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Fitch-Style Modal Necessity as a Substructural Sequent-Style System

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Abstract. The question of defining a Jaśkowski-Fitch natural deduction system for modal logic has been settled since the very introduction of such formalisms in the middle of the last century. In contrast, a sequent-style formulation of this approach has only been discussed since the turn of this century but exclusively from the point of view of type theories. In this paper we propose a substructural sequent-style deductive system, based on previous ideas by Borghuis and Clouston, which captures Fitch-style for modal logic in a faithful way, meaning that the features of the original diagrammatic proofs are enforced by the sequent rules. This answers the question of what is a sequent-style version of Fitch-style natural deduction in the case of the necessity fragment of minimal modal logic $\mathsf{S4}$.

Keywords. Natural deduction, fitch-style, modal necessity, substructural logics, sequents.

1 Introduction

Fitch-style natural deduction [5], originally introduced by Jaśkowski [8], is a style of proof characterized by the use of so-called subordinate proofs, encompassing the idea of a mathematical proof which depends on temporal assumptions, such as a conditional proof, an indirect proof or a proof by cases. The actual construction of a Fitch-style proof usually requires a diagrammatic mechanism to indicate the beginning and end of a subordinate proof. For instance the use of indentation, lines or rectangles (text boxes). These kinds of diagrammatic features make difficult to develop meta-theoretical results for such systems but also to pursue formal verification tasks with the help of modern proof-assistants.

An important and well-known solution to this challenge is to avoid the use of diagrams by using a sequent-style system, one that manipulates sequents instead of formulas.

Such formalisms already appear in the seminal work of Gentzen [7] and their relevant feature here is that all assumptions are local and therefore there is no need for a visual aid to keep track of a temporal assumption.

The problem of defining a natural deduction system in Fitch-style for modal logic has been settled since the middle of the last century. In his seminal book [5], Fitch himself proposes such a formalism by introducing a second notion of subordinate proof, called strict.

Those proofs follow a semantical intuition: the opening of a strict subordinate proof corresponds to going to an arbitrary world in the usual Kripke semantics, though neither Fitch nor us employ formal semantical concepts. Such proofs appear spontaneously, for they do not require an assumption to be triggered.

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Later, during the 1980's Fitting [6], using the ideas of Siemens [10], gives a systematic treatment of Fitch-style systems for normal modal logics, using signed formulas common to tableaux procedures, and Borghuis [2, 3] proposes an extension to higher-order logic, by means of modal pure type systems.

The absence of a formula indicating the beginning of a strict subordinate proof represents a challenge to avoid the use of diagrams, not to mention mechanization issues.

Perhaps this is the reason why the work on sequent systems for Fitch-style modal logic dates only from the turn of this century:

Borghuis [1, 2] uses a so-called structural connective to indicate the beginning of a strict subordinate proof; a similar idea occurrs in Ritter and De Paiva [12], who sketch a multi-context sequent system dictated by the maximum modal depth of formulas in a derivation; and Clouston [4] presents modal type systems using a lock sign to indicate that a box has been opened thus allowing access to its content.

However, all these works do not discuss the logic itself, instead they are interested in type systems and categorical semantics (see also Kakutani et al. [9] for a more recent approach).

Moreover, even when they mention that their systems capture Fitch-style, they provide no proof sustaining this claim.

To this purpose, we provide a formal notion of Fitch-proof which captures the visual intuition using proof translations and avoiding the direct use of diagrammatic reasoning.

For the time being, we consider only the box operator $\Box$ and their $\mathrm{S4}$ axioms, since it is well-known that weaker systems like K or T require further restrictions on the diagrams and stronger systems like $S5$ are not easily captured by an ordinary sequent systems, for we either require labelled systems or pure systems handling not sequents but hypersequents or hypersequent trees.

Furthermore, we consider only the minimal logic, that is, we do not handle a negation operator, either constructive or classical, since such extensions are straightforward.

On the other hand, the study of minimal systems alone, is important in Computer Science where the classical versions are not so well behaved with respect to algorithmic content.

The paper is organized as follows: in Section 2 we present $\mathfrak{F}^{\Box}$, an ordinary Fitch-style logic but featuring a formal definition of proof, enhancing the original definition of [3], which allows us to depart from the diagrammatic reasoning while keeping its intuition.

In Section 3 we introduce a sequent-style version of Fitch-style for modal logic, called $\mathsf{NS4}^{\Box}$, and emphasize the substructural nature of the system. The equivalence between both systems is proved in detail in Section 4.

1.1 Syntax

We dedicate this brief section to set up the important syntactic notions that accompany the rest of the work, namely formulas and contexts. Modal propositions are generated by the following grammar:

$$\mathsf{Prop} \ni A, B ::= p_n \mid A \to B \mid \Box A, \tag{1}$$

where p_n denotes an element taken from an infinite supply of propositional variables, indexed by a natural number. Let us observe that we consider neither negation nor the constant $\bot$.

Thus we will be dealing with modal logic obtained from minimal implicative propositional logic, extended with the modal operator of necessity.

In this paper contexts are implemented as finite lists built from the empty list, denoted here by $\cdot$, and a constructor that generates a new list from a given one Γ by adding a new item Q to its right-end.

The elements of such a list are formulas in the case of $\mathfrak{F}^{\Box}$ but they also include the symbol 🔓 for $\mathsf{NS4}^{\Box}$. Contexts are formally defined as follows:

$$\Gamma ::= \cdot \mid \Gamma, \mathsf{Q}. \tag{2}$$

The operation Γ, Q is usually called *snoc*. Furthermore, the append operation of two contexts Γ and Γ' is recursively defined as usual and denoted with a semicolon $\Gamma; \Gamma'$.

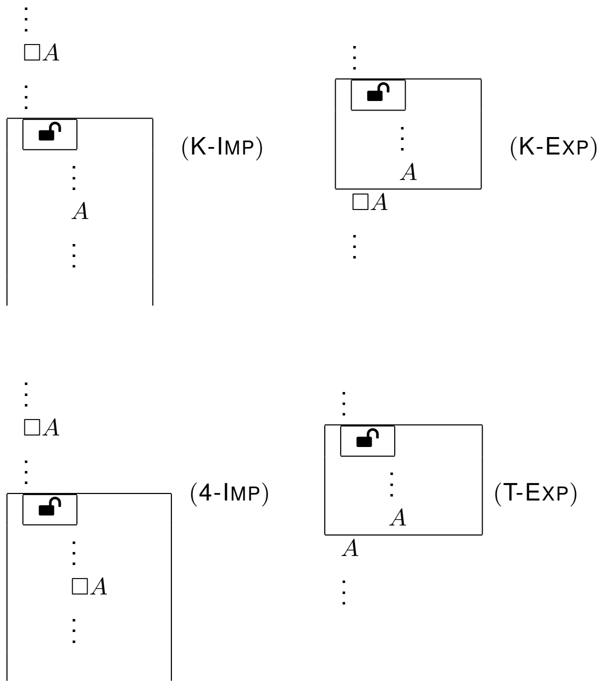

Fig. 1. Modal rules for system $\mathfrak{F}^{\Box}$

2 Fitch-Style for Modal Necessity

Here we define a Fitch-style natural deduction system for the $\mathrm{S4}$ necessity fragment.

The emphasis is in the behavior of this operator alone, but since we do not consider here neither $\bot$ nor $\neg$ it can be said that we are dealing with the minimal modal logic $\mathrm{S4}$.

We follow the approach of [2]. All modal rules involve the use of a strict subordinate proof, whose start is indicated by an opened lock 🔓.

The modal rules of system $\mathfrak{F}^{\Box}$ are in Figure 1. It is worth noting that the rules (K-IMP) and (4-IMP) can be considered as specialized reiteration rules allowing to introduce already known information, (A as consequence of $\Box A$ or $\Box A$ itself) inside a strict subordinate proof.

On the other hand, rules (K-EXP) and (T-EXP) allow us to close a strict subordinate proof. It is important to remark that the general reiteration rule common in Fitch-style systems, listed below, is not present here for it is not necessary [1], see [13, Section 2.4], and departs from the natural mathematical reasoning where any already proved information can be used at any stage in a proof without state it again.

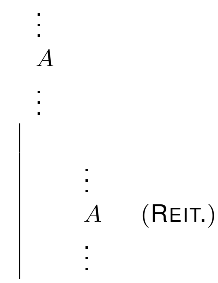

Figure 2 shows an example of a proof in the system $\mathfrak{F}^{\Box}$. This diagram corresponds to a proof of $\Box(\Box A \to \Box B)$ from the hypothesis $\Box(\Box A \to \Box(\Box A \to \Box B))$.

The visual aids provided by the diagram originate from Fitch's work. In this diagram, the vertical line indicates a subordinate proof while the boxes labelled with 🔓 designate strict subordinate proofs.

We observe that the unique hypothesis is at step 1; a temporal assumption starting a subordinate proof starting at step 2 which is closed at step 8; a strict subordinate proof starting at step 2 and finished at step 9 and a strict subordinate proof from steps 5 to 7 The meaning of the last column in the above proof will be explained later.

Our aim here is to give a formal definition of these diagrammatic proofs departing from the diagram but somehow keeping the visual intuition.

This was originally done by Van Westrhenen et al. [11], and extended to modal logic by Borghuis in [3], by means of a mathematical structure called proof scheme or proof figure.

This notion is enhanced by us here in order to achieve our main goal, the definition of a sequent-style calculus for Fitch-style in the case of modal logic $\mathrm{S4}$ and the proof of its equivalence with $\mathfrak{F}^{\Box}$.

Definition 2.1. A proof scheme or proof figure is a mathematical structure $\mathcal{P} = \langle \mathcal{D}, F, \mathcal{I} \rangle$ such that:

– $\mathcal{D} = \{1, 2 \ldots, n\} \subset \mathbb{N}$ is a (discrete) interval hereafter denoted as $\mathcal{D} = [1, n]$.

[1] An interesting related discussion can be found at www.logicmatters.net/2017/07/20/reiterating-in-fitch-style-proofs/

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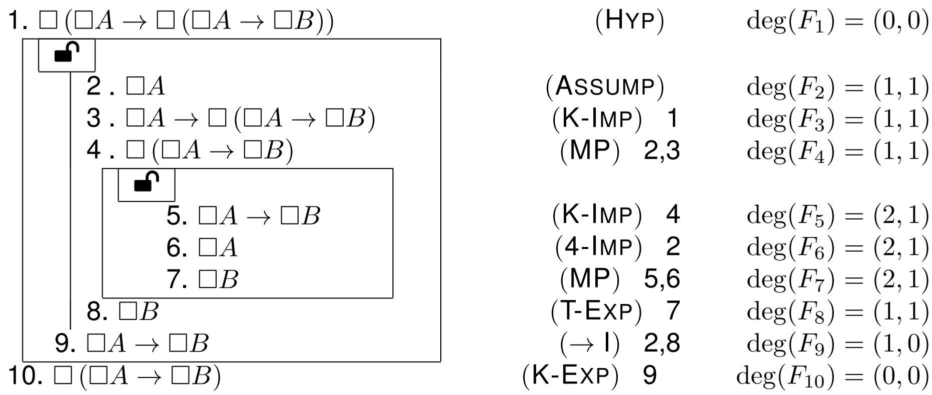

Fig. 2. Example including degree of steps

- $F : \mathcal{D} \to$ Prop is a function assigning propositions to the elements of $\mathcal{D}$.
- $\mathcal{I}$ is a collection of intervals of one of the following forms:
  - $[i, j] \subseteq \mathcal{D}$.
  - $[i, \infty)$ with $i \in \mathcal{D}$, called an indeterminate interval.

  Any two intervals $I, J \in \mathcal{I}$ are either disjoint $I \cap J = \varnothing$ or included one in the other $I \subset J$ or $J \subset I$.
- There exists a decomposition $\mathcal{I} = \mathcal{O} \cup \mathcal{M}$ where:
  - $\mathcal{O}$ is the collection of *ordinary* intervals such that if $J = [i, j] \in \mathcal{O}$ then the proposition $F(i)$ is called the assumption of the interval J. It is possible that $\mathcal{D} \in \mathcal{O}$.
  - $\mathcal{M}$ is the collection of *modal* intervals. If $J = [i, j] \in \mathcal{M}$ then the proposition $F(i)$ is not an assumption of J. Moreover $\mathcal{D} \notin \mathcal{M}$.

Further, if $\mathcal{P}$ has no indeterminate intervals and the decomposition $\mathcal{I} = \mathcal{O} \cup \mathcal{M}$ is a partition of $\mathcal{I}$ we say that the proof figure $\mathcal{P}$ is closed.

We can see that the above definition captures the features of Fitch diagrammatic proofs, the interval $\mathcal{D}$ represents the total number of steps in a derivation; the function F models the sequence of formulas, that is $F(i)$ is the formula in the ith step of the proof; an ordinary interval $[i, j] \in \mathcal{O}$ captures an ordinary subordinate proof, one that introduces $F(i)$ as an assumption which is discharged at step $j + 1$.

Moreover, $[i, j] \in \mathcal{M}$ models an strict or modal subordinate proof, one that starts at step i and closes at step j.

Further, an indeterminate interval $[i, \infty)$ represents the opening of a subordinate proof, either ordinary or strict, which was never closed.

For instance the diagrammatic proof of the above example corresponds to the proof figure $\mathcal{P} = \langle [1, 10], F, \mathcal{I} \rangle$ where $\mathcal{O} = \{[2, 8]\}$ and $\mathcal{M} = \{[2, 9], [5, 7]\}$.

Of course, not every proof figure represents a valid proof, in particular those which have indeterminate intervals are not valid proofs.

This will be made precise in a moment by introducing a formal notion of proof depending on proof figures, but before let us introduce some technical requirements.

Definition 2.2. Let $\mathcal{P} = \langle \mathcal{D}, F, \mathcal{I} \rangle$ be a proof figure. If $i \in \mathcal{D}$, the proposition $F(i)$ will be denoted by F_i. We say that F_i precedes F_j exactly when $i < j$.

For $I \in \mathcal{I}$, if $I = [i, j]$ we say that $k \in I$ if and only if and $i \leq k \leq j$ and in case $I = [i, \infty)$ we say that $k \in I$ if and only if $i \leq k \leq |\mathcal{D}|$.

Further, if $i \in I$ for some interval $I \in \mathcal{I} \cup \{\mathcal{D}\}$ and there is no $J \in \mathcal{I}$ such that $i \in J$ and $J \subset I$, then we say that the proposition F_i lies in I, which will be denoted as $F_i \in I$.

If $J \subseteq K \subset I$ we say that I precedes J, in particular if $J = K \subset I$ we say that J lies in I.

The usual definition of derivation in Fitch-style systems requires that all subordinate proofs become closed.

In our case we have two kinds of such proofs and at each step we need to keep track of the number of subordinate proofs, boxes in a diagrammatic proof, that need to be closed after the current step.

To this purpose we introduce the concept of degree of a (step) formula F_i in a derivation, which is a pair of numbers (n, m), meaning that F_i lies in the nth nested modal interval, say $\mathcal{M}_n$, and that there are m ordinary intervals opened inside $\mathcal{M}_n$ and before the next, i.e., the $n+1$th modal interval.

This means that in order to finish the proof we first must close m ordinary intervals to being able to close the n nested modal intervals containing F_i.

This way, by using the degree, we will ensure that subordinate proofs are closed in the proper order, guaranteeing for example that if a strict proof is opened inside an ordinary subordinate proof, the former has to be closed before the latter.

Furthermore, we will know that a derivation of a formula A is an actual proof if it does not contain unclosed subordinate proofs, that is if its degree is $(0, 0)$. Let us give the formal definition of degree.

Definition 2.3. Let $\mathcal{P} = \langle \mathcal{D}, F, \mathcal{I} \rangle$ be a proof figure. We define the degree functions of $\mathcal{P}$ as follows:

– $\deg_{\mathcal{M}} : \mathcal{D} \to \mathbb{N}$ is the modal degree function, defined as:

$$\deg_{\mathcal{M}}(i) = |\{J \in \mathcal{M} \mid i \in J\}|. \quad (3)$$

– $\deg_{\mathcal{O}} : \mathcal{D} \to \mathbb{N}$ is the ordinary degree function, defined as:

$$\deg_{\mathcal{O}}(i) = |\{K \in \mathcal{O}' \mid i \in K\}, \quad (4)$$

where:

$$\mathcal{O}' = \{I \in \mathcal{O} \mid i \in I \wedge \nexists J \in \mathcal{M}(i \in J \subset I)\}. \quad (5)$$

– $\deg : \mathcal{D} \to \mathbb{N} \times \mathbb{N}$ is the general degree function, defined as:

$$\deg(i) = (\deg_{\mathcal{M}}(i),\ \deg_{\mathcal{O}}(i)). \quad (6)$$

Further, a degree of a formula F_i in $\mathcal{P}$ is defined as the corresponding degree of i.

The reader can now verify that the last column in the above example renders the degree of each formula in a correct way.

To completely capture the notion of correct derivation we need to verify that every formula F_i is introduced in a sound way, that is, it is either a hypothesis, an assumption or it is the result of applying an inference rule to previous formulas in the sequence and taking care of the intervals involved.

In order to verify this question we give now a precise definition of rule application.

Definition 2.4. Let $\mathcal{P} = \langle \mathcal{D}, F, \mathcal{I} \rangle$ be a proof scheme. A formula E is the result of an application of the deduction rule $\mathfrak{R}$ if E is the conclusion of $\mathfrak{R}$, the premises of $\mathfrak{R}$ precede E in $\mathcal{P}$ and one and only one of the following conditions holds according to the shape of $\mathfrak{R}$.

– (MP). There are $j, k, l \in \mathcal{D}$, a proposition A and intervals $J, K, L \in \mathcal{I}$ such that $F_k = A \to E \in J, F_l = A \in K, F_j = E \in L$ and either $L \subseteq K \subseteq J$ or $L \subseteq J \subseteq K$. Further, the modal degree of $A \to E, A$ and E remains constant, that is $\deg_{\mathcal{M}}(F_j) = \deg_{\mathcal{M}}(F_k) = \deg_{\mathcal{M}}(F_l)$. This means that no modal intervals are opened in between.

We are now in position to give the formal definition of proof but since these objects are built step by step it is more adequate to introduce first a notion of pseudoproof corresponding to an unfinished proof, one possibly having indeterminate intervals.

Definition 2.5. A pseudoproof of a formula C from a finite collection of hypotheses $H_1, \ldots, H_k$ is a proof figure $\mathcal{P} = \langle \mathcal{D}, F, \mathcal{I} \rangle$ such that:

– $\mathcal{D} = [1, n]$ with $n \geq k$.

– If $1 \leq i \leq k$ then $F_i = H_i$ is a hypothesis.

– If $k < i \leq n$ then one and only one of the following cases hold:

  – F_i is an assumption and there is an interval $I \in \mathcal{O}$ starting at i.

  – F_i is the result of the application of an inference rule whose premises belong to the image of F and precede F_i. If the applied rule was K-IMP or 4-IMP then there is an interval $I \in \mathcal{M}$ starting at i.

– $F_n = C$.

If there is a pseudoproof $\mathcal{P}$ of C where $\Gamma = \{A_1, \ldots, A_m\}$ is the list of hypotheses and assumptions in $\mathcal{P}$ then we write $\Gamma \mid\!\sim_{\mathfrak{F}} C$.

It is easy to discern if a formula F_i in $\mathcal{P}$ is a hypothesis or an assumption: hypotheses are given a priori and are placed only at the beginning of the proof figure, which means that there are no intervals starting at a $j < i$ and therefore $\deg(F_i) = (0,0)$, whereas assumptions are generated by intervals: if $I \in \mathcal{O}$ starts at i then F_i is an assumption and thus $\deg(F_i) \neq (0,0)$.

Proofs are a special case of pseudoproofs, according to the following.

Definition 2.6. A proof of C from a given collection of hypotheses Γ is a closed proof-figure $\mathcal{P} = \langle \mathcal{D}, F, \mathcal{I} \rangle$ such that $\Gamma \mid\!\sim_{\mathfrak{F}} C$. In such case we write $\Gamma \vdash_{\mathfrak{F}} C$. Moreover, if $\cdot \vdash_{\mathfrak{F}} C$, that is if Γ is empty, then we say that C is a theorem of that C is derivable in $\mathfrak{F}^{\Box}$.

It is easy to prove that if $\mathcal{P} = \langle [1,n], F, \mathcal{I} \rangle$ is a proof figure witnessing $\Gamma \vdash_{\mathfrak{F}} C$ then there is no assumptions, that is Γ consists of given hypotheses only and $\deg(F_n) = (0,0)$.

Let us finish this section with a important property of pseudoproofs that will be needed later, namely the composition or substitution principle.

Proposition 2.1. The substitution principle holds for pseudoproofs in $\mathfrak{F}^{\Box}$, that is if $\Gamma, D \mid\!\sim_{\mathfrak{F}} C$ and $\Gamma \mid\!\sim_{\mathfrak{F}} D$ then $\Gamma \mid\!\sim_{\mathfrak{F}} C$.

Proof. This is a direct consequence of modus ponens.

It is well known that natural deduction systems have no structural rules, in the sense that none such rule is explicitly stated as primitive. In several logics this means that such rules are sound and used tacitly, which implies that they are admissible in the corresponding sequent-style versions.

However, in our case the manage of hypotheses and assumptions is more delicate due to the presence of strict subordinate proofs. This implies that the unrestricted use of structural reasoning is not sound thus generating a substructural sequent-style system discussed next.

3 $\mathsf{NS4}^{\Box}$: A Substructural Sequent-Style System for Modal Necessity

In this section we present the main contribution of this work, a sequent-style system capturing the ideas of $\mathfrak{F}^{\Box}$ but without any reference neither to the diagrams nor to the proof-figures.

Our system follows the idea of Clouston's modal lambda calculi [4], which introduces the open lock 🔓 as a sign indicating the opening of a new strict subordinate proof, suggesting the action of travelling to an arbitrary world in the Kripke semantics intuition.

This element can occurr anywhere in a context thus causing the lack of general structural rules. However, we will see that these rules are still valid under some restrictions.

System $\mathsf{NS4}^{\Box}$ is given in Figure 3 where the rules have sequents with explicit contexts including the lock symbol 🔓 to emphasize a subordinate modal proof. Recall that contexts are lists of formulae and locks.

The starting rule (HYP) allows us to derive a formula in the context, only if it was introduced within the last strict subordinate proof, that is, after the last 🔓 in the context.

For implication, the introduction rule discharges the last formula in the context while the modus ponens is additive, requiring the same context to derive the conclusion. The rules regarding modal formulas use a lock in contexts to emphasize the modal scope in the proof.

$$\frac{}{\Gamma, A, \Gamma' \vdash_{\mathsf{NS4}^{\Box}} A}\ \text{🔓} \notin \Gamma'\ (\textsc{Hyp}) \qquad \frac{\Gamma, A \vdash_{\mathsf{NS4}^{\Box}} B}{\Gamma \vdash_{\mathsf{NS4}^{\Box}} A \to B}\ (\to \mathsf{I}) \qquad \frac{\Gamma \vdash_{\mathsf{NS4}^{\Box}} A \to B \qquad \Gamma \vdash_{\mathsf{NS4}^{\Box}} A}{\Gamma \vdash_{\mathsf{NS4}^{\Box}} B}\ (\mathsf{MP}_{+})$$

$$\frac{\Gamma, \text{🔓} \vdash_{\mathsf{NS4}^{\Box}} A}{\Gamma \vdash_{\mathsf{NS4}^{\Box}} \Box A}\ (\textsc{K-Exp}) \qquad \frac{\Gamma \vdash_{\mathsf{NS4}^{\Box}} \Box A}{\Gamma, \Gamma', \text{🔓}, \Gamma'' \vdash_{\mathsf{NS4}^{\Box}} A}\ (\textsc{K-Imp})$$

$$\frac{\Gamma \vdash_{\mathsf{NS4}^{\Box}} \Box A}{\Gamma, \Gamma', \text{🔓}, \Gamma'' \vdash_{\mathsf{NS4}^{\Box}} \Box A}\ (\textsc{4-Imp}) \qquad \frac{\Gamma, \text{🔓} \vdash_{\mathsf{NS4}^{\Box}} A}{\Gamma \vdash_{\mathsf{NS4}^{\Box}} A}\ (\textsc{T-Exp})$$

Fig. 3. Rules of system $\mathsf{NS4}^{\Box}$

Rule (K-EXP) mimics closing a strict subordinate proof, that is whenever a formula A is derived using a context where the last hypothesis is a lock then a boxed formula $\Box A$ may be derived discharging this 🔓.

Rule (K-IMP) indicates the opening of a subordinate proof by adding a 🔓, this is stated in the most general way using contexts Γ' and Γ''. The last two rules (4-IMP) and (T-EXP) behave in a similar way and characterize $\mathsf{S4}$.

It is very important to note that the modus ponens must be stated in the additive or context sharing way, meaning that the hypotheses context for both premises is one and the same (as opposed to the multiplicative or independent contexts formulation, meaning that the hypotheses contexts of each premise are different).

The use of a context sharing formulation of modus ponens departs from the intuition of natural deduction but is the price to pay here to get a sound system.

Let us exhibit a counterexample showing that the use of independent contexts in modus ponens would allow us to derive the unsound sequent $A,$ 🔓 $\vdash_{\mathsf{NS4}^{\Box}} A$.

This sequent is unsound since it would allow us to conclude $\cdot \vdash_{\mathsf{NS4}^{\Box}} A \to \Box A$, which is invalid in all usual semantics for $\mathsf{S4}$.

Let B any formula such that $\cdot \vdash_{\mathsf{NS4}^{\Box}} \Box B$ (for instance B may be any propositional tautology), by (K-IMP) we get 🔓 $\vdash_{\mathsf{NS4}^{\Box}} B$.

On the other hand we can derive $A \vdash_{\mathsf{NS4}^{\Box}} B \to A$ and the multiplicative modus ponens would allow us to get $A,$ 🔓 $\vdash_{\mathsf{NS4}^{\Box}} A$.

Moreover, if we state the rule concatenating the contexts in the inverse way we can again derive the same unsound sequent: since 🔓 $\vdash_{\mathsf{NS4}^{\Box}} A \to A$ and $A \vdash_{\mathsf{NS4}^{\Box}} A$, modus ponens would allow us to derive $A,$ 🔓 $\vdash_{\mathsf{NS4}^{\Box}} A$.

A derivation of our running example in system $\mathsf{NS4}^{\Box}$ is in Figure 4.

Let us see why our logic is substructural in a strong way, since all three rules weakening, exchange and contraction are unsound when the lock is involved.

As counterexample, starting with an initial sequent (given by (HYP)) we derive again the unsound sequent $A,$ 🔓 $\vdash_{\mathsf{NS4}^{\Box}} A$ in any case of structural rule:

– Weakening: from $A \vdash_{\mathsf{NS4}^{\Box}} A$ by weakening we conclude $A,$ 🔓 $\vdash_{\mathsf{NS4}^{\Box}} A$.

– Exchange: since 🔓$, A \vdash_{\mathsf{NS4}^{\Box}} A$ by exchange we conclude $A,$ 🔓 $\vdash_{\mathsf{NS4}^{\Box}} A$.

– Contraction: from $A,$ 🔓$, A \vdash_{\mathsf{NS4}^{\Box}} A$ by contraction we conclude $A,$ 🔓 $\vdash_{\mathsf{NS4}^{\Box}} A$.

This shows that, if unrestricted, all three structural rules will generate an unsound system. However, the rules are sound under some restrictions according to the following.

Proposition 3.1 (Restricted structural rules)**.** The following rules are admissible:

– Weakening:

$$\frac{\Gamma, \Gamma' \vdash_{\mathsf{NS4}^{\Box}} C}{\Gamma, A, \Gamma' \vdash_{\mathsf{NS4}^{\Box}} C} \tag{7}$$

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1.	$\Box(\Box A \to \Box(\Box A \to \Box B))$	$\vdash_{\mathsf{NS4}^{\Box}}$	$\Box(\Box A \to \Box(\Box A \to \Box B))$	(Hyp)	
2.	$\Box(\Box A \to \Box(\Box A \to \Box B)), \text{🔓}, \Box A$	$\vdash_{\mathsf{NS4}^{\Box}}$	$\Box A$	(Hyp)	
3.	$\Box(\Box A \to \Box(\Box A \to \Box B)), \text{🔓}, \Box A$	$\vdash_{\mathsf{NS4}^{\Box}}$	$\Box A \to \Box(\Box A \to \Box B)$	(K-Imp)	1
4.	$\Box(\Box A \to \Box(\Box A \to \Box B)), \text{🔓}, \Box A$	$\vdash_{\mathsf{NS4}^{\Box}}$	$\Box(\Box A \to \Box B)$	(MP$_+$)	$2, 3$
5.	$\Box(\Box A \to \Box(\Box A \to \Box B)), \text{🔓}, \Box A, \text{🔓}$	$\vdash_{\mathsf{NS4}^{\Box}}$	$\Box A \to \Box B$	(K-Imp)	4
6.	$\Box(\Box A \to \Box(\Box A \to \Box B)), \text{🔓}, \Box A, \text{🔓}$	$\vdash_{\mathsf{NS4}^{\Box}}$	$\Box A$	(4-Imp)	2
7.	$\Box(\Box A \to \Box(\Box A \to \Box B)), \text{🔓}, \Box A, \text{🔓}$	$\vdash_{\mathsf{NS4}^{\Box}}$	$\Box B$	(MP$_+$)	$5, 6$
8.	$\Box(\Box A \to \Box(\Box A \to \Box B)), \text{🔓}, \Box A$	$\vdash_{\mathsf{NS4}^{\Box}}$	$\Box B$	(T-Exp)	7
9.	$\Box(\Box A \to \Box(\Box A \to \Box B)), \text{🔓}$	$\vdash_{\mathsf{NS4}^{\Box}}$	$\Box A \to \Box B$	$(\to I)$	8
10.	$\Box(\Box A \to \Box(\Box A \to \Box B))$	$\vdash_{\mathsf{NS4}^{\Box}}$	$\Box(\Box A \to \Box B)$	(K-Exp)	9

Fig. 4. Example derived in $\mathsf{NS4}^{\Box}$

– Exchange:

$$\frac{\Gamma, A, B, \Gamma' \vdash_{\mathsf{NS4}^{\Box}} C}{\Gamma, B, A, \Gamma' \vdash_{\mathsf{NS4}^{\Box}} C} \tag{8}$$

– Contraction:

$$\frac{\Gamma, A, A, \Gamma' \vdash_{\mathsf{NS4}^{\Box}} C}{\Gamma, A, \Gamma' \vdash_{\mathsf{NS4}^{\Box}} C} \tag{9}$$

Proof. Each rule is proved admissible by structural induction on its premise.

It is worth noting that although these structural patterns are stated in the usual way, and there are not conditions about the presence or absence of locks, they are still restricted since our contexts are lists and not (multi)sets.

For instance, it is not possible to use the exchange rule above to get $\Gamma, B, \text{🔓}, A \vdash_{\mathsf{NS4}^{\Box}} C$ from $\Gamma, A, \text{🔓}, B \vdash_{\mathsf{NS4}^{\Box}} C$ since the formulas A and B are not together.

This would not be the case if the contexts were (multi)sets, which obviously would generate unsound rules.

As stated the rules ensure that structural reasoning is locally safe, that is it can be applied as long as the assumptions involved appear together in the context.

But observe that for the case of weakening it is safe to add an assumption anywhere in the context.

Let us next show some structural reasoning patterns involving modal formulas and locks.

These rules emphasize the unrestricted alteration of contexts when a derived formula is somehow encapsulated by necessity (explicitly or by the presence of a lock).

Proposition 3.2 (Specialized modal structural rules)**.** The next rules are admissible:

$$\frac{\Gamma \vdash_{\mathsf{NS4}^{\Box}} \Box A}{\Gamma; \Gamma' \vdash_{\mathsf{NS4}^{\Box}} \Box A} \text{ (Weak-Modal)} \tag{10}$$

$$\frac{\Gamma \vdash_{\mathsf{NS4}^{\Box}} \Box A}{\Gamma; \Gamma' \vdash_{\mathsf{NS4}^{\Box}} A} \text{ (Open)} \tag{11}$$

$$\frac{\Gamma, \text{🔓}; \Gamma' \vdash_{\mathsf{NS4}^{\Box}} A}{\Gamma; \Gamma''; \Gamma' \vdash_{\mathsf{NS4}^{\Box}} A} \text{ (Lock-Replace)} \tag{12}$$

Proof. (Weak-Modal) is proved by induction on Γ'; (Open) is a direct consequence of (K-Imp) and (T-Exp). Finally (Lock-Replace) is proved by structural induction on its premise.

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Proposition 3.3 (Substitution principle)**.** The following rule is admissible:

$$\frac{\Gamma, A \vdash_{\mathsf{NS4}^{\Box}} B \qquad \Gamma \vdash_{\mathsf{NS4}^{\Box}} A}{\Gamma \vdash_{\mathsf{NS4}^{\Box}} B} \tag{13}$$

Proof. The rule is even derivable from modus ponens.

This finishes the section. In the following section we show that the proposed sequent-style system $\mathsf{NS4}^{\Box}$ is equivalent to the Fitch modal logic $\mathfrak{F}^{\Box}$.

4 Equivalence between $\mathfrak{F}^{\Box}$and $\mathsf{NS4}^{\Box}$

We present in this section a detailed equivalence between the Fitch-style logic $\mathfrak{F}^{\Box}$ and our sequent-style system $\mathsf{NS4}^{\Box}$, to the best of our knowledge this has not been done before.

The idea is intuitively clear: any (pseudo)proof π in $\mathfrak{F}^{\Box}$ corresponds to a sequent in $\mathsf{NS4}^{\Box}$ obtained by translating every step F_i in π to a sequent $\Gamma_i \vdash_{\mathsf{NS4}^{\Box}} F_i$ where Γ_i keeps trace of all subordinate proofs opened until step i.

In the other direction we construct a Fitch pseudoproof of C from any given sequent $\Gamma \vdash_{\mathsf{NS4}^{\Box}} C$ in $\mathsf{NS4}^{\Box}$ using Γ to open all required subordinate proofs.

Theorem 4.1. If $\Gamma \mathrel{|\!\sim}_{\mathfrak{F}} C$ then there exists a context Γ' such that $\Gamma' \vdash_{\mathsf{NS4}^{\Box}} C$.

Proof. Let $\mathcal{P} = \langle \mathcal{D}, F, \mathcal{I} \rangle$ witnessing the fact that $\Gamma \mathrel{|\!\sim}_{\mathfrak{F}} C$ where $\Gamma = \{A_1, \ldots, A_q, \ldots, A_m\}$ and A_i is a hypothesis for $1 \leq i \leq q$.

Let $\mathcal{D} = [1, n]$ with $q \leq n$. We prove a stronger statement, namely that for every $1 \leq i \leq n$, there is a context Γ_i such that $\Gamma_i \vdash_{\mathsf{NS4}^{\Box}} F_i$.

The idea behind the construction of Γ_i is that this context will trace in order all intervals opened from 1 to i in the proof figure $\mathcal{P}$: a formula $A \in \Gamma_i$ denotes the opening of an ordinary interval with assumption A, whereas a 🔓 $\in \Gamma_i$ denotes the opening of a modal interval.

Further, the elimination of the last element in a context corresponds to the closing of the last opened interval. Since $\mathcal{P}$ is given we know all intervals in the proof and are able to construct Γ_i.

The proof is by strong induction on n. If $1 \leq i \leq q$ then we take $\Gamma_n = \Gamma$. This way $\Gamma_i \vdash_{\mathsf{NS4}^{\Box}} F_i$ by rule (HYP). This covers the base cases of the induction.

Let us assume now that for every $q < i < n$ there is a Γ_i such that $\Gamma_i \vdash_{\mathsf{NS4}^{\Box}} F_i$.

For the inductive step we consider first the case when F_n is an assumption: If $\deg_M(F_n) = \deg_M(F_{n-1})$ then we take $\Gamma_n = \Gamma_{n-1}, F_n$.

This way $\Gamma_n \vdash_{\mathsf{NS4}^{\Box}} F_n$ by the (HYP) rule. Otherwise we necessarily have $\deg_M(F_n) = \deg_M(F_{n-1}) + 1$ and take $\Gamma_n = \Gamma_{n-1}$, 🔓, F_n, which, by the (HYP) rule implies $\Gamma_n \vdash_{\mathsf{NS4}^{\Box}} F_n$.

Next we make a case analysis on the rule applied to get F_n. In the following argumentation we heavily use Definition 2.4.

– Modus ponens: We have $F_k = A \to E,\ F_l = A,\ F_n = E$ where w.l.o.g $k < l < n$. By IH we have $\Gamma_k \vdash_{\mathsf{NS4}^{\Box}} F_k$ and $\Gamma_l \vdash_{\mathsf{NS4}^{\Box}} F_l$.

  If $\deg(F_k) = \deg(F_l)$ then necessarily $\Gamma_k = \Gamma_l$ and we take $\Gamma_n = \Gamma_k$. Thus by (MP) we get $\Gamma_n \vdash_{\mathsf{NS4}^{\Box}} F_n$.

  In other case we have $o = \deg_O(F_k) < \deg_O(F_l) = o'$ which means that there are $o' - o$ new assumptions added from Γ_k to Γ_l, that is $\Gamma_l = \Gamma_k, C_1, \ldots, C_{o'-o}$.

  By weakening we have $\Gamma_l \vdash_{\mathsf{NS4}^{\Box}} F_k$ thus we take $\Gamma_n = \Gamma_l$ and by (MP) we get $\Gamma_n \vdash_{\mathsf{NS4}^{\Box}} F_n$.

– $(\to I)$: We have $F_k = A,\ F_{n-1} = B$ and $F_n = A \to B$. By I.H. $\Gamma_k \vdash_{\mathsf{NS4}^{\Box}} F_k$ and since F_k is an assumption $\Gamma_k = \Gamma_{k-1}, F_k$.

  Moreover, since $\deg(F_k) = \deg(F_l)$ then $\Gamma_l = \Gamma_k$ and by I.H. $\Gamma_{k-1}, F_k \vdash_{\mathsf{NS4}^{\Box}} F_l$.

  Then rule $(\to\ I)$ yields $\Gamma_{k-1} \vdash_{\mathsf{NS4}^{\Box}} F_k \to F_l$. Taking $\Gamma_n = \Gamma_{k-1}$ we have shown that $\Gamma_n \vdash_{\mathsf{NS4}^{\Box}} F_n$.

– (K-IMP): We have $F_n = E,\ F_j = \Box E$ with $j < n$. By I.H. we have $\Gamma_j \vdash_{\mathsf{NS4}^{\Box}} F_j$ with $\deg(F_j) = (m, o)$.

  Since $m = \deg_M(F_j) < \deg_M(F_n)$ at least one strict subordinate proof (modal interval) was started after j, say the first of such intervals starts at step l with $j < l \leq n$. This means that $\Gamma_l = \Gamma_j, \Gamma'$, 🔓.

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Thus we can take $\Gamma_n = \Gamma_l, \Gamma'' = \Gamma_j, \Gamma', 🔓$ where Γ'' witnesses the fact that maybe more intervals were opened after step l and before step n. We can now apply rule $Open$ to get $\Gamma_n \vdash_{\mathsf{NS4}^\Box} F_n$.

- (4-IMP) is analogous to the previous case.
- (K-EXP) is analogous to the next case.
- (T-EXP): In this situation there is a $j \in \mathcal{D}$ a proposition A and intervals $J \in M$ and $K \in \mathcal{I}$ such that $F_{n-1} = A = F_n,\ F_{n-1} \in J,\ F_n \in K$.

  By I.H. we have that $\Gamma_{n-1} \vdash_{\mathsf{NS4}^\Box} F_{n-1}$ and since $\deg_M(F_n) + 1 = \deg_M(F_{n-1})$ then the last open interval was J, which is a modal interval.

  This necessarily means that $\Gamma_{n-1} = \Gamma', 🔓$. Taking $\Gamma_n = \Gamma'$ we get $\Gamma_n \vdash_{\mathsf{NS4}^\Box} F_n$ by rule (T-EXP).

This finishes the proof. As a corollary we obtain the desired proof translation from $\mathfrak{F}^\Box$ to $\mathsf{NS4}^\Box$.

Corollary 4.1. If $\Gamma \vdash_{\mathfrak{F}} C$ then there exists a context Γ' such that $\Gamma' \vdash_{\mathsf{NS4}^\Box} C$.

Now we deal with the other direction of the equivalence. To this purpose let us first define, given a $\mathsf{NS4}^\Box$-context Γ, the context Γ^- in $\mathfrak{F}^\Box$ as $\Gamma^- = \Gamma \cap \mathsf{Prop}$ that is Γ^- is obtained from Γ eliminating all locks 🔓.

Next we show how to construct a $\mathfrak{F}^\Box$-pseudoproof of a formula A declared as hypothesis or assumption in $\mathsf{NS4}^\Box$. This task is not quite trivial since the unrestricted reiteration rule is not available in $\mathfrak{F}^\Box$.

Lemma 4.1. If $\Gamma, A; \Gamma' \vdash_{\mathsf{NS4}^\Box} A$ is an instance of the HYP rule then there is a pseudoproof $\mathcal{P}$ such that $\Gamma^-, A; \Gamma' \vdash\!\sim_{\mathfrak{F}} A$.

Proof. We show that $\Gamma^-, A; \Gamma' \vdash\!\sim_{\mathfrak{F}} A$ by induction on Γ'. In the base case $\Gamma' = \cdot$ and we need to show that $\Gamma^-, A \vdash\!\sim_{\mathfrak{F}} A$. Let $\Gamma = \{Q_1, \ldots, Q_n\}$ and define $\mathcal{D} = [1, n+1]$.

For each $1 \leq i \leq n$, if $Q_i \in \mathsf{Prop}$ then $Q_i \in \Gamma^-$ and we define $F_i = Q_i$ and add the interval $[i, \infty)$ to $\mathcal{O}$, otherwise $Q_i = 🔓$ and we add the interval $[i+1, \infty)$ to M. Finally we define $F_{n+1} = A$ and add the interval $[n+1, \infty)$ to $\mathcal{O}$.

This way we have built a proof figure that testifies that $\Gamma^-, A \vdash\!\sim_{\mathfrak{F}} A$. Next, the I.H. yields a proof figure $\mathcal{P} = \langle [1, n], F, \mathcal{I} \rangle$ testifying that $\Gamma^-, A; \Gamma' \vdash\!\sim_{\mathfrak{F}} A$.

For the inductive step we need to show that if $B \in \mathsf{Prop}$ then $\Gamma^-, A; \Gamma', B \vdash\!\sim_{\mathfrak{F}} A$. We do this by defining a proof figure $\mathcal{R} = \langle \mathcal{D}^+, F^+, \mathcal{I}^+ \rangle$, extending the proof figure $\mathcal{P}$, as follows:

- $\mathcal{D}^+ = [1, n+4]$.
- $F_i^+ = F_i$, if $1 \leq i \leq n$. $F_{n+1}^+ = A,\ F_{n+2}^+ = B \to A, F_{n+3}^+ = B, F_{n+4}^+ = A$, where F_{n+1}^+, F_{n+3}^+ are assumptions; F_{n+2}^+ is obtained by $(\to I)$ from F_{n+1}^+ (a vacuous discharge of B) and F_{n+4}^+ is obtained by MP of F_{n+2}^+ and F_{n+3}^+.
- $\mathcal{I}^+ = \mathcal{M}^+ \cup \mathcal{O}^+$ where:
- $\mathcal{M}^+ = \mathcal{M}$.
- $\mathcal{O}^+ = \mathcal{O} \cup \{[n+1, n+1],\ [n+3, \infty)\}$.

This finishes the proof. We are now in position of provide the desired proof translation from $\mathsf{NS4}^\Box$ to $\mathfrak{F}^\Box$.

Theorem 4.2. *If* $\Gamma \vdash_{\mathsf{NS4}^\Box} A$ *then* $\Gamma^- \vdash\!\sim_{\mathfrak{F}} A$.

Proof. By structural induction on $\Gamma \vdash_{\mathsf{NS4}^\Box} A$. Lemma 4.1 takes care of the base case of the induction. Next we prove the inductive steps:

- Modus Ponens: By I.H. there are proof figures $\mathcal{P}_1 = \langle [1, n_1], F^1, \mathcal{I}{+}1 \rangle$, $\mathcal{P}_2 = \langle [1, n_2], F^2, \mathcal{I}_2 \rangle$ witnessing $\Gamma^- \vdash\!\sim_{\mathfrak{F}} A \to B$ and $\Gamma^- \vdash\!\sim_{\mathfrak{F}} A$ respectively.

  We extend $\mathcal{P}_2$ to the interval $[1, n_2+2]$ with the assumption $F_{n_2+1}^2 = A \to B$, adding $[n_2+1, \infty)$ to $\mathcal{O}_2$, and with $F_{n_2+2}^2 = B$ justified by modus ponens of $F_{n_2}^2$ and $F_{n_2+1}^2$.

  This yields a proof figure for $\Gamma^-, A \to B \vdash\!\sim_{\mathfrak{F}} B$. From this and $\mathcal{P}_1$ we can now apply the substitution principle (Proposition 2.1) to conclude $\Gamma^- \vdash\!\sim_{\mathfrak{F}} B$.

– $(\to \ I)$: By I.H. there is a proof figure $\mathcal{P} = \langle [1, n], F, I \rangle$ such that $\Gamma^-, A \vdash_{\mathfrak{F}} B$.

  Moreover, if the A in the context, which is the last assumption in the figure, was introduced by $F_j = A$ then $deg(F_j) = deg(F_n)$ and $[j, \infty) \in \mathcal{O}$.

  We extend $\mathcal{P}$ to $[1, n+1]$ with $F_{n+1} = A \to B$ justified by $(\to \ I)$ and replacing the interval $[j, \infty)$ with $[j, n+1] \in \mathcal{O}$. This yields $\Gamma^- \vdash_{\mathfrak{F}} A \to B$.

– (K-IMP): By I.H. there is a proof figure $\mathcal{P} = \langle [1, n], F, I \rangle$ such that $\Gamma^- \vdash_{\mathfrak{F}} \Box A$. We need to show that $\Gamma^-, \Gamma'^-, \Gamma''^- \vdash_{\mathfrak{F}} A$. Let Γ', 🔓; $\Gamma'' = \{Q_1, \ldots, Q_p\}$.

  We extend $\mathcal{P}$ to the interval $[1, n+r+1]$ where $r = |(\Gamma'; \Gamma'')^-|$, as follows: let $1 \leq i \leq p$, if $Q_i \in$ Prop then define $F_{n+i} = Q_i$ and add $[n+i, \infty)$ to $\mathcal{O}$.

  Otherwise define $F_{n+i} = Q_j$ where $j = \min\{k \mid k > i, Q_k \in \text{Prop}\}$ and add $[n+i, \infty)$ to $\mathcal{M}$.

  Finally we define $F_{n+r+1} = A$ justified by (K-IMP), which is correct since we add at least one $I \in \mathcal{M}$, which implies $\deg_M(F_n) < \deg_M(F_{n+r+1})$ as required. This way we have built a proof figure showing $\Gamma^-, \Gamma'^-, \Gamma''^- \vdash_{\mathfrak{F}} A$.

– (4-IMP). Is analogous to the previous case.

– (SHUT): By I.H. there is a proof figure $\mathcal{P} = \langle [1, n], F, I \rangle$ such that $(\Gamma,$ 🔓$)^- \vdash_{\mathfrak{F}} A$.

  We need to show that $\Gamma^- \vdash_{\mathfrak{F}} \Box A$. Since the original context in $\mathsf{NS4}^\Box$ has a lock as its last element the last opened interval in $\mathcal{I}$ is modal and indeterminate, say $[j, \infty) \in \mathcal{M}$.

  We extend $\mathcal{P}$ to $[1, n+1]$ replacing $[j, \infty)$ with $[j, n+1] \in \mathcal{M}$ and setting $F_{n+1} = \Box A$. This is justified by the rule (K-EXP) and yields a proof figure for $\Gamma^- \vdash_{\mathfrak{F}} \Box A$.

– (T-EXP) is analogous to the previous case.

This finishes our exposition. Let us close this paper briefly mentioning some futures lines of research such as the development of completeness results, which in the case of a classical extension of our formalisms seems to be straightforward, but in the case of the here developed constructive systems provides an interesting challenge.

An important future task is the actual mechanization of our results in a modern proof assistant, which is the main reason to depart here from the diagrammatic reasoning and giving detailed proofs.

Another topic of interest is the extension of our work to cover other modal logics, in particular the full $\mathsf{S4}$, including the possibility operator $\Diamond$, both in the classical and constructive variants.

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