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Observability and Observer Design for Continuous-Time Perturbed Switched Linear Systems under Unknown Switchings

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Abstract. In this work we address the observability and observer design problem for perturbed switched linear systems (*SLS*) subject to an unknown switching signal, where the continuous state and the evolving linear system (*LS*) are estimated from the continuous output in spite of the unknown disturbance. The proposed observer is composed of a collection of finite-time observers, one for each *LS* composing the *SLS*. Based on the observability results hereinafter derived and in the observer's output estimation error, the evolving *LS* and its continuous state are inferred. Illustrative examples are presented in detail.

Keywords. Switched systems, observability, finite time observers.

1 Introduction

Switched Linear Systems (*SLS*) are hybrid dynamical systems whose dynamics are represented by a collection of Linear Systems (*LS*), together with an exogenous switching signal determining at each time instant the evolving *LS*. Hybrid systems capture the continuous and discrete interaction that appears in complex systems ranging from biological systems [14] to automotive systems [7]. For *SLS*, many contributions have been reported concerning such basic system properties as stability, controllability, and observability.

Since the conditions for the observability with unknown inputs of the individual *LS* have been already established [4], the main problem for the observability of *SLS* is then to infer, from the knowledge of the continuous measurements, which *LS* is evolving. This is usually referred to as the *distinguishability* problem. This problem, together with the *SLS* observability, has been extensively studied in the literature for the case of known inputs [1, 9, 13, 18, 20]. However, the observability of continuous-time *SLS* subject to unknown disturbances and unknown switching signals has received less attention.

In the context of unknown switching signals, the concept of observability of *SLS* becomes more

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complex than in the *LS* case. The complexity arises from the two following reasons: first, because the autonomous case [20] becomes non-equivalent to the non-autonomous case [1, 9, 13], as the input plays a central role; second, because the unknown switching signal may make the system's trajectory to be observable even if it is evolving in a unobservable *LS* [1].

A geometric characterization of the main results presented in [1, 9, 13, 20] for *SLS* with unknown switching signal and no maximum dwell time was reported in a previous work [11].

If on the contrary a maximum dwell time is considered, then the *SLS* state does not need to be recovered before the first switching time, but after a finite number of switches, either by taking advantage of the underlying discrete event system [9] or by investigating the distinguishability between *LS* [13]. This problem has been considered in [8] for autonomous systems and in [9] and [13] for the known input case.

Concerning the observer design, if the switching signal is unknown then the observer for *SLS* requires to estimate the switching signal from the continuous measurements via a *location observer*. In [2], the location observer uses a residual generator to infer a change in the continuous dynamics. In [10], an algorithm for computing the switching signal has been proposed for monovariable *SLS* with structured perturbations (i.e. disturbances with known derivatives). In [3], a super twisting based observer for unperturbed switched autonomous nonlinear systems has been presented. In [6], the location observer is formed by a set of Luenberger observers with an associated robust differentiator used to obtain the exact error signal which updates the estimate.

1.1 Main Contribution

In this paper we derive new observabilty and observer design results for perturbed *SLS* subject to an unknown switching signal. Based on the framework proposed in [11], here we consider a new observability notion that requires less restrictive conditions and is useful in practical applications.

The proposed observer, which extends the results of [12] to the perturbed case, is based on a collection of high order sliding mode based observers, one for each *LS* forming the *SLS*. In [12], the evolving *LS* was decided based on the output estimation error and an additional variable of the observer, used to measure the observer's effort to maintain a zero output estimation error (since more than one observer may give zero output estimation error). However, in the perturbed case addressed in this paper, the evolving *LS* must be inferred from the output estimation error only.

Based on the observability analysis here derived, we show that if the perturbed *SLS* is observable, then for "almost every" input only one observer in the collection will converge to a zero output estimation error (the one associated to the evolving *LS*), thus inferring the evolving *LS* and obtaining the associated continuous state. Compared to [1–3, 6, 9, 12, 12, 13, 20, 22], the proposed approach addresses a wider class of *SLS* as unknown disturbances are considered. Moreover, unlike the recent results [19, 22] where the switching signal is known, in our approach the continuous state together with the switching signal are inferred from the continuous output.

2 Preliminaries

2.1 On the Concept of "Almost Every" or "Almost Everywhere"

In this paper, the concept from measure theory of "almost everywhere" or for "almost every" is used to express that the observability property is practically certain to hold, except on a proper subspace of the complete state space (which is known as "shy set" or "Haar null set" [15]).

2.2 Preliminaries on Linear Systems

The next lines review some of the basic geometric concepts on *LS* which are mainly taken from [21].

A Linear System (*LS*) is represented by the dynamic equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Sd(t), \quad x(t_0) = x_0 \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x \in \mathcal{X} = \mathbb{R}^n$ is the state vector, $u \in \mathcal{U} = \mathbb{R}^g$ is the control input, $y \in \mathcal{Y} = \mathbb{R}^q$ is the output signal,

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$d \in \mathcal{D} = \mathbb{R}^m$ is the disturbance and A, B, C, and S are constant matrices of appropriate dimensions.

Throughout this work $\Sigma(A, B, C, S)$ denotes the *LS* (1). When the matrices are clear from the context, a *LS* is simply denoted by Σ.

For an initial condition $x(t_0) = x_0$, without disturbance ($d(t) = 0$), the solution of (1) is given by

$$y(t) = Ce^{At}\left[x_0 + \int_0^t e^{-A\tau}Bu(\tau)d\tau\right]. \quad (2)$$

The input function space, denoted by $\mathcal{U}_f$, is considered to be $L_p(\mathcal{U})$,[1] i.e. it contains piecewise continuous functions. Throughout the paper, $\mathcal{B}$ stands for $\mathrm{Im}\, B$ (image or column space of B), $\mathcal{S}$ for $\mathrm{Im}\, S$ and $\mathcal{K}$ for $\ker C$ (kernel or null-space of C).

A subspace $\mathcal{T} \subset \mathcal{X}$ is called A–invariant if $A\mathcal{T} \subset \mathcal{T}$. The supremal A–invariant subspace contained in $\mathcal{K}$ is $\mathcal{N} = \bigcap_{i=1}^{n} \ker(CA^{i-1})$. The subspace $\mathcal{N}$ is known as the unobservable subspace of the *LS* Σ.

A subspace $\mathcal{V} \subset \mathcal{X}$ is said to be (A, B)–invariant if there exists a state feedback $u = Fx$ such that $(A + BF)\mathcal{V} \subset \mathcal{V}$ or, equivalently, if $A\mathcal{V} \subset \mathcal{V} + \mathcal{B}$. The set of maps F for which $(A + BF)\mathcal{V} \subset \mathcal{V}$ holds is denoted as $\mathbf{F}(\mathcal{V})$.

The set of (A, B)–invariant subspaces contained in a subspace $\mathcal{L} \subset \mathcal{X}$ is denoted by $\Im(A, B; \mathcal{L})$. This set is closed under addition, then it contains a supremal element [5], [21] denoted as $\sup \Im(A, B; \mathcal{L})$. Furthermore, $\sup \Im(A, B; \mathcal{L})$ can be computed with the following algorithm.

Algorithm 1. *[5], [21] The subspace* $\sup \Im(A, B; \mathcal{L}) = \mathcal{V}^{(k)}$*, where* $\mathcal{V}^{(k)}$ *is the last term of the sequence*

$$\begin{aligned} &\mathcal{V}^{(0)} = \mathcal{L}, \\ &\mathcal{V}^{(i)} = \mathcal{L} \cap A^{-1}(\mathcal{B} + \mathcal{V}^{(i-1)}), \quad i = 1, \cdots, k, \end{aligned} \quad (3)$$

where the value of $k \leq n - 1$ *is determined by the condition* $\mathcal{V}^{(k+1)} = \mathcal{V}^{(k)}$.

Lemma 2. *[5] Any state trajectory* $x(t)$, $t \in [t_0, \tau]$ *of* Σ *belongs to a subspace* $\mathcal{L} \subseteq \mathcal{X}$ *if and only if* $x(t_0) \in \mathcal{L}$ *and* $\dot{x}(t) \in \mathcal{L}$ *almost everywhere in* $[t_0, \tau]$.

[1] $L_p(\mathcal{U})$, $1 \leq p \leq \infty$ is the set of all piecewise continuous functions $u : \mathbb{R}_{\geq 0} \to \mathcal{U}$ satisfying $\|u(\cdot)\|_p = \left(\int_0^\infty \|u(t)\|_p^p \, dt\right)^{\frac{1}{p}} < \infty$

A *LS* is observable under unknown inputs iff a non-zero state trajectory producing a zero output does not exist. This is formally stated below.

Theorem 3. *[5] Let* $\Sigma(A, B, C, S)$ *be A LS. Then the LS* Σ *is observable under partially unknown inputs if and only if the* $\sup \Im(A, S; \mathcal{K})$ *is trivial.*

2.3 The Switched Linear System's Model

a *SLS* is described as a tuple $\Sigma_\sigma = \langle \mathcal{F}, \sigma \rangle$, where $\mathcal{F} = \{\Sigma_1, \ldots, \Sigma_m\}$ is a collection of *LS* and $\sigma \colon [t_0, \infty) \to \{1, \ldots, m\}$ is the switching signal determining, at each time instant, the evolving *LS* $\Sigma_\sigma \in \mathcal{F}$. The *SLS*'s state equation is represented by

$$\begin{aligned} &\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + S_{\sigma(t)}d(t), \\ &y(t) = C_{\sigma(t)}x(t), \\ &x(t_0) = x_0, \quad \sigma(t_0) = \sigma_0. \end{aligned} \quad (4)$$

We use the notation $x_i(t, x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]})$, to emphasize that the state trajectory $x(t)$ is obtained when $\sigma(t) = i$ and the inputs $u(t)$, $d(t)$ are applied since the initial condition is $x(t_0) = x_0$. In a similar way, $y_i(t, x_0, u(t), d(t))$ represents the output trajectory of $x_i(t, x_0, u(t), d(t))$, i.e.

$$\begin{aligned} x_i(t, x_0, &u(t), d(t)) = \\ &e^{A_i t}\left[x_0 + \int_0^t e^{-A_i\tau}[B_i u(\tau) + S_i d(\tau)]d\tau\right], \end{aligned}$$

$$y_i(t, x_0, u(t), d(t)) = C_i x_i(t, x_0, u(t), d(t)).$$

When we want to emphasize that the state trajectory is restricted to an interval $[\tau_1, \tau_2]$, we write $y_i(t, x_0, u_{[\tau_1,\tau_2]}, d_{[\tau_1,\tau_2]})$, where $u_{[\tau_1,\tau_2]}$, $d_{[\tau_1,\tau_2]}$ are the restriction of the functions $u(t)$, $d(t)$ to $[\tau_1, \tau_2]$. When $d(t) = 0$, we write $y_i(t, x_0, u(t))$, and on autonomous systems we write $y_i(t, x_0)$.

Let us remark that even though a *SLS* is formed by a collection of *LS*, the classical results on the fundamental properties of *LS*, such as stability, observability, and controllability, do not hold straightforwardly in the switching case.

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2.4 Assumptions

Unless otherwise is stated, the following assumptions on the *SLS* are considered.

A. 1. *Only a finite number of switches can occur in a finite interval, i.e. Zeno behavior is not possible.*

A. 2. *The initial condition of the SLS is bounded, i.e. $\|x_0\| < \delta$ with a known constant δ.*

A. 3. *A minimum dwell time in each discrete state is assumed, i.e. if t_{k-1} and t_k are two consecutive switching times, then $t_k - t_{k-1} > \tau_{d_k}$, where $\tau_{d_k} > 0$ is fixed. However, only the dwell time for the first switching time τ_{d_1} is assumed to be known. No maximum dwell time is set.*

A. 4. *The state $x(t)$ is assumed to be continuous, i.e. at each switching time t_l, $x(t_l) = x(t_l^-)$.*

2.5 Distinguishability in Perturbed Switched Linear Systems

In the following we review some of the results in observability using a geometric approach. This review is mainly taken from [11], where it was shown that many observability notions arise in the perturbed case of *SLS*. Next, using the same framework of [11], in Section 3 we will extend the results on a new (and less restrictive) observability notion that is more meaningful for the estimation using the proposed observer.

In the unknown switching signal setting, it is fundamental to infer the evolving linear system, and thus estimate the switching signal from the continuous input-output information. That problem is known as the *distinguishability* problem.

Formally, the *LS* Σ_i is said *indistinguishable* from another *LS* Σ_j if there exist x_0, $d_{[t_0,\tau]}$, x_0', and $d'_{[t_0,\tau]}$ s.t.

$$y_i(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}) = y_j(x_0', u_{[t_0,\tau]}, d'_{[t_0,\tau]}). \tag{5}$$

If (5) holds for some x_0, $d_{[t_0,\tau]}$, x_0', and $d'_{[t_0,\tau]}$, it is impossible to determine from the measured signals, $u_{[t_0,\tau]}$ and $y_{[t_0,\tau]}$, if the state trajectory was generated by Σ_i or by Σ_j, consequently, it is impossible to determine if the continuous initial state is x_0 or x_0'. On the contrary, if (5) does not hold for all x_0, $d_{[t_0,\tau]}$, x_0', and $d'_{[t_0,\tau]}$, then the pair of *LS* is said *distinguishable*, since it is possible to determine if the state is generated by Σ_i or by Σ_j from $u_{[t_0,\tau]}$ and $y_{[t_0,\tau]}$. The indistinguishability subspace $\hat{\mathcal{W}}_{ij}$ of Σ_i, Σ_j is defined as

$$\hat{\mathcal{W}}_{ij} = \left\{ \begin{bmatrix} x_0 \\ x_0' \end{bmatrix} : \begin{array}{l} \exists u_{[t_0,\tau]}, d_{[t_0,\tau]}, d'_{[t_0,\tau]}, \\ y_i(t, x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}) = \\ \qquad y_j(t, x_0', u_{[t_0,\tau]}, d'_{[t_0,\tau]}) \end{array} \right\}$$

and represents the set of initial conditions that under a particular input makes it impossible to determine which is the evolving system and the current discrete state. Then if the state trajectory $x_i(t, x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]})$ evolves inside $\mathcal{Q}_i \hat{\mathcal{W}}_{ij}$ then it is impossible to determine, from the measurements, if the evolving state trajectory is $x_i(t, x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]})$ or $x_j(t, x_0', u_{[t_0,\tau]}, d'_{[t_0,\tau]})$, thus it is not possible to infer either the continuous initial state which could be x_0 or x_0' or the discrete state which could be $\sigma(t) = i$ or $\sigma(t) = j$ $\forall t \in [t_0, \tau]$.

For a given pair of *LS*, Σ_i and Σ_j, the extended *LS* $\hat{\Sigma}_{ij}$ is defined as

$$\begin{array}{ll} \hat{A}_{ij} = \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} & \hat{B}_{ij} = \begin{bmatrix} B_i \\ B_j \end{bmatrix} \\ \hat{C}_{ij} = \begin{bmatrix} C_i & -C_j \end{bmatrix} & \hat{S}_{ij} = \begin{bmatrix} S_i & 0 \\ 0 & S_j \end{bmatrix}. \end{array} \tag{6}$$

The state space of $\hat{\Sigma}_{ij}$ is denoted by $\hat{\mathcal{X}}_{ij}$. We denote by $\hat{x}_0$ the initial state, by $\hat{x}(t)$ the state trajectory, and by $\hat{d}(t)$ the disturbance signal of the extended *LS* $\hat{\Sigma}_{ij}$.

Let us denote the natural projection of $\hat{\mathcal{X}}_{ij}$ over $\mathcal{X}$ as $\mathcal{Q}_i : \hat{\mathcal{X}}_{ij} \to \mathcal{X}$, i.e., $\mathcal{Q}_i(\begin{bmatrix} x^T & x'^T \end{bmatrix}^T) = x$.

Lemma 4. *Let Σ_i and Σ_j be two LS. Equation (5) holds if and only if the extended LS $\hat{\Sigma}_{ij}$ produces a zero output for all $t \in [t_0, \tau]$, i.e. $y_{ij}(\hat{x}_0, u_{[t_0,\tau]}, \hat{d}_{[t_0,\tau]}) = 0$ with $\hat{x}_0 = \begin{bmatrix} x_0^T & x_0'^T \end{bmatrix}^T$, $u(t)$ and $\hat{d}(t) = \begin{bmatrix} d^T(t) & d'^T(t) \end{bmatrix}^T$.*

Proof. The proof can be found in [11]. □

The following result characterizes the indistinguishability subspace.

Lemma 5. *Let Σ_i and Σ_j be two LS and let $\mathbf{B}_{ij} = [\hat{B}_{ij}\ \hat{S}_{ij}]$. Then the indistinguishability subspace $\hat{\mathcal{W}}_{ij}$ is equal to the supremal $\left(\hat{A}_{ij}, \mathbf{B}_{ij}\right)$-invariant subspace contained in $\hat{\mathcal{K}}_{ij} = \operatorname{Ker} \hat{C}_{ij}$, denoted as $\sup \Im(\hat{A}_{ij}, \mathbf{B}_{ij}; \hat{\mathcal{K}}_{ij})$.*

Proof. The proof can be found in [11]. □

By means of Algorithm 1 that computes (A, B)-invariant subspaces, the indistinguishability subspace can be obtained. This subspace is fundamental, since the distinguishability and the observability can be derived from it.

It is worth noting that control inputs play a central role in inferring the evolving *LS*. In order to use such input to distinguish Σ_i from Σ_j, these input has to be capable of steering the state trajectory outside $Q_i\hat{\mathcal{W}}_{ij}$, i.e. the projection of the indistinguishability subspace $\hat{\mathcal{W}}_{ij}$ over $\mathcal{X}$. The conditions for this case are presented below.

Proposition 6. *Let Σ_i and Σ_j be two LS. Then for almost every input Σ_i and Σ_j are distinguishable, i.e.*

$$y_i(t, x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}) \neq y_j(t, x_0', u_{[t_0,\tau]}, d'_{[t_0,\tau]}) \quad (7)$$

if and only if

$$\hat{\mathcal{B}}_{ij} \nsubseteq \hat{\mathcal{W}}_{ij} + \hat{\mathcal{S}}_{ij}. \quad (8)$$

Proof. The proof can be found in [11]. □

3 New Observability Conditions Meaningful in Practical Cases

In this section we derive less restrictive conditions for the distinguishability (hence, for the observability), that can be used in practical applications.

If condition (8) is not satisfied for a pair of *LS* Σ_i and Σ_j then, by Lemma 4, for every $u(t)$ there exist $\hat{x}_0$ and $\hat{d}(t) = [d(t)\ \ d'(t)]^T$ such that Σ_{ij} produces a zero output for all $t \in [t_0, \tau]$. As recalled in Subsection 2.2, an input $\hat{d}(t)$ that makes the extended system unobservable (equivalently, Σ_i and Σ_j indistinguishable) can be written as a state feedback $\hat{d}(t) = F\hat{x}$, where $F \in \mathbf{F}(\sup \Im(\hat{A}_{ij}, \hat{S}_{ij}; \hat{\mathcal{K}}_{ij}))$. Moreover, condition (8) gives the possibility for the input to be in $\hat{\mathcal{S}}_{ij}$. Therefore, regardless of the input, we can write any disturbance making the *LS* Σ_i and Σ_j indistinguishable as $\hat{d}(t) = F\hat{x} + v(t)$, where $v(t) \in \hat{\mathcal{S}}_{ij}$. Now, by Lemma 2

$$\dot{\hat{x}}(t) = (\hat{A}_{ij} + \hat{S}_{ij}F)\hat{x}(t) + \hat{B}_{ij}u(t) + \hat{S}_{ij}v(t) \in \hat{\mathcal{W}}_{ij}.$$

This means that given $u(t)$ there exists $v(t)$ such that $\hat{B}_{ij}u(t) + \hat{S}_{ij}v(t) \in \hat{\mathcal{W}}_{ij}$ in order to maintain the trajectory in the indistinguishable subspace $\hat{\mathcal{W}}_{ij}$.

Remark 7. *Although the conditions of Proposition 8 guarantee that the control input can make the LS Σ_i and Σ_j distinguishable regardless of the disturbance applied, in general, if $u(t)$ is discontinuous then the corresponding disturbance d and d', making Σ_i and Σ_j indistinguishable, must be discontinuous as well with discontinuities at the same time. However, since $u(t)$ and $d(t)$ (the control input and disturbance driving Σ_i) are independent (in the sense that they are not functions of the same set of variables), the probability that their discontinuities are synchronized can be neglected.*

A more convenient distinguishability notion for our setting (with less restrictive conditions), which can be derived using the same framework as in [11], can be stated as the following problem.

Problem 8. *(Distinguishability for almost every control input when $u(t)$ and $d(t)$ are independent, and the latter is continuous)*

Assume that $d(t)$ and $d'(t)$ are continuous on $t \in [t_0, \tau]$, under which conditions is

$$\{u_{[t_0,\tau]} \in \mathcal{U}_f : \forall d_{[t_0,\tau]} \exists x_0, x_0', d'_{[t_0,\tau]}, \\ y_i(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}) = y_j(x_0', u_{[t_0,\tau]}, d'_{[t_0,\tau]})\} \quad (9)$$

a shy set w.r.t $\mathcal{U}_f$? ◇

Notice than requiring $d(t)$ and $d'(t)$ to be continuous on $t \in [t_0, \tau]$ is not a restriction over the class of disturbance that we consider in our setting, but rather it avoids the unlikely case where $d(t)$ has discontinuities at the same time as $u(t)$.

Lemma 9. *Let Σ_i and Σ_j be two LS, and assume that $d(t)$ and $d'(t)$ are continuous on $t \in [t_0, \tau]$. Then*

$$\{u_{[t_0,\tau]} \in \mathcal{U}_f : \forall d_{[t_0,\tau]} \exists x_0, x_0', d'_{[t_0,\tau]}, \\ y_i(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}) = y_j(x_0', u_{[t_0,\tau]}, d'_{[t_0,\tau]})\} \quad (10)$$

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is a shy set, i.e. Σ_i and Σ_j are distinguishable for almost every input with $u(t)$ and $d(t)$ independent and $d(t)$ continuous, if and only if either

$$\hat{\mathcal{B}}_{ij} \nsubseteq \hat{\mathcal{W}}_{ij} \tag{11}$$

or

$$\hat{\mathcal{S}}_i \nsubseteq \hat{\mathcal{W}}_{ij} + \hat{\mathcal{S}}_j \tag{12}$$

holds, where $\hat{\mathcal{S}}_i = Im\,\hat{S}_i$ and $\hat{\mathcal{S}}_j = Im\,\hat{S}_j$ with $\hat{S}_i = \begin{bmatrix} S_i^T & 0^T \end{bmatrix}^T \in \mathbb{R}^{2n}$ and $\hat{S}_j = \begin{bmatrix} 0^T & S_j^T \end{bmatrix}^T \in \mathbb{R}^{2n}$.

Proof. (Necessity) Assume that $\hat{\mathcal{B}}_{ij} \subseteq \hat{\mathcal{W}}_{ij}$ and $\hat{\mathcal{S}}_i \subseteq \hat{\mathcal{W}}_{ij} + \hat{\mathcal{S}}_j$, then, by [11, Lemma 13], $\hat{\mathcal{W}}_{ij} = \sup \Im(\hat{A}_{ij}, \hat{S}_j; \hat{\mathcal{K}}_{ij})$. Now let $d'(t) = F\hat{x} + v$ with $F \in \mathbf{F}(\sup \Im(\hat{A}_{ij}, \hat{S}_j; \hat{\mathcal{K}}_{ij}))$ and v such that $\hat{S}_i d(t) + \hat{S}_j v(t) \in \hat{\mathcal{W}}_{ij}$. Then notice that, if $\hat{x}_0 \in \hat{\mathcal{W}}_{ij}$ and $\dot{\hat{x}}(t) \in \hat{\mathcal{W}}_{ij}$, thus by Lemma 2, $\hat{x}(t) \in \hat{\mathcal{W}}_{ij}$ regardless of $u(t)$. Hence, (10) is equal to $\mathcal{U}_f$ and (10) is not a shy set.

(Sufficiency) Assume that (10) is not a shy set. In [11] it was shown that if an input exists to distinguish two *LS* then almost every input can be used. Thus, (10) being a shy set implies that (10) is equal to $\mathcal{U}_f$ i.e. $\forall u_{[t_0,\tau]}, d_{[t_0,\tau]} \exists\, x_0, x_0', d'_{[t_0,\tau]}$

$$y_i(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}) = y_j(x_0', u_{[t_0,\tau]}, d'_{[t_0,\tau]}).$$

Let $d'(t) = F\hat{x} + v$ (notice that if v is continuous then $d'(t)$ is continuous as well) with $F \in \mathbf{F}(\sup \Im(\hat{A}_{ij}, S_j; \hat{\mathcal{K}}_{ij}))$. Thus, by Lemma 2

$$\begin{aligned} \dot{\hat{x}}(t) = (\hat{A}_{ij} + &\hat{S}_j F)\hat{x}(t) + \\ &+ \hat{B}_{ij} u(t) + \hat{S}_i d(t) + \hat{S}_j v(t) \in \hat{\mathcal{W}}_{ij}. \end{aligned} \tag{13}$$

In particular consider $d(t) = 0$. Since $u(t)$ can be discontinuous on $[t_0, \tau]$ and $d'(t)$ cannot, then $\hat{B}_{ij} u(t)$ evolves inside $\hat{\mathcal{W}}_{ij}$ for every $u(t)$, i.e. $\hat{\mathcal{B}}_{ij} \subseteq \hat{\mathcal{W}}_{ij}$. Now, consider $d(t) \neq 0$. Since $(\hat{A}_{ij} + \hat{S}_j F)\hat{x}(t) + \hat{B}_{ij} u(t)$ evolves inside $\hat{\mathcal{W}}_{ij}$, then (13) implies that for every $d(t)$ there exists $v(t)$ such that $\hat{S}_i d(t) + \hat{S}_j v(t) \in \hat{\mathcal{W}}_{ij}$, i.e. $\hat{\mathcal{S}}_i \subseteq \hat{\mathcal{W}}_{ij} + \hat{\mathcal{S}}_j$, which completes the proof. □

Based on the distinguishability result we can state the observability for *SLS*.

Theorem 10. *Let $\mathcal{G} = \langle \mathcal{F}, \sigma \rangle$ be a SLS with partially unknown inputs and maximum dwell time $\tau = \infty$. Then, the continuous state x_0 and $x(t)$ and the discrete state σ_0 and $\sigma(t)$ can be uniquely computed (said observable) for almost every control input, with $u(t)$ and $d(t)$ independent and $d(t)$ continuous, if and only if every LS $\Sigma_i \in \mathcal{F}$ is observable with partially unknown inputs (i.e. according to Theorem 3) and $\forall \Sigma_i, \Sigma_j \in \mathcal{F}$ $i \neq j$, either $\hat{\mathcal{B}}_{ij} \nsubseteq \hat{\mathcal{W}}_{ij}$ or $\hat{\mathcal{S}}_i \nsubseteq \hat{\mathcal{W}}_{ij} + \hat{\mathcal{S}}_j$.*

Proof. The proof follows from the previous argument. □

4 Observer Design for Perturbed *SLS*

In this section we show that if the continuous and the discrete states of the *SLS* are observable according to Theorem 10, then the *SLS* admits an observer based on a multi-observer structure, in which a finite-time observer is designed for each *LS* composing the *SLS*. It will be shown that, based on the observability results previously presented, a set of finite-time observers allow us to decide the evolving *LS* from the output estimation error. The observability of the *SLS* implies that the observer associated with the evolving *LS* will produce a zero output estimation error whereas the rest of the observers associated to *LS* that are distinguishable from the evolving one, cannot produce a zero output estimation error. Thus, the evolving *LS* is the one associated to the observer with zero output estimation error, whereas an exact estimation of the continuous state is provided by such observer. This observer extends the design proposed in [12] in order to consider unknown inputs.

The construction of the observer for each *LS* is based on the *multivariable observability form* (see, for instance, [16]). Let us firstly introduce some results about the transformation of an observable *LS* into such form.

Lemma 11. *If the LS is observable under partially unknown inputs, i.e. if $\sup \Im(A, S; \mathcal{K})$ is the trivial subspace, then there exists a set of integers $\{r_1, \ldots, r_m\}$ such that $\forall i \in \{1, \ldots, m\}$ $\forall k \in \{0, \ldots, r_i - 2\}$, $c_i A^k S = 0$ and $rank(\mathcal{O}_{\{r_1 \ldots, r_m\}}) = n$ where*

$$\mathcal{O}_{\{r_1\ldots,r_m\}} = \begin{pmatrix} c_1 \\ \vdots \\ c_1A^{r_1-1} \\ \vdots \\ c_m \\ \vdots \\ c_mA^{r_m-1} \end{pmatrix}.$$

Proof. The proof is presented in Appendix A. □

The values $\{r_1\ldots,r_m\}$ are known as the observability indices of $\Sigma(A,B,C,S)$ [16]. A *LS* observable under unknown inputs can be transformed by means of the similarity transformation T, defined such that $T^{-1} = \mathcal{O}_{\{r_1\ldots,r_m\}}$, into the multivariable observability form[2] (see [16]).The resulting *LS* is given by the following matrices.

$$\bar{A} = \left[\begin{array}{ccccc:c:ccccc} * & 1 & 0 & \cdots & 0 & & * & & & & \\ * & 0 & 1 & \cdots & 0 & & * & & & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & & \bigcirc & & \\ * & 0 & 0 & \cdots & 1 & & * & & & & \\ * & 0 & 0 & \cdots & 0 & & * & & & & \\ \hdashline \vdots & & & & & \ddots & \vdots & & & & \\ \hdashline * & & & & & & * & 1 & 0 & \cdots & 0 \\ * & & & & & & * & 0 & 1 & \cdots & 0 \\ * & & \bigcirc & & & & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & & & & & & * & 0 & 0 & \cdots & 1 \\ * & & & & & \cdots & * & 0 & 0 & \cdots & 0 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} r_1 \\ \vdots \\ \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} r_m \end{array} \quad \bar{S} = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ \bar{S}_1 \\ \hdashline \vdots \\ \hdashline 0 \\ 0 \\ \vdots \\ 0 \\ \bar{S}_m \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} r_1 \\ \vdots \\ \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} r_m \end{array}$$

$$\bar{C} = \left[\begin{array}{ccccc:c:ccccc} 1 & 0 & 0 & \cdots & 0 & & 0 & & \bigcirc & & \\ \hdashline * & & \bigcirc & & & \ddots & \vdots & & & & \\ \hdashline * & & & & & & 1 & 0 & 0 & \cdots & 0 \end{array}\right] \Bigg\} m$$

$$\bar{B} = \left[\begin{array}{ccc} \bar{B}_1 & \cdots & \bar{B}_n \end{array}\right]^T$$

Notice that the transformed *LS* is composed of blocks, each one in the observer canonical form (see [16]), but coupled only through the measured variables ($\bar{x}_1,\ \bar{x}_{r_1+1},\ \ldots$). Each i-th block is of dimension r_i and is of the form

$$\begin{array}{rcl} \dot{\bar{x}}_1^i &=& a_1^i(y) + \bar{x}_2^i + \bar{B}_1^i u(t), \\ \dot{\bar{x}}_2^i &=& a_2^i(y) + \bar{x}_3^i + \bar{B}_2^i u(t), \\ & \vdots & \\ \dot{\bar{x}}_{r_i}^i &=& a_{r_i}^i(y) + \bar{B}_{r_i}^i u(t) + \bar{d}^i(t), \end{array} \tag{14}$$

[2]The observer form can be obtained taking $T^{-1} = \mathcal{O}_{\{r_1\ldots,r_m\}}$ as a baseline. The procedure for computing this transformation can be obtained as the dual of the controller form presented in [16, Example 6.4-7, Pag. 436].

where the $a_j^i(y)$ $j = 1,\cdots,r_i$ is a linear combination of the output variables $\bar{x}_1,\ \bar{x}_{r_1+1},\ \ldots$ and $\bar{d}^i(t) = \bar{S}_i d(t)$

In what follows, let us consider the following assumption.

A. 5. *The disturbances $\bar{d}^i(t)$ are differentiable and satisfy $L_i \geq |\dot{\bar{d}}_i(t)|$ with known constants L_i, $i = 1,\ldots,m$.*

Thus, we can add to (14) the dynamics of $\bar{d}^i$, i.e.

$$\dot{\bar{d}}^i = \upsilon_i(t).$$

Then, an observer based in the robust differentiator described in [17] can be designed for each block in order to estimate the state of the system in the presence of disturbances. The observer is designed in such a way that its error dynamics coincides with the differentiation error of [17], for which convergence has already been demonstrated in [17].

Thus, each block admits the following observer:

$$\begin{array}{rcl} \dot{\tilde{x}}_1^i &=& a_1^i(y) + \tilde{x}_2^i + \bar{B}_1^i u, \\ && +l_1\rho|x_1^i - \tilde{x}_1^i|^{r_i/(r_i+1)} sign(x_1^i - \tilde{x}_1^i), \\ \dot{\tilde{x}}_2^i &=& a_2^i(y) + \tilde{x}_3^i + \bar{B}_2^i u, \\ && +l_2\rho^2|x_1^i - \tilde{x}_1^i|^{(r_i-1)/(r_i+1)} sign(x_1^i - \tilde{x}_1^i), \\ & \vdots & \\ \dot{\tilde{x}}_{r_i}^i &=& a_{r_i}^i(y) + \bar{B}_{r_i}^i u(t) + \xi_i, \\ && +l_{r_i}\rho^{r_i}|x_1^i - \tilde{x}_1^i|^{1/(r_i+1)} sign(x_1^i - \tilde{x}_1^i), \\ \dot{\xi}_i &=& l_{r_i+1}\rho^{r_i+1} sign(x_1^i - \tilde{x}_1^i). \end{array} \tag{15}$$

Defining the error dynamics as $e_j^i = x_j^i - \tilde{x}_j^i$ $j = 1,\cdots,m$ and $e_{r_i+1}^i = \bar{d}^i - \xi$, yields to

$$\begin{array}{c} \dot{e}_1^i = e_2^i - l_1\rho|e_1^i|^{r_i/(r_i+1)} sign(e_1^i), \\ \dot{e}_2^i = e_3^i - l_2\rho^2|e_1^i|^{(r_i-1)/(r_i+1)} sign(e_1^i), \\ \vdots \\ \dot{e}_{r_i}^i = e_{r_i+1}^i - l_{r_i\rho^{r_i}}|e_1^i|^{1/(r_i+1)} sign(e_1^i), \\ \dot{e}_{r_i+1}^i = \upsilon_i(t) - l_{r_i+1}\rho^{r_i+1} sign(e_1^i). \end{array} \tag{16}$$

Since the m subsystems are only coupled through the measured variables, which are fed into the observer by output injection through the terms $a_j^i(y)$, the error dynamics of the subsystems are independent from each other and coincide with the form of the differentiation error of the high order sliding mode differentiation presented in [17]. Thus, with the proper choice of the observer parameters ρ and l_i a finite-time convergence to the continuous variables can be obtained in the presence of the disturbance $d(t)$ [17] with arbitrary small convergence time. Thus, an estimation of the variables $\bar{x}_1^i, \ldots, x_{r_i}^i$ is obtained in finite-time. By designing an observer (15) for each block, the continuous state $x(t)$ of the *LS* can be estimated in finite-time.

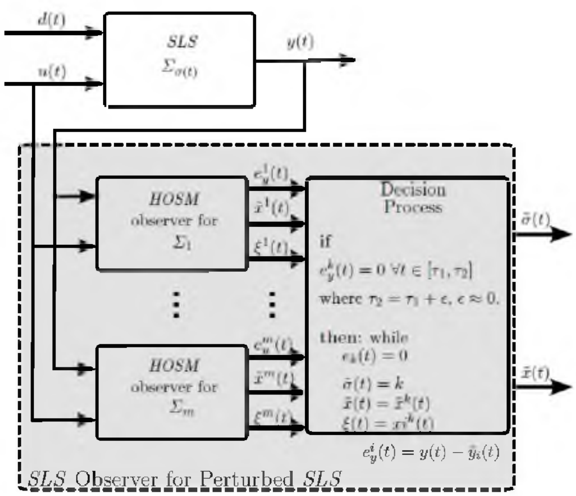

Fig. 1. Observer for perturbed *SLS*

Finally, assuming that each pair of *LS* Σ_i and Σ_j is $(u_{[\tau_1,\tau_2]}, y_{[\tau_1,\tau_2]})$-distinguishable under unknown inputs, if an observer is designed for each *LS* with time convergence bound $\tau < \tau_1 < \tau_2$, then the observer associated to the evolving *LS* will be the only one maintaining $e_y(t) = y(t) - \tilde{y}(t) = 0\ \forall t \in [\tau_1, \tau_2]$ (unlike the known input case where the ξ variable would be also maintained at $\xi = 0$, in the unknown input case $\xi \neq 0$ [12]).

Lemma 12. *Let $\Sigma_{\sigma(t)}$ be a SLS and $\hat{\Sigma}_{\hat{\sigma}(t)}$ its observer with time convergence bound τ, and let $\sigma(t) = i$, $\forall t \in [t_0, t_1)$. Then, if Σ_i and Σ_j are distinguishable under unknown inputs according to Lemma 9, then for almost every input $u(t)$, $e_y^j(t) \neq 0$ almost everywhere in $[\tau, t_1)$ (where $e_y^j(t)$ is the output error signal $e_y(t) = y - \tilde{y}_j$ of the j-th observer).*

Proof. The proof follows by noticing, from (15), that whenever the output error signal e_y^j satisfies $e_y^j = 0$, then the observer becomes a copy of its associated *LS*, driven by a perturbation signal ξ and producing the same output of Σ_i, because due to finite-time convergence of (15) the error correction terms in (15) associated to each block become equal to zero. However, as the Σ_i and Σ_j are distinguishable, then $e_y^j = 0$ cannot occur on a nonzero interval.

Thus, it follows by the assumption that Σ_i and Σ_j are distinguishable under unknown inputs that whenever $\sigma(t) = i$, $\forall t \in [\tau, t_1)$ only the observer associated to Σ_i satisfies $e_y^i = 0$ in the interval $[\tau, t_1)$. □

Hence, a multi-observer structure depicted in Fig. 1 can be used to estimate the continuous state $x(t)$ of the *SLS* and to compute the switching signal $\sigma(t)$. Notice that, since by Lemma 12 only the observer associated to the evolving *LS* gives $e_y = 0$, then by analysing this error signal the evolving *LS* can be ascertain. In a similar way, once the evolving *LS* Σ_j has been detected, the switching occurrence to another *LS*, say Σ_j, can be detected by the time when the error signal no longer satisfies $e_y = 0$, because by the pairwise distinguishability when switching from Σ_i to Σ_j the signal e_y^i can no longer be maintained zero from a nonzero interval.

Remark 13. *Let Σ_i and Σ_j be two LS observable under unknown inputs. Suppose the distinguishability conditions of Proposition 6 do not hold but those of Lemma 9 hold. In such case, as pointed out in Remark 7, if we make $u(t)$ discontinuous then two cases may occur. The first one, no disturbance exists in Σ_j such that* (5) *holds, in which case by Lemma 12 only the output estimation error of the observer associated to Σ_i will be zero. The second one, the disturbance in Σ_j required to make Σ_i and Σ_j indistinguishable is discontinuous as well, in which case the observer associated to Σ_j may estimate $x'(t)$ and $d'(t)$ such that* (5) *holds, but the discontinuities of $d'(t)$ will be synchronized with those of $u(t)$ with a transient with $e_y \neq 0$. However, since $u(t)$ and $d(t)$ are independent then the synchronization of their discontinuities given by the observer associated to Σ_j are identified as unlikely,* *thus discarding Σ_j as the evolving system. In this way, discontinuous inputs in $\mathcal{U}_p$, are useful to make the LS distinguishable and the evolving LS can be inferred from the observers' output estimation error.*

Proposition 14. *Let the continuous and discrete state of $\Sigma_{\sigma(x)}$ be observable under unknown inputs, i.e. each LS $\Sigma_i \in \mathcal{F}$ is observable under unknown inputs and pairwise the LS in $\mathcal{F}$ are distinguishable under unknown inputs, and let T_k be the similarity transformation taking the LS Σ_k into the multivariable observer form. Then the state $x(t)$ of $\Sigma_{\sigma(x)}$ and the switching signal $\sigma(x)$ can be estimated by the following procedure:*

1. *Design an observer (15) with time convergence bound $\tau \ll \tau_d$ for each $\Sigma_i \in \mathcal{F}$.*
2. *The current state of $\sigma(x)$ is k if $e_y^k(t) = 0$, $\forall t \in [\tau, \tau + \delta]$ with $\tau + \delta \ll \tau_d$. That is, the evolving LS is Σ_k if its associated observer is the only one satisfying $e_y = 0$ for a small time interval. If the LS Σ_k is detected $\tilde{x}(t) = T_k x(t)$ with $\tilde{\xi}(t) = \tilde{\xi}^k(t)$.*
3. *A switching is detected when the observer associated to Σ_k no longer satisfies $e_y = 0$.*
4. *After the switching t_j is detected, the state of the observer i, $i \in \{1, \dots, m\}$, is reinitialized as $\tilde{x}^i(t_j) = T_i^{-1} T_k \tilde{x}^k(t_j^-)$, where $x^i(t_j^-)$ is the estimated state of the LS Σ_i at the switching time t_j^-.*
5. *The next state of $\sigma(x)$ is l such that the observer associated to l is the only one satisfying $e_y^l(t) = 0$, $\forall t \in [t_j, t_j + \delta]$.*

Proof. The proof follows by the previous argument and from the observability result which requires each pair of *LS* to be distinguishable for almost every control input. □

5 Illustrative Examples and Simulations

Example 15. ***(Observability under unknown inputs)***

Consider the perturbed SLS $\Sigma_{\sigma(t)}$ where $\sigma(t)$ is an exogenous switching signal and $\mathcal{F} = \{\Sigma_1, \Sigma_2\}$ is the continuous dynamics composed of LS with system matrices as in Table 1, input matrices $B_1 = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}^T$ and $B_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$, output matrices $C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$, and disturbance matrices $S_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and $S_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$.

Table 1. System matrix of linear systems

	Σ_1	Σ_2
A	$\begin{bmatrix} -2 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & 0 & -3 \end{bmatrix}$	$\begin{bmatrix} -1 & -1 & 0 \\ 6 & -5 & 0 \\ 5 & -2 & -3 \end{bmatrix}$

The indistinguishability subspace of Σ_1 and Σ_2 is

$$\hat{\mathcal{W}}_{12} = Im \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -5 & 1 \end{bmatrix}.$$

Since $\hat{\mathcal{W}}_{12}$ is not trivial for every $x_0 \in \mathcal{Q}_1 \hat{\mathcal{W}}_{12} = \mathcal{X}$, there exist $u(t)$ and $d(t)$ such that (5) *holds for some x_0' and $d'(t)$, making it impossible to infer the evolving LS.*

For instance, suppose that the evolving LS is Σ_1, with

$$x_0 = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}^T, \ u(t) = \frac{4}{3}e^{-3t} + \frac{2}{3}$$

and $d(t) = 0$, then the output is

$$y_1(t, x_0, u(t), d(t)) = \frac{4}{3}e^{-3t} + \frac{2}{3},$$

since the LS Σ_j produces this same output with

$$x_0' = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T, u(t) = \frac{4}{3}e^{-3t} + \frac{2}{3},$$

and

$$d'(t) = -\frac{20}{9} + \frac{38}{9}e^{-3t} - \frac{16}{3}te^{-3t},$$

then it is impossible to determine, from the input-output behavior, the evolving LS and the continuous initial state for the current state trajectory. Hence, the LS Σ_1 and Σ_2 cannot be distinguished for every state trajectory.

Moreover, since $\hat{\mathcal{B}}_{12} \subset \hat{\mathcal{W}}_{12}$, then according to Proposition 6, there is no input $u(t)$ allowing to distinguish Σ_1

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from Σ_2, i.e. the effect of the input $u(t)$ does not show up at the output of the extended system.

Furthermore, since $\hat{\mathcal{S}}_1 \subset \hat{\mathcal{W}}_{12} + \hat{\mathcal{S}}_2$ and $\mathcal{Q}_1\hat{\mathcal{W}}_{12} = \mathcal{X}$ then for every state trajectory of Σ_i there exists a state trajectory of Σ_2 producing the same output, therefore it is always impossible to distinguish the LS Σ_1 from the LS Σ_2, i.e. $\forall x_0, u(t), d(t) \exists x_0', d'(t)$ such that (5) *holds, in fact with x_0' such that $\begin{bmatrix} x_0 \\ x_0' \end{bmatrix} \in \hat{\mathcal{W}}_{12}$ and $d'(t) = F\hat{x}(t)$ such that $F \in \mathbf{F}(\hat{\mathcal{W}}_{12})$ (e.g $F = [1\ -4\ 4\ 0\ 2\ 0]$), then* (5) *holds.*

The existence of this state feedback implies that for every state trajectory of the LS Σ_1, there exist x_0' and $d'(t)$ (not necessarily constructed as a state feedback) in Σ_2 such that it is impossible to determine the evolving system and the initial continuous state since there exist state trajectories in both LS producing the same input-output information.

Example 16. ***(Observability and Observer Design)***
Now, if we consider the SLS composed of LS with system matrices as in Table 1, input matrices $B_1 = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^T$ and $B_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, output matrices $C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$, and disturbance matrices $S_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and $S_2 = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}^T$, then the indistinguishability subspace $\hat{\mathcal{W}}_{12}$ is

$$\hat{\mathcal{W}}_{12} = Im \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.8165 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0.4082 & 0 & 0.7071 \\ 0 & 0 & -0.4082 & 0 & 0.7071 \end{bmatrix}.$$

It is easy to verify that, both Σ_1 and Σ_2 are observable under unknown inputs, in addition, Σ_1 and Σ_2 are distinguishable from each other according to Lemma 9 as $\hat{\mathcal{B}}_{12} \not\subseteq \hat{\mathcal{W}}_{12}$. Thus, according to Theorem 10, the continuous and discrete states of the SLS $\Sigma_{\sigma(t)}$ are observable, in infinitesimal time, for almost every control input $u(t)$, and the proposed observer design can be applied to estimate the continuous and discrete states of $\Sigma_{\sigma(t)}$.

Fig. 2 illustrates the application of Lemma 12 to infer the evolving LS. It can be seen that the output estimation error e_y^1, related to the observer of Σ_1, converges to zero in finite-time, whereas the output estimation error e_y^2, related to the observer of Σ_2, cannot be zero (when the system evolves in Σ_1) for a nonzero interval, allowing to infer the evolving LS and the discrete state σ. In this example, the detection of the switching time is trivial, as the continuous output is discontinuous. At the switching instants, the observers are reinitialized as described in Proposition 14. After the reinitialization at 3.5s, only the output estimation error of the observer associated to the evolving LS can be maintained as zero, in this case the signal e_y^2 is the only one that remains at zero.

Hence, the proposed observer determines the evolving LS and the switching signal using only the output information $y(t)$ in spite of the unknown disturbance affecting the system.

Fig. 3 shows the estimation of the continuous state using the procedure described in Proposition 14. The continuous and discrete states of the SLS are estimated in finite-time by the proposed observer, using only the output information $y(t)$ in spite of the unknown disturbance $d(t)$.

It is worth noticing that, in the case where the disturbance is scalar, the proposed observer also estimates the unknown disturbance $d(t)$. However, in general, the unknown disturbance is not estimated. The design of an unknown input observer for SLS can be derived straightforwardly if in addition to the observability under unknown inputs we require each LS to be invertible (i.e. two different disturbance signals cannot produce the same output).

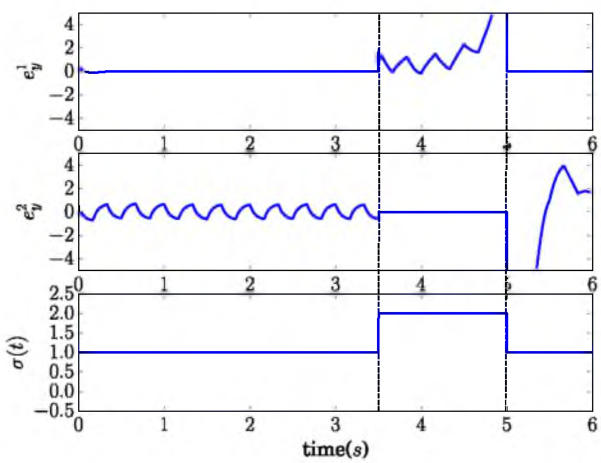

Fig. 2. Inferring the evolving *LS* from the output estimation error

6 Conclusions

Necessary and sufficient conditions for a new observability notion for perturbed *SLS*, meaningful for

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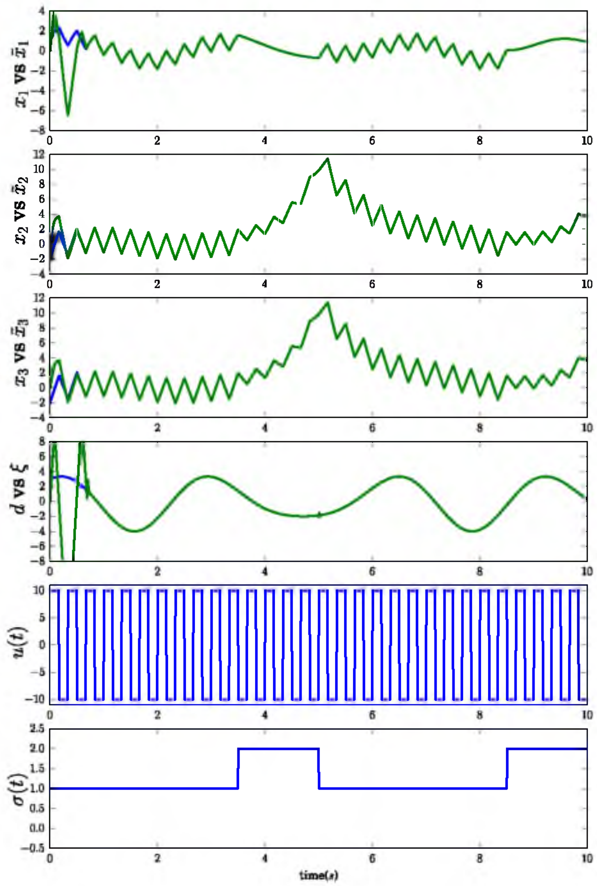

Fig. 3. Estimation of the continuous state $x(t)$

practical applications, have been derived, which result in less restrictive conditions than those reported previously in the literature. Based on the observability analysis we derive the conditions for detecting the evolving *LS* and estimating its continuous state from a collection of finite-time observers in spite of the acting disturbance.

As future work, we consider to model the discrete dynamic by Petri nets to reconstruct the continuous and discrete states after a finite number of switching.

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A Proof of Lemma 11

Proof. The proof follows a similar development as the one shown in [5, Section 4.3.1]. Consider the following iterative process, for simplicity let $B = 0$ as the control input does not affect the observability under unknown inputs in *LS*. Let

$$q_0(t) = Y_0 x(t) \tag{17}$$

with $q_0(t) = y(t)$ and $Y_0 = C$. Differentiating (17) yields $\dot{q}_0(t) = Y_0 A x(t) + Y_0 S d(t)$. Let P_0 be a projection matrix such that $\ker P_0 = \mathrm{Im}(Y_0 S)$. Thus

$$P_0 \dot{q}_0(t) = P_0 Y_0 A x(t). \tag{18}$$

Let $q_1 = \begin{bmatrix} q_0 \\ P_0 \dot{q}_0 \end{bmatrix}$ and $Y_1 = \begin{bmatrix} Y_0 \\ P_0 Y_0 A \end{bmatrix}$ then (17) and (18) can be combined to get $q_1(t) = Y_1 x(t)$. It is easy to see that

$$\begin{aligned} \ker Y_1 &= \ker Y_0 \cap \ker(P_0 Y_0 A) \\ &= \mathcal{K} \cap A^{-1}(\mathcal{K} + \mathcal{S}). \end{aligned}$$

The k-th iteration of the procedure yields $q_k(t) = Y_k x(t)$ such that

$$\begin{aligned} \ker Y_k &= \ker Y_{k-1} \cap A^{-1}(\ker Y_{k-1} + \mathcal{S}) \\ &= \mathcal{K} \cap A^{-1}(\ker Y_{k-1} + \mathcal{S}). \end{aligned}$$

Notice that such iteration process coincides with Algorithm 1 and therefore converges to $\sup \Im(A, S; \mathcal{K})$. Reordering Y_k yields the matrix $\mathcal{O}_{\{r_1 \ldots, r_m\}}$. Since, after a sufficient number of iterations k, $\ker Y_k = \sup \Im(A, S; \mathcal{K}) = 0$, then $rank(\mathcal{O}_{\{r_1 \ldots, r_m\}}) = n$. □

References

1. **Babaali, M. & Pappas, G. J. (2005).** Observability of switched linear systems in continuous time. *Hybrid Systems: Computation and Control*, Springer-Verlag, pp. 103–117.
2. **Balluchi, A., Benvenuti, L., Benedetto, M. D. D., & Sangiovanni-vincentelli, A. L. (2002).** Design of observers for hybrid systems. *In Hybrid Systems: Computation and Control*, Springer-Verlag, pp. 76–89.

3. **Barbot, J., Saadaoui, H., Djemaï, M., & Manamanni, N.** (**2007**). Nonlinear observer for autonomous switching systems with jumps. *Nonlinear Analysis: Hybrid Systems*, Vol. 1, No. 4, pp. 537 – 547.
4. **Basile, G. & Marro, G.** (**1969**). On the observability of linear, time-invariant systems with unknown inputs. *Journal of Optimization Theory and Applications*, Vol. 3, No. 6, pp. 410–415.
5. **Basile, G. & Marro, G.** (**1992**). *Controlled and Conditioned Invariants in Linear System Theory*. Prentice Hall.
6. **Bejarano, F. J. & Fridman, L.** (**2011**). State exact reconstruction for switched linear systems via a super-twisting algorithm. *Intern. J. Syst. Sci.*, Vol. 42, pp. 717–724.
7. **Benvenuti, L., Balluchi, A., Bemporad, A., Cairano, S. D., Johansson, B., Johansson, R., Vincentelli, A. S., & Tunestål, P.** (**2009**). Automotive control. In **Lunze, J. & Lamnabhi-Lagarrigue, F.**, editors, *Handbook of Hybrid Systems Control – Theory, Tools, Applications*, chapter 5. Cambridge University Press, Cambridge, UK, pp. 139–192.
8. **Collins, P. & Van Schuppen, J. H.** (**2004**). Observability of piecewise-affine hybrid systems. In *Hybrid Systems: Computation and Control*. Springer-Verlag, pp. 265–279.
9. **De Santis, E., Di Benedetto, M. D., & Pola, G.** (**2003**). On observability and detectability of continuous-time linear switching systems. *Proceedings of the 42nd IEEE Conference on Decision and Control*, pp. 5777–5782.
10. **Fliess, M., Join, C., & Perruquetti, W.** (**2009**). Real-time estimation of the switching signal for perturbed switched linear systems. *Analysis and Design of Hybrid Systems*, pp. 409–414.
11. **Gómez-Gutiérrez, D., Ramírez-Treviño, A., Ruiz-León, J., & Di Gennaro, S.** (**2012**). On the observability of continuous-time switched linear systems under partially unknown inputs. *IEEE Transactions on Automatic Control*, Vol. 57, No. 3, pp. 732–738.
12. **Gómez-Gutiérrez, D., Čelikovský, S., Ramírez-Treviño, A., & Castillo-Toledo, B.** (**2015**). On the observer design problem for continuous-time switched linear systems with unknown switchings. *Journal of the Franklin Institute*, Vol. 352, pp. 1595–1612.
13. **Gómez-Gutiérrez, D., Ramírez-Prado, G., Ramírez-Treviño, A., & Ruiz-León, J.** (**2010**). Observability of switched linear systems. *IEEE Transactions on Industrial Informatics*, Vol. 6, No. 2, pp. 127 – 135.
14. **Hernandez-Vargas, E. A., Mehta, D., & Middleton, R. H.** (**2011**). Towards modeling HIV long term behavior. *Proceedings of the 2011 IFAC World Congress*.
15. **Hunt, B. R. & Kaloshin, V. Y.** (**2010**). Prevalence. In **Broer, H., Takens, F., & Hasselblatt, B.**, editors, *Handbook of Dynamical Systems*, volume 3. Elsevier Science, pp. 43 – 87.
16. **Kailath, T.** (**1980**). *Linear Systems*. Prentice-Hall.
17. **Levant, A.** (**2003**). Higher-order sliding modes, differentiation and output-feedback controls. *International Journal of Control*, Vol. 76, No. 9-10, pp. 924–941.
18. **Lou, H. & Yang, R.** (**2011**). Conditions for distinguishability and observability of switched linear systems. *Nonlinear Analysis: Hybrid Systems*, Vol. 5, No. 3, pp. 427–445.
19. **Manaa, I., Barhoumi, N., & M'Sahli, F.** (**2015**). Unknown inputs observers design for a class of nonlinear switched systems. *International Journal of Modelling, Identification and Control*, Vol. 23, No. 1, pp. 45–54.
20. **Vidal, R., Chiuso, A., Soatto, S., & Sastry, S.** (**2003**). Observability of linear hybrid systems. In *Hybrid systems: Computation and control*. Springer-Verlag, pp. 526–539.
21. **Wonham, W. M.** (**1979**). *Linear Multivariable Control: A Geometric Approach*. Springer-Verlag.
22. **Yang, J., Chen, Y., Zhu, F., Yu, K., & Bu, X.** (**2015**). Synchronous switching observer for nonlinear switched systems with minimum dwell time constraint. *Journal of the Franklin Institute*.

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