Observability and Observer Design for Continuous-Time Perturbed Switched Linear Systems under Unknown Switchings

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Abstract. In this work we address the observability and observer design problem for perturbed switched linear systems (SLS) subject to an unknown switching signal, where the continuous state and the evolving linear system (LS) are estimated from the continuous output in spite of the unknown disturbance. The proposed observer is composed of a collection of finite-time observers, one for each LS composing the SLS. Based on the observability results hereinafter derived and in the observer’s output estimation error, the evolving LS and its continuous state are inferred. Illustrative examples are presented in detail.

Keywords. Switched systems, observability, finite time observers.

1 Introduction

Switched Linear Systems (SLS) are hybrid dynamical systems whose dynamics are represented by a collection of Linear Systems (LS), together with an exogenous switching signal determining at each time instant the evolving LS. Hybrid systems capture the continuous and discrete interaction that appears in complex systems ranging from biological systems [14] to automotive systems [7]. For SLS, many contributions have been reported concerning such basic system properties as stability, controllability, and observability.

Since the conditions for the observability with unknown inputs of the individual LS have been already established [4], the main problem for the observability of SLS is then to infer, from the knowledge of the continuous measurements, which LS is evolving. This is usually referred to as the distinguishability problem. This problem, together with the SLS observability, has been extensively studied in the literature for the case of known inputs [1, 9, 13, 18, 20]. However, the observability of continuous-time SLS subject to unknown disturbances and unknown switching signals has received less attention.

In the context of unknown switching signals, the concept of observability of SLS becomes more
complex than in the LS case. The complexity arises from the two following reasons: first, because the autonomous case [20] becomes non-equivalent to the non-autonomous case [1, 9, 13], as the input plays a central role; second, because the unknown switching signal may make the system’s trajectory to be observable even if it is evolving in a unobservable LS [1].

A geometric characterization of the main results presented in [1, 9, 13, 20] for SLS with unknown switching signal and no maximum dwell time was reported in a previous work [11].

If on the contrary a maximum dwell time is considered, then the SLS state does not need to be recovered before the first switching time, but after a finite number of switches, either by taking advantage of the underlying discrete event system [9] or by investigating the distinguishability between LS [13]. This problem has been considered in [8] for autonomous systems and in [9] and [13] for the known input case.

Concerning the observer design, if the switching signal is unknown then the observer for SLS requires to estimate the switching signal from the continuous measurements via a location observer. In [2], the location observer uses a residual generator to infer a change in the continuous dynamics. In [10], an algorithm for computing the switching signal has been proposed for monovariable SLS with structured perturbations (i.e. disturbances with known derivatives). In [3], a super twisting based observer for unperturbed switched autonomous nonlinear systems has been presented. In [6], the location observer is formed by a set of Luenberger observers with an associated robust differentiator used to obtain the exact error signal which updates the estimate.

1.1 Main Contribution

In this paper we derive new observability and observer design results for perturbed SLS subject to an unknown switching signal. Based on the framework proposed in [11], here we consider a new observability notion that requires less restrictive conditions and is useful in practical applications.

The proposed observer, which extends the results of [12] to the perturbed case, is based on a collection of high order sliding mode based observers, one for each LS forming the SLS. In [12], the evolving LS was decided based on the output estimation error and an additional variable of the observer, used to measure the observer's effort to maintain a zero output estimation error (since more than one observer may give zero output estimation error). However, in the perturbed case addressed in this paper, the evolving LS must be inferred from the output estimation error only.

Based on the observability analysis here derived, we show that if the perturbed SLS is observable, then for “almost every” input only one observer in the collection will converge to a zero output estimation error (the one associated to the evolving LS), thus inferring the evolving LS and obtaining the associated continuous state. Compared to [1-3, 6, 9, 12, 13, 20, 22], the proposed approach addresses a wider class of SLS as unknown disturbances are considered. Moreover, unlike the recent results [19, 22] where the switching signal is known, in our approach the continuous state together with the switching signal are inferred from the continuous output.

2 Preliminaries

2.1 On the Concept of “Almost Every” or “Almost Everywhere”

In this paper, the concept from measure theory of “almost everywhere” or for “almost every” is used to express that the observability property is practically certain to hold, except on a proper subspace of the complete state space (which is known as “shy set” or “Haar null set” [15]).

2.2 Preliminaries on Linear Systems

The next lines review some of the basic geometric concepts on LS which are mainly taken from [21].

A Linear System (LS) is represented by the dynamic equation

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Sd(t), \quad x(t_0) = x_0 \\
y(t) &= Cx(t),
\end{align*}
\]

where \( x \in \mathcal{X} = \mathbb{R}^n \) is the state vector, \( u \in \mathcal{U} = \mathbb{R}^q \) is the control input, \( y \in \mathcal{Y} = \mathbb{R}^q \) is the output signal,
A LS is observable under unknown inputs if a non-zero state trajectory producing a zero output does not exist. This is formally stated below.

Theorem 3. [5] Let Σ(A, B, C, S) be a LS. Then the LS Σ is observable under partially unknown inputs if and only if the sup Ξ(A, S; K) is trivial.

2.3 The Switched Linear System’s Model

A SLS is described as a tuple Μσ = (F, a), where F = {Σ₁,...,Σₙ} is a collection of LS and σ: [0,∞) → {1,...,m} is the switching signal determining, at each time instant, the evolving LS Σ_σ ∈ F. The SLS’s state equation is represented by

\[
\begin{aligned}
x(t) &= A_a(t)x(t) + B_a(t)u(t) + S_a(t)d(t), \\
y(t) &= C_a(t)x(t),
\end{aligned}
\]

x(t₀) = x₀, a(t₀) = σ₀.

We use the notation X_j(t, x₀, u[t₀,T], d[t₀,T]), to emphasize that the state trajectory x(t) is obtained when σ(t) = i and the inputs u(t), d(t) are applied since the initial condition is x(t₀) = x₀. In a similar way, y_i(t, x₀, u(t), d(t)) represents the output trajectory of x_i(t, x₀, u(t), d(t)), i.e.

\[
y_i(t, x₀, u(t), d(t)) = C_i x_i(t, x₀, u(t), d(t)).
\]

When we want to emphasize that the state trajectory is restricted to an interval [τ₁, τ₂], we write y_i(t, x₀, u[t₁,T₂], d[t₁,T₂]), which is the restriction of the functions u(t), d(t) to [τ₁, τ₂]. When d(t) = 0, we write y_i(t, x₀, u(t)), and on autonomous systems we write y_i(t, x₀).

Let us remark that even though a SLS is formed by a collection of LS, the classical results on the fundamental properties of LS, such as stability, observability, and controllability, do not hold straightforwardly in the switching case.
2.4 Assumptions

Unless otherwise is stated, the following assumptions on the SLS are considered.

**A.1.** Only a finite number of switches can occur in a finite interval, i.e. Zeno behavior is not possible.

**A.2.** The initial condition of the SLS is bounded, i.e. \( \|x_0\| < \delta \) with a known constant \( \delta \).

**A.3.** A minimum dwell time in each discrete state is assumed, i.e. if \( t_{k-1} \) and \( t_k \) are two consecutive switching times, then \( t_k - t_{k-1} > \tau_{d_k} \), where \( \tau_{d_k} > 0 \) is fixed. However, only the dwell time for the first switching time \( \tau_{d_1} \) is assumed to be known. No maximum dwell time is set.

**A.4.** The state \( x(t) \) is assumed to be continuous, i.e. at each switching time \( t_s \), \( x(t_s) = x(t_{s-}) \).

2.5 Distinguishability in Perturbed Switched Linear Systems

In the following we review some of the results in observability using a geometric approach. This review is mainly taken from [11], where it was shown that many observability notions arise in the perturbed case of SLS. Next, using the same framework of [11], in Section 3 we will extend the results on a new (and less restrictive) observability notion that is more meaningful for the estimation using the proposed observer.

In the unknown switching signal setting, it is fundamental to infer the evolving linear system, and thus estimate the switching signal from the continuous input-output information. That problem is known as the distinguishability problem.

Formally, the LS \( \Sigma_i \) is said indistinguishable from another LS \( \Sigma_j \) if there exist \( x_0, d_{[t_0,\tau]}, x_0', \) and \( d'_{[t_0,\tau]} \) s.t.

\[
y_i(x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}) = y_j(x_0', u_{[t_0,\tau]}, d'_{[t_0,\tau]}). \tag{5}
\]

If (5) holds for some \( x_0, d_{[t_0,\tau]}, x_0', \) and \( d'_{[t_0,\tau]} \), it is impossible to determine from the measured signals, \( u_{[t_0,\tau]} \) and \( y_{[t_0,\tau]} \), if the state trajectory was generated by \( \Sigma_i \) or by \( \Sigma_j \), consequently, it is impossible to determine if the continuous initial state is \( x_0 \) or \( x_0' \). On the contrary, if (5) does not hold for all \( x_0, d_{[t_0,\tau]}, x_0', \) and \( d'_{[t_0,\tau]} \), then the pair of LS is said distinguishable, since it is possible to determine if the state is generated by \( \Sigma_i \) or by \( \Sigma_j \) from \( y_{[t_0,\tau]} \) and \( u_{[t_0,\tau]} \). The indistinguishability subspace \( \mathcal{W}_{ij} \) of \( \Sigma_i, \Sigma_j \) is defined as

\[
\mathcal{W}_{ij} = \left\{ \left[ \begin{array}{c} x_0 \\ x_0' \end{array} \right] : y_i(t, x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}) = y_j(t, x_0', u_{[t_0,\tau]}, d'_{[t_0,\tau]}) \right\}
\]

and represents the set of initial conditions that under a particular input makes it impossible to determine which is the evolving system and the current discrete state. Then if the state trajectory \( x_i(t, x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}) \) evolves inside \( Q_i \mathcal{W}_{ij} \) then it is impossible to determine, from the measurements, if the evolving state trajectory is \( x_i(t, x_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}) \) or \( x_j(t, x_0', u_{[t_0,\tau]}, d'_{[t_0,\tau]}) \), thus it is not possible to infer either the continuous initial state which could be \( x_0 \) or \( x_0' \), or the discrete state which could be \( \sigma(t) = i \) or \( \sigma(t) = j \) \( \forall t \in [t_0,\tau] \).

For a given pair of LS, \( \Sigma_i \) and \( \Sigma_j \), the extended LS \( \Sigma_{ij} \) is defined as

\[
\begin{align*}
\dot{A}_{ij} &= \begin{bmatrix}
A_i & 0 \\
0 & A_j
\end{bmatrix}, & \dot{B}_{ij} &= \begin{bmatrix}
B_i \\
B_j
\end{bmatrix}, \\
\dot{C}_{ij} &= \begin{bmatrix}
C_i & -C_j
\end{bmatrix}, & \dot{S}_{ij} &= \begin{bmatrix}
S_i & 0 \\
0 & S_j
\end{bmatrix}.
\end{align*}
\tag{6}
\]

The state space of \( \Sigma_{ij} \) is denoted by \( \hat{\chi}_{ij} \). We denote by \( \hat{x}_0 \) the initial state, by \( \hat{x}(t) \) the state trajectory, and by \( \hat{d}(t) \) the disturbance signal of the extended LS \( \Sigma_{ij} \).

Let us denote the natural projection of \( \hat{\chi}_{ij} \) over \( \chi \) as \( Q_i : \hat{\chi}_{ij} \to \chi \), i.e., \( Q_i([x^T, x'^T]^T) = x \).

**Lemma 4.** Let \( \Sigma_i \) and \( \Sigma_j \) be two LS. Equation (5) holds if and only if the extended LS \( \Sigma_{ij} \) produces a zero output for all \( t \in [t_0,\tau] \), i.e. \( y_{ij}(\hat{x}_0, u_{[t_0,\tau]}, d_{[t_0,\tau]}) = 0 \) with \( \hat{x}_0 = [x_0^T, x_0'^T]^T, u(t) \) and \( \hat{d}(t) = [d^T(t), d'^T(t)]^T \).

**Proof.** The proof can be found in [11]. \( \square \)

The following result characterizes the indistinguishability subspace.
Lemma 5. Let $\Sigma_i$ and $\Sigma_j$ be two LS and let $B_{ij} = [\hat{B}_{ij} \hat{S}_{ij}]$. Then the indistinguishability subspace $W_{ij}$ is equal to the supremal $(\hat{A}_{ij}, B_{ij})$-invariant subspace contained in $\hat{K}_{ij} = \text{Ker} \hat{C}_{ij}$, denoted as $\sup (\hat{A}_{ij}, B_{ij}; \hat{K}_{ij})$.

**Proof.** The proof can be found in [11].

By means of Algorithm 1 that computes $(A, B)$-invariant subspaces, the indistinguishability subspace can be obtained. This subspace is fundamental, since the distinguishability and the observability can be derived from it.

It is worth noting that control inputs play a central role in inferring the evolving LS. In order to use such input to distinguish $\Sigma_i$ from $\Sigma_j$, this input has to be capable of steering the state trajectory outside $Q_j W_{ij}$, i.e. the projection of the indistinguishability subspace $W_{ij}$ over $X$. The conditions for this case are presented below.

**Proposition 6.** Let $\Sigma_i$ and $\Sigma_j$ be two LS. Then for almost every input $\Sigma_i$ and $\Sigma_j$ are distinguishable, i.e.
\[ y_i(t, x_0, u_{[t_0, \tau]}, d_{[t_0, \tau]}) \neq y_j(t, x_0', u_{[t_0, \tau]}, d'_{[t_0, \tau]}) \] if and only if
\[ \hat{B}_{ij} \notin \hat{W}_{ij} + \hat{S}_{ij}. \]

**Proof.** The proof can be found in [11].

3 New Observability Conditions Meaningful in Practical Cases

In this section we derive less restrictive conditions for the distinguishability (hence, for the observability), that can be used in practical applications.

If condition (8) is not satisfied for a pair of LS $\Sigma_i$ and $\Sigma_j$ then, by Lemma 4, for every $u(t)$ there exist $\hat{x}_0$ and $\hat{d}(t) = [d(t) \ d'(t)]^T$ such that $\Sigma_{ij}$ produces a zero output for all $t \in [t_0, \tau]$. As recalled in Subsection 2.2, an input $\hat{d}(t)$ that makes the extended system unobservable (equivalently, $\Sigma_i$ and $\Sigma_j$ indistinguishable) can be written as a state feedback $\hat{d}(t) = F \hat{x}$, where $F \in \mathbf{F}(\sup \delta(\hat{A}_{ij}, \hat{S}_{ij}; \hat{K}_{ij}))$. Moreover, condition (8) gives the possibility for the input to be in $\hat{S}_{ij}$. Therefore, regardless of the input, we can write any disturbance making the LS $\Sigma_i$ and $\Sigma_j$ indistinguishable as $\hat{d}(t) = F \hat{x} + \nu(t)$, where $\nu(t) \in \hat{S}_{ij}$. Now, by Lemma 2
\[ \hat{x}(t) = (\hat{A}_{ij} + \hat{S}_{ij} F) \hat{x}(t) + \hat{B}_{ij} u(t) + \hat{S}_{ij} \nu(t) \in \hat{W}_{ij}. \]

This means that given $u(t)$ there exists $\nu(t)$ such that $\hat{B}_{ij} u(t) + \hat{S}_{ij} \nu(t) \in \hat{W}_{ij}$ in order to maintain the trajectory in the indistinguishable subspace $W_{ij}$.

Remark 7. Although the conditions of Proposition 8 guarantee that the control input can make the LS $\Sigma_i$ and $\Sigma_j$ distinguishable regardless of the disturbance applied, in general, if $u(t)$ is discontinuous then the corresponding disturbance $\nu$ and $\nu'$, making $\Sigma_i$ and $\Sigma_j$ indistinguishable, must be discontinuous as well with discontinuities at the same time. However, since $u(t)$ and $d(t)$ (the control input and disturbance driving $\Sigma_i$) are independent (in the sense that they are not functions of the same set of variables), the probability that their discontinuities are synchronized can be neglected.

A more convenient distinguishability notion for our setting (with less restrictive conditions), which can be derived using the same framework as in [11], can be stated as the following problem.

**Problem 8.** (Distinguishability for almost every control input when $u(t)$ and $d(t)$ are independent, and the latter is continuous)

Assume that $d(t)$ and $d'(t)$ are continuous on $t \in [t_0, \tau]$, under which conditions is
\[ \{u_{[t_0, \tau]} \in U_f : \forall d_{[t_0, \tau]} \exists x_0, x_0', d_{[t_0, \tau]}, \]
\[ y_i(x_0, u_{[t_0, \tau]}, d_{[t_0, \tau]}) = y_j(x_0', u_{[t_0, \tau]}, d'_{[t_0, \tau]}) \}
\]
a shy set w.r.t $U_f$?

Notice that requiring $d(t)$ and $d'(t)$ to be continuous on $t \in [t_0, \tau]$ is not a restriction over the class of disturbance that we consider in our setting, but rather it avoids the unlikely case where $d(t)$ has discontinuities at the same time as $u(t)$.

**Lemma 9.** Let $\Sigma_i$ and $\Sigma_j$ be two LS, and assume that $d(t)$ and $d'(t)$ are continuous on $t \in [t_0, \tau]$. Then
\[ \{u_{[t_0, \tau]} \in U_f : \forall d_{[t_0, \tau]} \exists x_0, x_0', d_{[t_0, \tau]}, \]
\[ y_i(x_0, u_{[t_0, \tau]}, d_{[t_0, \tau]}) = y_j(x_0', u_{[t_0, \tau]}, d'_{[t_0, \tau]}) \}
\]
is a shy set, i.e. \( \Sigma_i \) and \( \Sigma_j \) are distinguishable for almost every input with \( u(t) \) and \( d(t) \) independent and \( d(t) \) continuous, if and only if either

\[
\hat{B}_{ij} \notin \hat{W}_{ij}
\]

or

\[
\hat{S}_i \notin \hat{W}_{ij} + \hat{S}_j
\]

holds, where \( \hat{S}_i = \text{Im} \hat{S}_i \) and \( \hat{S}_j = \text{Im} \hat{S}_j \) with \( \hat{S}_i = [S_i^T \ 0^T]^T \in \mathbb{R}^{2n} \) and \( \hat{S}_j = [0^T \ S_j^T]^T \in \mathbb{R}^{2n} \).

**Proof.** (Necessity) Assume that \( \hat{B}_{ij} \subset \hat{W}_{ij} \) and \( \hat{S}_i \subset \hat{W}_{ij} + \hat{S}_j \), then, by [11, Lemma 13], \( \hat{W}_{ij} = \sup \exists(A_{ij}, \hat{S}_i; K_{ij}) \). Now let \( \hat{d}'(t) = F\hat{x} + v \) with \( F \in F(\sup \exists(A_{ij}, \hat{S}_i; K_{ij})) \) and \( v \) such that \( \hat{S}_i d(t) + \hat{S}_j v(t) \in \hat{W}_{ij} \). Then notice that, if \( \hat{x}_0 \in \hat{W}_{ij} \) and \( \hat{x}(t) \in \hat{W}_{ij} \), thus by Lemma 2, \( \hat{x}(t) \in \hat{W}_{ij} \) regardless of \( u(t) \). Hence, (10) is equal to \( \hat{U}_j \) and (10) is not a shy set.

(Sufficiency) Assume that (10) is not a shy set. In [11] it was shown that if an input exists to distinguish two LS then almost every input can be used. Thus, (10) being a shy set implies that (10) is equal to \( \hat{U}_j \) i.e. \( \forall u(t_0, \tau], d(t_0, \tau], \exists x_0, d_0(t_0, \tau] \)

\[
y_j(x_0, u(t_0, \tau], d(t_0, \tau]) = y_j(x_0, u(t_0, \tau], d(t_0, \tau]).
\]

Let \( \hat{d}'(t) = F\hat{x} + v \) (notice that if \( v \) is continuous then \( d'(t) \) is continuous as well) with \( F \in F(\sup \exists(A_{ij}, \hat{S}_j; K_{ij})) \). Thus, by Lemma 2

\[
\hat{x}(t) = (\hat{A}_{ij} + \hat{S}_j F)\hat{x}(t) + \hat{B}_{ij} u(t) + \hat{S}_i d(t) + \hat{S}_j v(t) \in \hat{W}_{ij}.
\]

In particular consider \( d(t) = 0 \). Since \( u(t) \) can be discontinuous on \( [t_0, \tau] \) and \( d'(t) \) cannot, then \( \hat{B}_{ij} u(t) \) evolves inside \( \hat{W}_{ij} \) for every \( u(t) \), i.e. \( \hat{B}_{ij} \subset \hat{W}_{ij} \). Now, consider \( d(t) \neq 0 \). Since \( (\hat{A}_{ij} + \hat{S}_j F)\hat{x}(t) + \hat{B}_{ij} u(t) \) evolves inside \( \hat{W}_{ij} \), then (13) implies that for every \( d(t) \) there exists \( v(t) \) such that \( \hat{S}_i d(t) + \hat{S}_j v(t) \in \hat{W}_{ij} \), i.e. \( \hat{S}_i \subset \hat{W}_{ij} + \hat{S}_j \), which completes the proof. \( \square \)

Based on the distinguishability result we can state the observability for SLS.

**Theorem 10.** Let \( G = (F, \sigma) \) be a SLS with partially unknown inputs and maximum dwell time \( \tau = \infty \). Then, the continuous state \( x_0 \) and \( x(t) \) and the discrete state \( \sigma_0 \) and \( \sigma(t) \) can be uniquely computed (said observable) for almost every control input, with \( u(t) \) and \( d(t) \) independent and \( d(t) \) continuous, if and only if every LS \( \Sigma_i \in F \) is observable with partially unknown inputs (i.e. according to Theorem 3) and \( \forall \Sigma_j, \Sigma_j \in F \ i \neq j \), either \( \hat{B}_{ij} \notin \hat{W}_{ij} \) or \( \hat{S}_i \notin \hat{W}_{ij} + \hat{S}_j \).

**Proof.** The proof follows from the previous argument. \( \square \)

### 4 Observer Design for Perturbed SLS

In this section we show that if the continuous and the discrete states of the SLS are observable according to Theorem 10, then the SLS admits an observer based on a multi-observer structure, in which a finite-time observer is designed for each LS composing the SLS. It will be shown that, based on the observability results previously presented, a set of finite-time observers allow us to decide the evolving LS from the output estimation error. The observability of the SLS implies that the observer associated with the evolving LS will produce a zero output estimation error whereas the rest of the observers associated to LS that are distinguishable from the evolving one, cannot produce a zero output estimation error. Thus, the evolving LS is the one associated to the observer with zero output estimation error, whereas an exact estimation of the continuous state is provided by such observer. This observer extends the design proposed in [12] in order to consider unknown inputs.

The construction of the observer for each LS is based on the multivariable observability form (see, for instance, [16]). Let us firstly introduce some results about the transformation of an observable LS into such form.

**Lemma 11.** If the LS is observable under partially unknown inputs, i.e. if \( \sup \exists(A, S; K) = \) the trivial subspace, then there exists a set of integers \( \{r_1, \ldots, r_m\} \) such that \( \forall i \in \{1, \ldots, m\} \ \forall k \in \{0, \ldots, r_i - 2\}, c_i A^k S = 0 \) and \( \text{rank}(C_{(r_1, \ldots, r_m)}) = n \) where
where the $a_i^j(y) j = 1, \ldots, r_i$ is a linear combination of the output variables $x_1, x_{r_i+1}, \ldots$ and $d_i^r(t) = S_i d(t)$

In what follows, let us consider the following assumption.

A. 5. The disturbances $d_i^r(t)$ are differentiable and satisfy $L_i \geq |d_i^r(t)|$ with known constants $L_i$, $i = 1, \ldots, m$.

Thus, we can add to (14) the dynamics of $d_i$, i.e.

$$\dot{d}_i^r = v_i(t).$$

Then, an observer based in the robust differentiator described in [17] can be designed for each block in order to estimate the state of the system in the presence of disturbances. The observer is designed in such a way that its error dynamics coincides with the differentiation error of [17], for which convergence has already been demonstrated in [17].

Thus, each block admits the following observer:

$$\dot{\hat{x}}_1^i = a_1^i(y) + \bar{x}_2^i + B_1^i u(t),$$

$$\dot{\hat{x}}_2^i = a_2^i(y) + \bar{x}_3^i + B_2^i u(t),$$

$$\vdots$$

$$\dot{\hat{x}}_{r_i}^i = a_{r_i}^i(y) + B_{r_i}^i u(t) + d_i^r(t),$$

(14)

Defining the error dynamics as $e_i^j = x_i^j - \bar{x}_i^j$ $j = 1, \ldots, m$ and $e_{r_i+1}^j = \bar{d}_i^r - \delta$, yields to

$$e_1^j = e_1^j - l_1 \rho |x_1^j - x_{r_i}^j|^{(r_i+1)/r_i} \text{sign}(e_1^j),$$

$$e_2^j = e_2^j - l_2 \rho |e_1^j|^{(r_i-1)/(r_i+1)} \text{sign}(e_1^j),$$

$$\vdots$$

$$e_{r_i}^j = e_{r_i}^j - l_{r_i} \rho |e_1^j|^{(r_i+1)/r_i} \text{sign}(e_1^j),$$

$$\dot{\bar{d}}_i^r = v_i(t) - l_{r_i+1} \rho e_{r_i}^j \text{sign}(e_1^j).$$

(16)

Since the $m$ subsystems are only coupled through the measured variables, which are fed into the observer by output injection through the terms $a_i^j(y)$, the error dynamics of the subsystems are independent from each other and coincide with

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**Proof.** The proof is presented in Appendix A. □

The values $\{r_1, \ldots, r_m\}$ are known as the observability indices of $\Sigma(A, B, C, S)$ [16]. A LS observable under unknown inputs can be transformed by means of the similarity transformation $T$, defined such that $T^{-1} = O_{(r_1, \ldots, r_m)}$ into the multivariable observability form (see [16]). The resulting LS is given by the following matrices.

$$O_{(r_1, \ldots, r_m)} = \left( \begin{array}{c} c_1 \\ \vdots \\ c_1 A^{r_1-1} \\ \vdots \\ c_m \\ c_m A^{r_m-1} \end{array} \right).$$

Notice that the transformed LS is composed of blocks, each one in the observer canonical form (see [16]), but coupled only through the measured variables $(\bar{x}_1, \bar{x}_{r_i+1}, \ldots)$. Each $i$-th block is of dimension $r_i$ and is of the form

$$\dot{x}_1^i = a_1^i(y) + \bar{x}_2^i + B_1^i u(t),$$

$$\dot{x}_2^i = a_2^i(y) + \bar{x}_3^i + B_2^i u(t),$$

$$\vdots$$

$$\dot{x}_{r_i}^i = a_{r_i}^i(y) + B_{r_i}^i u(t) + \bar{d}_i^r(t),$$

(14)

The observer form can be obtained taking $T^{-1} = O_{(r_1, \ldots, r_m)}$ as a baseline. The procedure for computing this transformation can be obtained as the dual of the controller form presented in [16, Example 6.4-7, Pag. 436].
the form of the differentiation error of the high order sliding mode differentiation presented in [17]. Thus, with the proper choice of the observer parameters $\rho$ and $l$, a finite-time convergence to the continuous variables can be obtained in the presence of the disturbance $d(t)$ [17] with arbitrary small convergence time. Thus, an estimation of the variables $\tilde{x}_i^1, \ldots, \tilde{x}_i^h$ is obtained in finite-time. By designing an observer (15) for each block, the continuous state $x(t)$ of the LS can be estimated in finite-time.

**Proof.** The proof follows by noticing, from (15), that whenever the output error signal $e_y^j(t)$ satisfies $e_y^j(t) = 0$, then the observer becomes a copy of its associated LS, driven by a perturbation signal $\xi$ and producing the same output of $E_j$, because due to finite-time convergence of (15) the error correction terms in (15) associated to each block become equal to zero. However, as the $E_i$ and $E_j$ are distinguishable, then $e_y^j(t) = 0$ cannot occur on a nonzero interval. Thus, it follows by the assumption that $E_i$ and $E_j$ are distinguishable under unknown inputs that whenever $a(t) = i$, $V(t) = \tau$, $\al$ (where $\tau = \tau_1 + \tau_2$), then $e_y^j(t) = 0$ in the interval $[\tau, t_1)$.

Hence, a multi-observer structure depicted in Fig. 1 can be used to estimate the continuous state $x(t)$ of the SLS and to compute the switching signal $a(t)$. Notice that, since by Lemma 12 only the observer associated to the evolving LS gives $e_y^j(t) = 0$, then by analysing this error signal the evolving LS can be ascertain. In a similar way, once the evolving LS $E_j$ has been detected, the switching occurrence to another LS, say $E_j$, can be detected by the time when the error signal no longer satisfies $e_y^j(t) = 0$, because by the pairwise distinguishability when switching from $E_i$ to $E_j$ the signal $e_y^j(t)$ can no longer be maintained zero from a nonzero interval.

**Remark 13.** Let $E_i$ and $E_j$ be two LS observable under unknown inputs. Suppose the distinguishability conditions of Proposition 6 do not hold but those of Lemma 9 hold. In such case, as pointed out in Remark 7, if we make $u(t)$ discontinuous then two cases may occur. The first one, no disturbance exists in $E_j$ such that (5) holds, in which case by Lemma 12 only the output estimation error of the observer associated to $E_i$ will be zero. The second one, the disturbance in $E_j$ required to make $E_i$ and $E_j$ indistinguishable is discontinuous as well, in which case the observer associated to $E_j$ may estimate $x(t)$ and $d(t)$ such that (5) holds, but the discontinuities of $d(t)$ will be synchronized with those of $u(t)$ with a transient with $e_y^j(t) = 0$. However, since $u(t)$ and $d(t)$ are independent then the synchronization of their discontinuities given by the observer associated to $E_j$ are identified as unlikely,

---

Fig. 1. Observer for perturbed SLS

Finally, assuming that each pair of LS $\Sigma_i$ and $\Sigma_j$ is $(u_1(t_1, t_2), y_1(t_1, t_2))$-distinguishable under unknown inputs, if an observer is designed for each LS with time convergence bound $\tau < \tau_1 < \tau_2$, then the observer associated to the evolving LS will be the only one maintaining $e_y^j(t) = y(t) - \tilde{y}(t) = 0 \forall t \in [\tau, t_2]$ (unlike the known input case where the $\xi$ variable would be also maintained at $\xi = 0$, in the unknown input case $\xi \neq 0$ [12]).

**Lemma 12.** Let $\Sigma_{\sigma(t)}$ be a SLS and $\tilde{\Sigma}_{\sigma(t)}$ its observer with time convergence bound $\tau$, and let $\sigma(t) = i, \forall t \in [t_0, t_1)$. Then, if $\Sigma_i$ and $\Sigma_j$ are distinguishable under unknown inputs according to Lemma 9, then for almost every input $u(t)$, $e_y^j(t) \neq 0$ almost everywhere in $[\tau, t_1]$ (where $e_y^j(t)$ is the output error signal $e_y^j(t) = y - \tilde{y}_j$ of the $j$-th observer).
Consider the perturbed SLS $\Sigma_{\sigma(t)}$ where $\sigma(t)$ is an exogenous switching signal and $\mathcal{F} = \{\Sigma_1, \Sigma_2\}$ is the continuous dynamics composed of LS with system matrices as in Table 1, input matrices $B_1 = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}^T$ and $B_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$, output matrices $C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$, and disturbance matrices $S_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and $S_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$.

Table 1. System matrix of linear systems

<table>
<thead>
<tr>
<th></th>
<th>$\Sigma_1$</th>
<th>$\Sigma_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$\begin{bmatrix} -2 &amp; 0 &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} -1 &amp; -1 &amp; 0 \end{bmatrix}$</td>
</tr>
<tr>
<td></td>
<td>$\begin{bmatrix} 2 &amp; -1 &amp; 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 6 &amp; -5 &amp; 0 \end{bmatrix}$</td>
</tr>
<tr>
<td></td>
<td>$\begin{bmatrix} 3 &amp; 0 &amp; -3 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 5 &amp; -2 &amp; -3 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

The indistinguishability subspace of $\Sigma_1$ and $\Sigma_2$ is $\mathcal{W}_{12} = \text{Im} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -5 & 1 \end{bmatrix}$.

Since $\mathcal{W}_{12}$ is not trivial for every $x_0 \in Q_1\mathcal{W}_{12} = \mathcal{X}$, there exist $u(t)$ and $d(t)$ such that (5) holds for some $x_0$ and $d'(t)$, making it impossible to infer the evolving LS. For instance, suppose that the evolving LS is $\Sigma_1$, with $x_0 = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}^T$, $u(t) = -\frac{4}{3}e^{-3t} + \frac{2}{3}$

and $d(t) = 0$, then the output is $x_1(t, x_0, u(t), d(t)) = -\frac{4}{3}e^{-3t} + \frac{2}{3}$, since the LS $\Sigma_1$ produces this same output with $x_0 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$, $u(t) = -\frac{4}{3}e^{-3t} + \frac{2}{3}$

and

$d'(t) = -\frac{20}{9} + \frac{38}{9} e^{-3t} - \frac{16}{3} t e^{-3t}$, then it is impossible to determine, from the input-output behavior, the evolving LS and the continuous initial state for the current state trajectory. Hence, the LS $\Sigma_1$ and $\Sigma_2$ cannot be distinguished for every state trajectory.

Moreover, since $B_{12} \subset \mathcal{W}_{12}$, then according to Proposition 6, there is no input $u(t)$ allowing to distinguish $\Sigma_1$...
from $\Sigma_2$, i.e. the effect of the input $u(t)$ does not show up at the output of the extended system.

Furthermore, since $S_1 \subseteq W_{12} + S_2$ and $Q_1 W_{12} = X$ then for every state trajectory of $\Sigma_1$ there exists a state trajectory of $\Sigma_2$ producing the same output, therefore it is always impossible to distinguish the LS $\Sigma_1$ from the LS $\Sigma_2$, i.e. $\forall x_0, u(t), d(t) \exists x_0, d(t)$ such that (5) holds, in fact with $x_0$ such that

$$\begin{bmatrix} x_0 \\ x_0 \end{bmatrix} \in W_{12}$$

such that $F \in F(W_{12})$ (e.g. $F = [1 - 4 0 2 0]$), then (5) holds.

The existence of this state feedback implies that for every state trajectory of the LS $\Sigma_1$, there exist $x_0$ and $d'(t)$ (not necessarily constructed as a state feedback) in $\Sigma_2$ such that it is impossible to determine the evolving system and the initial continuous state since there exist state trajectories in both LS producing the same input-output information.

Example 16. (Observability and Observer Design)

Now, if we consider the SLS composed of LS with system matrices as in Table 1, input matrices $B_1 = \begin{bmatrix} 1 & 2 & 0 & 0 \end{bmatrix}^T$ and $B_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, output matrices $C_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$, and disturbance matrices $S_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and $S_2 = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}^T$, then the indistinguishability subspace $W_{12}$ is

$$W_{12} = \text{Im} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.8165 & 0 \\ 0 & 0 & 0 & 0.7071 \\ 0 & 0 & 0.4082 & 0.7071 \\ 0 & 0 & 0 & 0.4082 \end{bmatrix}$$

It is easy to verify that, both $\Sigma_1$ and $\Sigma_2$ are observable under unknown inputs, in addition, $\Sigma_1$ and $\Sigma_2$ are distinguishable from each other according to Lemma 9 as $B_{12} \not\subseteq W_{12}$. Thus, according to Theorem 10, the continuous and discrete states of the SLS $\Sigma_{c(t)}$ are observable, in infinitesimal time, for almost every control input $u(t)$, and the proposed observer design can be applied to estimate the continuous and discrete states of $\Sigma_{c(t)}$.

Fig. 2 illustrates the application of Lemma 12 to infer the evolving LS. It can be seen that the output estimation error $e_{x_1}$, related to the observer of $\Sigma_1$, converges to zero in finite-time, whereas the output estimation error $e_{x_2}$, related to the observer of $\Sigma_2$, cannot be zero (when the system evolves in $\Sigma_1$) for a nonzero interval, allowing to infer the evolving LS and the discrete state $\sigma$. In this example, the detection of the switching time is trivial, as the continuous output is discontinuous. At the switching instants, the observers are reinitialized as described in Proposition 14. After the reinitialization at 3.5s, only the output estimation error of the observer associated to the evolving LS can be maintained as zero, in this case the signal $e_{x_2}$ is the only one that remains at zero.

Hence, the proposed observer determines the evolving LS and the switching signal using only the output information $y(t)$ in spite of the unknown disturbance affecting the system.

Fig. 3 shows the estimation of the continuous state using the procedure described in Proposition 14. The continuous and discrete states of the SLS are estimated in finite-time by the proposed observer, using only the output information $y(t)$ in spite of the unknown disturbance $d(t)$.

It is worth noticing that, in the case where the disturbance is scalar, the proposed observer also estimates the unknown disturbance $d(t)$. However, in general, the unknown disturbance is not estimated. The design of an unknown input observer for SLS can be derived straightforwardly if in addition to the observability under unknown inputs we require each LS to be invertible (i.e., two different disturbance signals cannot produce the same output).

Fig. 2. Inferring the evolving LS from the output estimation error

6 Conclusions

Necessary and sufficient conditions for a new observability notion for perturbed SLS, meaningful for...
A Proof of Lemma 11

Proof. The proof follows a similar development as the one shown in [5, Section 4.3.1]. Consider the following iterative process, for simplicity let \( B = 0 \) as the control input does not affect the observability under unknown inputs in \( \mathcal{L} \mathcal{S} \). Let

\[
q(t) = \begin{bmatrix} q_0(t) \\ P_0q_0(t) \end{bmatrix}
\]

with \( q(t) = y(t) \) and \( Y_0 = C \). Differentiating (17) yields

\[
\dot{q}_0(t) = Y_0x(t) + Y_0Sd(t).
\]

Let \( P_0 \) be a projection matrix such that \( \ker P_0 = \text{Im}(Y_0S) \). Thus

\[
P_0q_0(t) = P_0Y_0x(t).
\]

(18)

Let \( q_1 = \begin{bmatrix} q_0 \\ P_0q_0 \end{bmatrix} \) and \( Y_1 = \begin{bmatrix} Y_0 \\ P_0Y_0A \end{bmatrix} \). Then (17) and (18) can be combined to get

\[
q_1(t) = Y_1x(t).
\]

It is easy to see that

\[
\ker Y_1 = \ker Y_0 \cap \ker(P_0Y_0A) = \mathcal{K} \cap A^{-1}(\mathcal{K} + S).
\]

The \(k\)-th iteration of the procedure yields \( q_k(t) = Y_kx(t) \) such that

\[
\ker Y_k = \ker Y_{k-1} \cap A^{-1}(\ker Y_{k-1} + S) = \mathcal{K} \cap A^{-1}(\ker Y_{k-1} + S).
\]

Notice that such iteration process coincides with Algorithm 1 and therefore converges to \( \sup \mathcal{K}(A, S; \mathcal{K}) \). Reordering \( Y_k \) yields the matrix \( O_{\{r_1, \ldots, r_m\}} \). Since, after a sufficient number of iterations \( k \), \( \ker Y_k = \sup \mathcal{K}(A, S; \mathcal{K}) = 0 \), then \( \text{rank}(O_{\{r_1, \ldots, r_m\}}) = n \). □

References


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