i\,S_1}} \phi(\bigvee_{a \in G} a) = \bigvee_{G \in \mathcal{U}_C^{i\,S_1}} \bigvee_{b \in f(G)} b. \qquad (25)$$

On the other hand:

$$\exists_i \phi(s_1) = \exists_i s_2 = \bigvee_{\substack{b \in I \\ I \in m_i^{S_2}(f(C))}} b. \qquad (26)$$

From $\phi(\exists_i s_1) = \exists_i \phi(s_1)$, (1) and (2), we get:

$$\bigvee_{G \in \mathcal{U}_C^{i\,S_1}} \bigvee_{b \in f(G)} b = \bigvee_{\substack{b \in I \\ I \in m_i^{S_2}(f(C))}} b. \qquad (27)$$

From (3), and properties of $\mathcal{U}_C^{i\,S_1}$ and $m_i^{S_2}(f(C))$, it results that:

$$\bigcup_{G \in f(\mathcal{U}_C^{i\,S_1})} G = \bigcup_{H \in m_i^{S_2}(f(C))} H. \qquad (28)$$

(ii) $\Rightarrow$ (i). Let $f : \mathcal{P}_{S_1} \to \mathcal{P}_{S_2}$ be a one-to-one and onto function such that:

$$\bigcup_{G \in f(\mathcal{U}_C^{i\,S_1})} G = \bigcup_{H \in m_i^{S_2}(f(C))} H \qquad (29)$$

for every $C \in \mathcal{P}_{S_1}$ and $i = 1, 2$. Let $\psi_f : S_1 \to S_2$ be the Boolean homomorphism defined by:

$$\psi_f(s) = \bigvee_{H \in \mathcal{P}_{S_1}(s)} \bigvee_{r \in f(H)} r. \qquad (30)$$

Since f is one-to-one and onto, it is easy to check that ψ_f is a Boolean isomorphism. Let us now check that $\psi_f(\exists_i s) = \exists_i \psi_f(s)$ for every $s \in \Pi(S_1)$. Let $s \in \Pi(S_1)$, then there is $C \in \mathcal{P}_{S_1}$ such that $s = \bigvee_{a \in C} a$. By Lemma 3, we have:

$$\begin{aligned} ccr\psi_f(\exists_i s) &= \bigvee_{H \in \mathcal{P}_{S_1}(\exists_i s)} \bigvee_{r \in f(H)} r \\ &= \bigvee_{H \in \mathcal{U}_C^{i\,S_1}} \bigvee_{r \in f(C)} r \\ &= \bigvee_{G \in f(\mathcal{U}_C^{i\,S_1})} \bigvee_{r \in G} r. \end{aligned} \qquad (31)$$

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On other hand, it is clear that:

$$\exists_i(\psi_f(s)) = \exists_i(\bigvee_{a\in f(C)} a) = \bigvee_{D\in m_i^{S_1}(f(C))} \bigvee_{a\in D} a. \tag{32}$$

From (4), (5) and (6), we get that $\psi_f(\exists_i s) = \exists_i(\psi_f(s))$. $\square$

Now, consider the binary relation Δ on $\mathscr{P}(\mathbb{A})$ defined as:

$$\mathcal{P}_1 \Delta \mathcal{P}_2, \tag{33}$$

iff

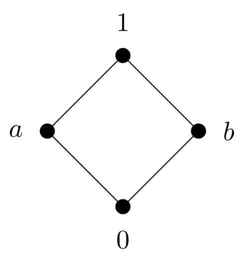
$$\mathcal{P}_1 \text{ and } \mathcal{P}_2 \text{ satisfy condition Lemma 8 (ii).} \tag{34}$$

Then, from all the results above stated, we have:

Theorem 2. The subalgebra lattice of the finite **Df**$_2-$algebra $\mathbb{A}$, $\mathscr{S}(\mathbb{A})$, is isomorphic to $(\mathscr{P}(\mathbb{A})/\Delta, \succeq)$.

Finally, we analyze some examples where we apply the result stated above.

Example 1. Let us consider the **Df**$_2-$algebra, $(\mathbb{B}_2, \exists_1, \exists_2)$ whose Hasse diagram is shown below and the quantifiers $\exists_1, \exists_2$ are defined by the next table.



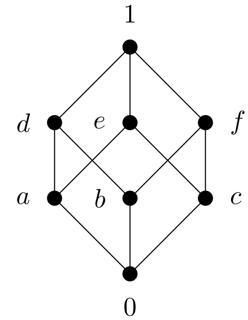
x	$\exists_1 x$	$\exists_2 x$
0	0	0
a	a	1
b	b	1
1	1	1

In this case $\mathcal{P}_1 = \{\{a\}, \{b\}\}$ and $\mathcal{P}_2 = \{\{a, b\}\}$ are the only partitions of $\Pi(\mathbb{B}_2)$ associated to quantifiers $\exists_1$ and $\exists_2$, respectively.

Then $\mathscr{P}(\mathbb{B}_2) = \{\mathcal{P}_1, \mathcal{P}_2\}$, hence it is clear that $(\mathscr{S}(\mathbb{B}_2), \subseteq)$ is the chain with two elements and $(\mathbb{B}_2, \exists_1, \exists_2)$ has two non-isomorphic subalgebras.

Example 2. Let $(\mathbb{B}_3, \exists_1, \exists_2)$ be the **Df**$_2-$algebra whose Hasse diagram is below and the quantifiers $\exists_1, \exists_2$ are given by the table:

Hence, $\mathcal{P}_1 = \{\{a\}, \{b, c\}\}$ and $\mathcal{P}_2 = \{\{a, b, c\}\}$ are the partitions of $\Pi(\mathbb{B}_3)$ associated to quantifiers $\exists_1$ and $\exists_2$, respectively.

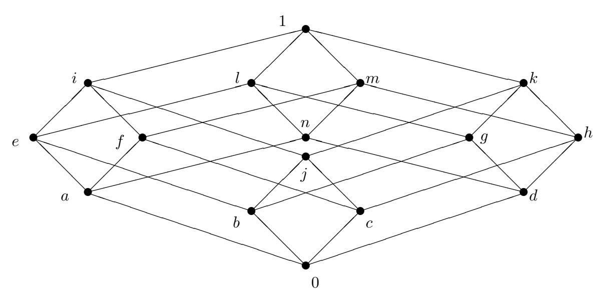


x	$\exists_1 x$	$\exists_2 x$
0	0	0
a	a	1
b	f	1
c	f	1
d	1	1
e	1	1
f	f	1
1	1	1

Then $\mathscr{P}(\mathbb{B}_3) = \{\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3\}$ where $\mathcal{P}^1 = \{\{a\}, \{b\}, \{c\}\}, \mathcal{P}^2 = \{\{a\}, \{b, c\}\}$ and $\mathcal{P}^3 = \{\{a, b, c\}\}$.

It is easy to verify that $\mathcal{P}^1 \preceq \mathcal{P}^2 \preceq \mathcal{P}^3$, hence $(\mathscr{S}(\mathbb{B}_3), \subseteq)$ is the chain with three elements and $(\mathbb{B}_3, \exists_1, \exists_2)$ has three non-isomorphic subalgebras.

Example 3. Finally, let us consider the **Df**$_2-$algebra, $(\mathbb{B}_4, \exists_1, \exists_2)$ whose Hasse diagram is below and the quantifiers $\exists_1$ and $\exists_2$ are defined by the partitions $\mathcal{P}_1 = \{\{a, b\}, \{c, d\}\}$ and $\mathcal{P}_2 = \{\{a, c\}, \{b, d\}\}$ of $\Pi(\mathbb{B}_4)$, respectively.



Then $\mathscr{P}(\mathbb{B}_4) = \{\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3, \mathcal{P}^4, \mathcal{P}^5\}$ where:

$$\mathcal{P}^1 = \{\{a\}, \{b\}, \{c\}, \{d\}\}, \tag{35}$$

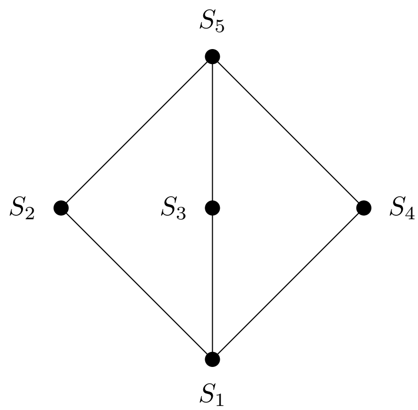
$$\mathcal{P}^2 = \{\{a, b\}, \{c, d\}\}, \tag{36}$$

$$\mathcal{P}^3 = \{\{a, c\}, \{b, d\}\}, \tag{37}$$

$$\mathcal{P}^4 = \{\{a, d\}, \{b, c\}\}, \tag{38}$$

$$\mathcal{P}^5 = \{\{a, b, c, d\}\}. \tag{39}$$

It can be seen that $\mathcal{P}^1 \preceq \mathcal{P}^2 \preceq \mathcal{P}^5; \mathcal{P}^1 \preceq \mathcal{P}^3 \preceq \mathcal{P}^5; \mathcal{P}^1 \preceq \mathcal{P}^4 \preceq \mathcal{P}^5$ and $\mathcal{P}^2, \mathcal{P}^3, \mathcal{P}^4$ are incomparable.



Then $(\mathscr{S}(\mathbb{B}_4), \subseteq)$ is the ordered set whose Hasse diagram is indicated next. Hence, the **Df**$_2-$algebra $(\mathbb{B}_4, \exists_1, \exists_2)$ has five non-isomorphic subalgebras.

References

1. **Bezhanishvili, N. (2002).** Varieties of two-dimensional cylindric algebras. Part I: Diagonal-free case. Algebra Universalis, Vol. 48, No. 1, pp. 11–42. DOI: 10.1007/s00012-002-8203-2.

2. **Birkhoff, G., Frink, O. (1948).** Representations of lattices by sets. Transactions of the American Mathematical Society, Vol. 64, pp. 299–316.

3. **Figallo, M. (2004).** Finite diagonal-free two-dimensional cylindric algebras. Logic Journal of IGPL, Vol. 12, No. 6, pp. 509–523. DOI: 10.1093/jigpal/12.6.509.

4. **Figallo, M. (2011).** Some results on diagonal-free two-dimensional cylindric algebras. Reports on Mathematical Logic, No. 46, pp. 3–15.

5. **Figallo, A. V., Gomes, C. M. (2019).** Monadic Boolean algebras with an automorphism and their relation to $\mathbf{Df}_2$-algebras. Soft Computing, vol. 24, no. 1. Springer Science and Business Media LLC, pp. 227–236. DOI: 10.1007/s00500-019-04317-4.

6. **Halmos, P. R. (1954-1956).** Algebraic logic, I. Monadic Boolean algebras. Compositio Mathematica, No. 12, pp. 217–249, numdam.org/item/CM_1954-1956__12__217_0/.

7. **Henkin, L., Donald-Monk, J., Tarski, A. (1971-1985).** Cylindric algebras. Part I, North-Holland, Elsevier Science.

8. **Iskander, A. A. (1965).** Correspondence lattices of universal algebras. Izvestiya Akademii Nauk SSSR, Vol. 29, No. 6, pp. 1357–1372.

9. **Sikorski, R. (1968).** Algebras de Boole, notas de lógica matemática 4, Instituto de Matemática, Universidad Nacional del Sur, Bahía Blanca, Argentina.

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