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Remedies for the Inconsistences in the Times of Execution of the Unsorted Database Search Algorithm within the Wave Approach

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Abstract. The typical semiclassical wave version of the unsorted database search algorithm based on a system of coupled simple harmonic oscillators does not consider an important ingredient of Grover's original algorithm as it is quantum entanglement. The role of entanglement in the wave version of the unsorted database search algorithm is explored and contradictions with the time of execution of Grover's algorithm are found. We remedy the contradictions by employing two arguments, one of them qualitative and the other quantitative. For the qualitative argument we employ the probabilistic nature of a legitimate quantum algorithm and remedy the above inconsistence. Within the quantitative argument we identify a parameter in the wave version of the unsorted database search algorithm which is related to entanglement. The contradiction with the time of execution of Grover's algorithm is solved by choosing an appropriate values of such a parameter which incorporates entanglement to the wave version of the unsorted database search algorithm. The utility of the present arguments are evident if the wave version of the unsorted data base search algorithm is experimentally implemented through a system of N quantum dots with a harmonic oscillator potential as a confinement potential for each of the quantum dots. Each of the above N vibrating quantum dots must be coupled to an extra single vibrating quantum dot which entangles to all of them. In order to obtain optimal results, the coupling constants of the mentioned quantum dots should be adjusted in the way described in the present work.

Keywords. Unsorted database search, Grover algorithm, wave, entanglement, queries time.

1 Introduction

In the past, database search for an unsorted database of N items has been object of intense investigation [1, 6]. The efficiency of the algorithm consists in finding the desired item in the minimum number as possible of queries made to the database. Within conventional approaches a query is a binary oracle with outcomes called YES and NOT. If one employs Boolean logic then to find the desired item in an unsorted database takes on the average $\langle Q \rangle = (N+1)/2$ queries providing the search mechanism has a memory with which an item rejected once is not picked up again.

With the help of the superposition of states of quantum mechanics, Grover discovered a search algorithm that reduces the number of necessary queries up to $Q = O(\sqrt{N})$ in a time $O(\sqrt{N})$ [1]. One can conclude that Grover's algorithm is strictly of quantum nature and constitutes one of the greatest achievements of Quantum Computation. Such an algorithm is the optimal one for an unsorted database search [3].

So far, several different experimental implementations of the algorithm have been done. For instance in Ref. [7] nuclear magnetic resonance techniques with a solution of chloroform molecules were employed. In [8] lasers techniques were used for the implementation of the algorithm while in [9] its optical implementation was made. The number of necessary queries (Q) in the Grover's algorithm is determined by the following equation [1]:

$$(2Q+1)\sin^{-1}\Big(\frac{1}{\sqrt{N}}\Big) = \frac{\pi}{2}. \tag{1}$$

Since Eq. (1) does not have an integer solution, the quantum algorithm stops when this is sufficiently close to the desired state which represents the target item. Thus, one says that the desired item is found with high *probability*. In the $n-$qubit implementation of the algorithm it is chosen $N = 2^n$ while the items in the unsorted database are labeled with binary digits. By the way, there exists a Hamiltonian version of the algorithm where the discrete unitary oracle is replaced with a continuous time interaction Hamiltonian that prevails along the entire duration of the algorithm being the respective number of queries represented by the time one has to wait for before finding the desired state encoding the target item [4]. According to Grover's algorithm [2] the starting and the target state are such that

$$|\langle a|s\rangle|^2 = 1/N, \qquad |\langle a|t\rangle|^2 = \delta_{at}, \tag{2}$$

where the index a labels the different items. Thus, the algorithm evolves the starting state $|s\rangle$ towards the target state $|t\rangle$ through the operators

$$U_t = 1 - 2|t\rangle\langle t|, \qquad U_s = 1 - 2|s\rangle\langle s|, \tag{3}$$

in such a way that

$$\Big(-U_s U_t\Big)^Q |s\rangle = |t\rangle. \tag{4}$$

The operator U_t plays the role of a binary oracle that flips the sign of the target state. On the other hand, $-U_s$ accomplishes the reflection-in-the-mean operation. The number of necessary queries of Eq. (4) is given by the solution of Eq. (1) [1].

It is worth emphasizing that there is a semiclassical version of Grover's algorithm called wave database search [6]. In the wave version of the algorithm, the n qubits are replaced by $N = 2^n$ distinct wave modes without involving quantum entanglement at any stage [5]. As we shall see lines below the absence of entanglement could be a serious deficiency of the wave model. The above is due that entanglement is a necessary ingredient for the quantum information processing [10]-[11]. By the way, the wave version of the unsorted database search has been already partially implemented from the experimental point of view through the use of classical optics where the respective oracle is constituted by a phase-shift plate [12].

An interesting version of the wave database search based on a coupled simple harmonic oscillators which is soluble within semiclassical domain was introduced in Ref. [6]. However, the approach of [6] does not include an essential quantum resource as it is entanglement.

The above contradicts the spirit of the original formulation of Grover's algorithm while in such algorithm the initialization process starts with the following uniform superposition state

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{x=1}^{N} |x\rangle, \tag{5}$$

which is entangled. In fact, in Ref. [13] the amount of multipartite entanglement associated to multiqubit quantum states employed in the Grover algorithm was calculated. Furthermore, the authors of [13] prove that genuine multipartite entanglement depending on $N = 2^n$ in a non trivial way is present at each computational step of Grover's algorithm.

In spite of the above, in the wave version of the unsorted database search algorithm based on a coupled simple harmonic oscillators introduced in [6], entanglement is not taken into account. In the present paper we find that a consequence of the absence of entanglement in the wave version of the database search algorithm based on a coupled simple harmonic oscillators is that its prediction for the time of execution of Grover's algorithm is $O\big(\frac{1}{\sqrt{N}}\big)$. However, Grover's algorithm predicts that such a time must be $O(\sqrt{N})$.

In order to be consistent with the time of execution of Grover's algorithm, in the present work we introduce both a qualitative argument based on probabilistic considerations and a quantitative argument where entanglement is incorporated in an effective way in the coupled simple harmonic oscillators approach. As a consequence of the above we eliminate the contradiction between the times of execution of the wave version of the unsorted database search algorithm with Grover's algorithm. We also remark that our effective approach is simpler than that of Ref. [13] and it gives good results. In our quantitative argument, within the wave version of the unsorted database search algorithm we identify a quantity that can be interpreted as an effective measure of entanglement and to explore its different values until a consistency with the original algorithm is found.

The present effective approach suggests methods and strategies for future experimental implementations of the wave version of the unsorted database search inspired in the coupled simple harmonic oscillators approach. Our suggestion is to experimentally implement such an approach through a system of N quantum dots with a harmonic oscillator potential of confinement for each one of them.

On the other hand, each of the N vibrating quantum dots must be coupled to a central vibrating quantum dot. In order to conciliate the value of the respective time of execution as predicted by the wave version of the unsorted database search algorithm with the value of the time of execution of Grover's algorithm, the parameters of the present wave model should be adjusted in the way indicated in the present work. The paper is organized as follows: In Section 1 we explain the wave version of the unsorted database search algorithm. Meanwhile, in Section 2 we discuss the validity of our arguments.

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Finally, in Section 3 we give a conclusions of the present formalism.

2 The Wave Version of the Database Search Algorithm Based on a Coupled Simple Harmonic Oscillators without Entanglement

According with the wave version of the data base search, the N different items of the unsorted database are associated with N identical harmonic of constant k and mass m. All of such N identical oscillators are coupled to a single main oscillator of mass M and constant K (see Figure 1 of [6]). The absence of multipartite entanglement of the semiclassical model is reflected in the fact that each oscillator is moving independently from the resting $N-1$ oscillators. However, it is worth observing that the N different oscillators are coupled through the driver oscillator of mass M and constant K. In absence of entanglement both quantities M and K do not depend on N (the total number of oscillators). The Lagrangian in the center-of-mass coordinates $R = \sum_{a=1}^{N} x_a/N$ is

$$\mathcal{L} = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}KX^2 + \frac{1}{2}Nm\dot{R}^2 - \frac{1}{2}Nk(R-X)^2 + \sum_{a=1}^{N}\Big[\frac{1}{2}m(\dot{x}_a - R)^2 - \frac{1}{2}k(x_a - R)^2\Big]. \quad (6)$$

In such an approach it is considered exclusively the dynamics of the tapped oscillator by assuming that the resting oscillators are coupled to this one through R. Furthermore, in such a work the $N-2$ resting oscillators are neglected in an effective way to assume that they decouple from the relevant modes $\{X, R, x_t\}$. The effective Lagrangian which does not consider entanglement is then [6]

$$\mathcal{L}^{ne} = \frac{1}{2}\dot{Y}^2 - \frac{1}{2}\frac{K}{M}Y^2 + \frac{1}{2}\dot{\rho}^2 - \frac{1}{2}\Big(\rho - \sqrt{\frac{N}{M}}Y\Big)^2 + \frac{N}{2(N-1)}\dot{y}_t^2 - \frac{N}{2(N-1)}y_t^2, \quad (7)$$

where $Y = \sqrt{M}X$, $\rho = \sqrt{N}R$, and $y_t = x_t - R$ [1]. Due to the decoupling to the relevant modes, in the above equation the modes different to the tapped one have been neglected.

[1] By doing the substitutions $\dot{X} = \dot{Y}/\sqrt{M}$, $X = Y/\sqrt{M}$, $\dot{R} = \dot{\rho}/\sqrt{N}$, and $R = \rho/\sqrt{N}$ in Rq. (6) it follows straightforward Eq. (7).

The eigenvalues ω_+, ω_-, and ω_t associated to $\mathcal{L}^{ne}$ satisfy

$$\omega_\pm = \sqrt{\frac{1}{2}\Big(1+\frac{K+N}{M}\Big) \pm \sqrt{\frac{1}{4}\Big(1+\frac{K+N}{M}\Big)^2 - \frac{K}{M}}}$$
$$\omega_+^2 + \omega_-^2 = 1 + \frac{K+N}{M}, \qquad \omega_t = 1. \quad (8)$$

From the above equation it can be concluded that in the limit of N large enough, the time of execution of Grover's algorithm is approximately

$$t_{\mathsf{G}}^{ne} \simeq \frac{2\pi}{\sqrt{\omega_+^2 + \omega_-^2 + \omega_t^2}} = \frac{2\pi}{\sqrt{2 + \frac{K+N}{M}}} \sim O\Big(\frac{1}{\sqrt{N}}\Big). \quad (9)$$

The above result contradicts the expected values of the time of execution of Grover's algorithm which are $t_{\mathsf{G}} \sim O(\sqrt{N})$ [2]. The contradiction in Eq. (9) consists in that this one predicts that for N small the times of execution would be very large and that for N large the time of execution would be very small.

By the way, Eq. (9) is a consequence of the absence of entanglement in the semiclassical wave version of the database search of Ref. [6]. In the present paper by considering multipartite entanglement in the wave version of the unsorted database search algorithm, we eliminate the above contradiction through a two different strategies, one of the them is qualitative and the other quantitative.

3 Consideration of Multipartite Entanglement in the Wave Version of the Unsorted Database Search Algorithm Based on a Coupled Simple Harmonic Oscillators

In this section we restore the correct values of the times of execution of Grover's algorithm within the semiclassical wave version of the unsorted database search algorithm by employing a two different approaches, one of them qualitative and the other quantitative.

3.1 Qualitative Approach

As we commented lines above, Grover's algorithm starts with the entangled state of Eq. (5). In other words, the uniform superposition of the different states plays a fundamental role for the efficient execution of the algorithm. In spite of the above, in the typical version of the wave model of the unsorted data base search it is considered exclusively one single mode (x_t) assuming that the $N-1$ remaining decouples from the relevant modes $\{X, R, x_t\}$. It is worth mentioning at this stage that the target mode can be any of the N modes with equal probability of occurrence.

Furthermore, multipartite entanglement demands that the modes $\{X, R, x_1, x_2, ..., x_N\}$ does not disentangle each other. In certain sense they are mutually coupled and all of the modes $\{x_1, x_2, ..., x_N\}$ participate on the same basis each of them in the evolution of the algorithm. In such a sense it is necessary a factor of N in Eq. (9). Thus, to multiply Eq. (9) by N we obtain that to consider entanglement in the wave version of the unsorted data base search, the predicted time of execution of Grover's algorithm is

$$T_{\mathsf{G}} \sim O(\sqrt{N}), \tag{10}$$

which is consistent with Grover's algorithm [1], [2]. In the above we have used the fact that Grover's algorithm is of a probabilistic nature. Likewise, we have assumed that all of the N different modes are entangled. Hence each of the different modes necessarily participate as tapped and do it with the same probability of occurrence.

3.2 Quantitative Approach

Within the wave version of the unsorted data base search algorithm all of the N different oscillators of mass m each are coupled to the mass M. We identify the constant K of the oscillator of mass M as the driver of the coordination of the movement of the N different oscillators. Therefore, we can think of the constant K/M as an indirect measurement of the multipartite "entanglement" of the different N oscillators of mass m. If we observe Eqs. (8), we can see that there is an effective time for the execution of the database search algorithm in its semiclassical wave approach. Namely,

$$T_{tot} \quad = \quad \frac{2\pi}{\omega_+} + \frac{2\pi}{\omega_-}, \tag{11}$$

where $\omega_\pm$ are given by Eq. (8). In the above, the times of execution depend on the significative driver parameter K/M which can be considered as an indirect measurement of the multipartite "entanglement" associated to the different N oscillators of mass m.

The other quantity appearing in Eq. (11) being N/M is related to the size of the database. By varying the driver parameter K/M in Eq. (11), we can find those range of values for which the times of Eq. (11) are consistent with the time of execution of Grover's algorithm, that is

$$T_{tot} \geq T_G = O(\sqrt{N}). \tag{12}$$

In Figure 1 it is plotted the quantity T_{tot} of Eq. (11) as a function of the two following parameters K/M and N/M. As we can appreciate from Figure 1, still there is a range of values of the parameter K/M for which it is satisfied Eq. (12). Due that in the present work the parameter K/M is identified as an indirect measure of the degree of entanglement of the N different oscillators, we can conclude that to incorporate entanglement in an effective way into the wave version of the unsorted database search algorithm it is restored the value of the time of execution of Grover's algorithm.

4 Conclusions

Along the present work we have assumed that the mass and the constants of all of the harmonic oscillators have the same values, that is $m = 1$ and $k = 1$. On the other hand, multipartite entanglement between the N harmonic oscillators and the control oscillator of mass M requires that the constant K must be large enough. The later imply that any change in the state of any of the N harmonic oscillator would modify the states of both the control oscillator and the other $N-1$ oscillators. Due that there are in total $N+1$ oscillators, it is assumed that $K \sim (N+1)$.

On the other hand, it is also assumed that all the $N+1$ oscillators have the same mass, that is, $M = m = 1$. With such an assumptions we found that in the limit of N large enough, the quantity $\sqrt{2+(K+N)/M}$ in Eq. (9) is $\sim \frac{1}{\sqrt{N}}$ which makes that the wave version of the unsorted data base search algorithm contradicts Grover's algorithm as it was already commented lines below to such equation. Such a fail of the wave version of the unsorted database search algorithm is eliminated if one observes that Grover's algorithm has an intrinsic probabilistic structure. Hence, the target mode which decouples from the rest of the harmonic oscillators can be any of the N different modes with the same probability. Therefore, it is necessary to include a factor of N in the Eq. (9). With that factor, it is restored the correct time of execution of Grover's algorithm which is $O(\sqrt{N})$. At this stage we note that there is an overhead

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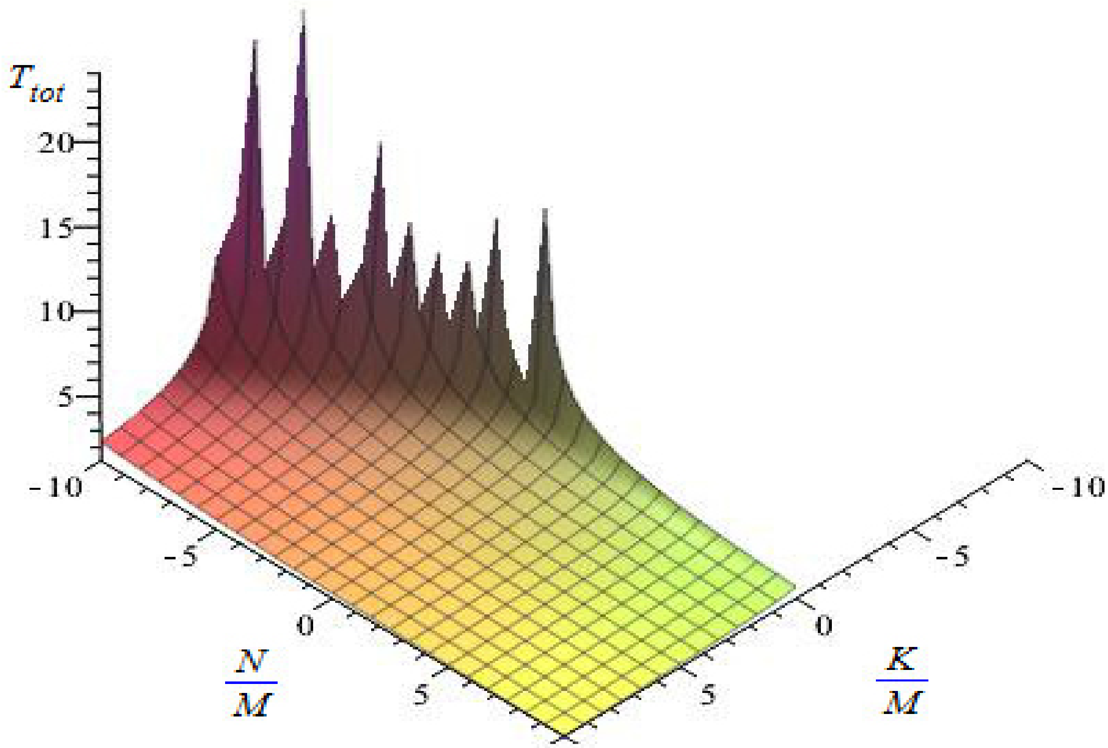

Fig. 1. Time of execution of the unsorted data base search according with the wave model of Eq. (11)

in terms of the number of oscillators with respect to the quantum case. The above reflects a problem of memory space in the wave model approach.

The approach considered in the present paper concludes that entanglement is a necessary ingredient for the simulation of Grover's algorithm through the wave version of the unsorted data base search algorithm based on a coupled classical simple harmonic oscillators. The above follows from our identification of the parameter K/M as and indirect measure of entanglement of the N harmonic oscillators. Thus, from Figure 1 we can appreciate that for certain values of such a parameter, Eq. (12) is satisfied, with which the time of execution of Grover's algorithm is restored from the wave version of the unsorted data base search algorithm.

In the present work, it is suggested that the our approach can be experimentally implemented through the use of a system of N vibrating quantum dots with a harmonic oscillator confinement potential for each of the dots. The N different vibrating quantum dots must be coupled to a main vibrating quantum dot. The value of the time of execution of the wave version of the unsorted database search algorithm is conciliated with the value of the time of execution of Grover's algorithm if the coupling constants of the vibrating quantum dots are adjusted in the way indicated in Section 2 of the present work. With the above, the wave version of the unsorted database search algorithm becomes both a legitimate and an efficient quantum algorithm.

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