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Disturbance Rejection Using SPR0 Substitutions

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Abstract. The disturbance rejection, defined as the problem of designing control laws that ensure, where possible, exogenous disturbances that do not affect the output of the perturbed system, has been resolved by means of algebraic and geometric techniques. This is a steady linear case by means of the static feedback state. Modifications of the Smith form through the SPR0 substitutions are presented which guarantees infinite zeros of a linear single input, single output (SISO) system.

Keywords. Algebraic technique, geometric technique, SPR0 perturbation, SISO.

1 Introduction

The disturbance rejection problem given the structural conditions is an interesting study concerning the robustness of such conditions in the presence of uncertainty in a mathematical system model.

In this paper a special case of uncertainty characterized in terms of strictly positive real substitutions with a relative zero degree is contemplated. Such substitutions correspond to non-linear uncorrelated uncertainty of system parameters under a single input, single output (SISO) system.

Under the action of a control law $u(t)=F\cdot x(t)+G\cdot v(t)$, with G invertible, the resulting system $(A+B\cdot F, B\cdot G, C)$ is such that for an input $v(t)=h\cdot e^{\omega\cdot t}$, with h and ω finite constants, the initial conditions and the values of *h* can always be found such that the trajectory of the system with feedback $x_b(t)$ is equal to $x(0)\cdot e^{\omega\cdot t}$ for all $t\geq 0$, and the output is zero for all $t\geq 0$, ω=1, and *ω=2*. Then {*1,2*} is the set of the "finite invariant zeros" of the system (A,B,C).

1.1 Zeros and Poles of a Stationary Linear System

A number *λ* (real or complex) is said to be a pole of a proper rational transfer function $\hat{g}(s)$ if $|\hat{g}(\lambda)|=\infty$. A zero of $\hat{g}(s)$ is such that $|\hat{g}(\lambda)|=0$.

Polynomial roots of equations $P(s)=0$ and $Q(s)=0$ are called zeros and poles, respectively.

1.2 Notion of Zeros at Infinity

Let $T(s)\in\mathbb{R}$, then if $\lim_{s\to\infty} t(s)=\alpha\in\mathbb{R}$, *t(s)* is said to be proper; if $\lim_{s\to\infty} t(s)=\infty\in\mathbb{R}$, then *t(s)* is said to be not proper. If $\alpha=0$, *t(s)* is strictly proper [6].

1.3 Basic Subspaces

Definition 1. Let (A, B, C) be a stationary linear system defined in (1):

$$\sum \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}, \qquad (1)$$

where vector x(•) $\in \mathcal{X} \approx \mathbb{R}^n$ denotes the state, vector u(•) $\in \mathcal{U} \approx \mathbb{R}^m$ denotes the input, and vector y(•) $\in \mathcal{Y} \approx \mathbb{R}^p$ denotes the output, with n, m, p positive integers elements. A: $\mathcal{X} \to \mathcal{X}$, B: $\mathcal{U} \to$ X, and C: $\mathcal{X} \to \mathcal{Y}$ are linear mappings represented by real constant matrices chosen from $\mathcal{X}$, $\mathcal{U}$, $\mathcal{Y}$, and $\mathcal{D}$. Then we have the following:

i. $\mathcal{V}^*(\mathcal{K})$ is the largest subspace (A, $\mathcal{B}$)-invariant containing a subspace $\mathcal{K} \subset \mathcal{X}$.

ii. $\mathcal{S}^*(\mathcal{B})$ is the smaller subspace ($\mathcal{C}$, A)-invariant containing a subspace $\mathcal{B} \subset \mathcal{X}$.

iii. $\mathcal{R}^*(\mathcal{K})$ is the largest subspace (A, $\mathcal{B}$)-controllability-invariant included in a subspace $\mathcal{K} \subset \mathcal{X}$.

iv. $\mathcal{V}(\mathcal{k})_{stab}{}^*$ denotes the largest internally stabilized subspace (A, B)-invariant containing $\mathcal{K} \subset \mathcal{X}$.

Remark 1. A subspace $\mathcal{V}$ is (A,$\mathcal{B}$)-invariant if A$\mathcal{V} \subset \mathcal{V}+\mathcal{B}$.

Remark 2. A subspace $\mathcal{S}$ is ($\mathcal{C}$,A)-invariant if A($\mathcal{C} \cap \mathcal{S}) \subset \mathcal{S}$.

Remark 3. A subspace $\mathcal{R}$ is (A,$\mathcal{B}$)-controllability-invariant if A$\mathcal{R} \subset \mathcal{R}+\mathcal{B}$ and furthermore σ($\mathcal{R}$|A + $\mathcal{B}$F|$\mathcal{R}$) =Λ, where Λ is a symmetric set in the complex plane $\mathbb{C}$ and card(Λ) = dim ($\mathcal{R}$). F: $\mathcal{X} \to \mathcal{U}$ is any linear application such that (A + $\mathcal{B}$F) $\subset \mathcal{R}$. $\mathcal{R}$|A + $\mathcal{B}$F|$\mathcal{R}$ denotes the double restriction of $\mathcal{R}$ to A + $\mathcal{B}$F.

Remark 4. A subspace $\mathcal{V}_g$ is (A,$\mathcal{B}$)-invariant $\mathbb{C}_g$ and internally stabilized if a linear mapping F:$\mathcal{X} \to \mathcal{U}$ can be found such that (A + $\mathcal{B}$F) $\mathcal{V}_g(\mathcal{K}) \subset \mathcal{V}_g(\mathcal{K})$ and *σ($\mathcal{V}_g$|A + $\mathcal{B}$F|$\mathcal{V}_g$) $\subset \mathbb{C}_g$. $\mathcal{R}$|A + $\mathcal{B}$F|$\mathcal{R}$ denotes the double restriction of $\mathcal{R}$ to A + $\mathcal{B}$F. $\mathcal{V}_g$|A + $\mathcal{B}$F|$\mathcal{V}_g$ denotes the double restriction of $\mathcal{V}_g$ to A + $\mathcal{B}$F.*

Subspaces $\mathcal{V}^*(\mathcal{K})$, $\mathcal{S}^*(\mathcal{B})$ can be computed using the following algorithms [1]:

Algorithm ISA

$$\begin{cases} V^0(K) = X \\ V^{\mu+1}(K) = K \cap A^{-1}\left\{V^{\mu}(K) + \operatorname{Im} B\right\}, \\ \text{for } \mu \geq 1, \end{cases} \qquad (2)$$

Algorithm CISA

$$\begin{cases} \mathrm{S}_K^0(Z) = 0 \\ S_K^{\mu+1}(Z) = Z + A\left(K \cap S_K^{\mu+1}\right), \\ \text{for } \mu \geq 1, \end{cases} \qquad (3)$$

and on the other hand,

$$R^*(K) = V^*(K) \cap S^*(B). \qquad (4)$$

Concerning $\mathcal{V}(\mathcal{k})_{stab}{}^*$, this is defined as follows.

Definition 2. *Let β(λ) be the minimal polynomial of $A_0|(\mathcal{V}^*(\mathcal{K}) / \mathcal{R}^*(\mathcal{K}))$, where $A_0 = A + B \cdot F_0$. The factor β(λ)= $\beta_g(\lambda) \cdot \beta_b(\lambda)$, where the zeros of $\beta_g(\lambda)$($\beta_b(\lambda)$) in $\mathbb{C}$ belong to $\mathbb{C}^-$ $\left(\mathbb{C}^+\right)$ and are defined as*

$$\begin{aligned} \mathrm{X}_g^* &:= \left(\frac{V^*(K)}{R^*(K)}\right) \cap Ker\beta_g(A_0), \\ \mathrm{X}_b^* &:= \dim\left(\frac{V^*(K)}{R^*(K)}\right) \cap Ker\beta_b(A_0), \end{aligned} \qquad (5)$$

$$V_{stab}^*(K) = P^{-1} \cdot X_g^*. \qquad (6)$$

Remark 5. The subspace $\mathcal{V}(\mathcal{k})_{stab}{}^*$ is the largest member of the family $\mathcal{B}$ defined as

$$B := \begin{Bmatrix} V : V \in S(K), \exists F \in F(V) \\ \sigma\left[\left(A + B\cdot F\right) | V\right] \subset \mathbb{C}^- \end{Bmatrix}. \quad (7)$$

1.4 Cumulus to Infinity

The total sum of the infinite system (A, B, C) is a positive integer number denoted as C_∞ (ABC) and called the Infinite Cumulus System (A, B, C). In geometric terms [18]:

$$C_\infty(A,B,C) := \sum n_i = \sum p_i = \dim\left(\frac{V^*(K)+S^*(B)}{V^*(K)}\right). \quad (8)$$

1.5 Unstable Cumulus

The total sum of the multiplicity of orders of unstable roots (i.e., roots located at $\mathbb{C}_-$) of the polynomials in Σ^f (A, B, C) is called unstable cumulus and is given in [8] as

$$C^+(A,B,C) := \dim\left(\frac{V^*(K)}{V^*_{stab}(K)}\right). \quad (9)$$

1.6 Total Cumulus

Total cumulus is the total sum of (A, B, C) and the sum of their infinite cumulus and unstable cumulus. This is a positive integer denoted as $C_\infty^+(A,B,C)$ and defined as

$$\begin{aligned} C_\infty^+(A,B,C) &:= C_\infty\left(A,B,C\right) + C^+\left(A,B,C\right) \\ &= \dim\left(\frac{V^*(K)+S^*(B)}{V^*_{stab}(K)}\right). \end{aligned} \quad (10)$$

Example 1. Consider a continuous stationary linear system (A, B, C) given as

$$\sum = \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}, \quad (11)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 1 \end{bmatrix}. \quad (12)$$

Expression (9) is minimal (i.e., controllable and observable) and the corresponding transfer function is given as

$$T(s) = \frac{s-1}{(s+1)^2}. \quad (13)$$

The relative degree system (11) is equal to 1, i.e., expression (10) has a zero at infinity of the order one. In addition, (10) has an instable zero of finite multiplicity equal to one in s = 1. Therefore,

$$\begin{aligned} &C_\infty\left(T(s)\right) = 1, \\ &C^+\left(T(s)\right) = 1, \\ &C_\infty^+\left(T(s)\right) = C_\infty\left(T(s)\right) + C^+\left(T(s)\right) = 2. \end{aligned} \quad (14)$$

2 SPR0 Substitutions

The (A, B, C) stationary system is a linear time invariant system (LTI) with minimal representation (13), i.e., it is controllable and observable in the time domain:

$$\sum = \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}, \quad (15)$$

whose transfer function is defined as

$$Y(s) = \left(D + C\cdot\left(sI - A\right)^{-1}\cdot B\right)\cdot U(s), \quad (16)$$

where $X(s) \in \mathbb{R}^n; U(s), Y(s) \in \mathbb{R}^m$.

Definition 3. Let **RH**$_\infty$ be the Euclidean domain of real functions, proper and stable rational transfer. It has the following form:

$$\left\|P(s)\right\|_\infty := \sup_\omega \left|P(j\omega)\right| \quad (17)$$

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It is a subspace of **H**∞, which include open right half-plane and bounded analytic functions, with the same standard space. The real number $\left\|P(s)\right\|_{\infty}$ is the H∞ norm of P (s).

Let $P(s)=N_p(s) / D_p(s)$ be a proper rational function, where $N_p(s)$ and $D_p(s)$ are real polynomials. *P(s)* has a relative degree zero, with $\deg\left(N_p(s)\right)=r$, $\deg\left(D_p(s)\right)=l$, and *r=l*:

$$\deg\left(N_p(s)\right)=\deg\left(D_p(s)\right). \tag{18}$$

Now we introduce a formal definition of a Strictly Positive Real Function of relative degree zero:

Definition 4. In agreement with [1] and [2], let P(s) be a rational function with a relative degree equal to zero. It is SPR0 if and only if

i. *P(s) is analytic in Re[s] ≥ 0,*

ii. *Re[P (jω)]> 0 for all ω ∈ $\mathbb{R}$.*

The set of functions SPR0 is denoted by SPR0.

Definition 5. A rational function is in SPR0* if it is described by a limit of sequence functions that is in SPR0.

Note 1. A rational function s ∈ SPR0* can be interpreted as the limit of SPR0 sequence of functions:

$$\lim_{a\to 0}\frac{s+a}{as+1}\to s\,, \tag{19}$$

where a > 0 and $a^2 \neq 1$.

Example 2. Let Z(s) be described as

$$Z(s)=\frac{s^2+4s+1}{s^2+s+2}. \tag{20}$$

i. *It is analytically verified taking into account that an F(s) function of the complex variable s is said to be analytic in an open set if it has a derivative at each point of this open set. Let Z, D (Z) be the domain in the range R_p and I (Z) in R_q. It is said that Z is differentiable at c if there exists a linear function L: $R_p \to R_q$ such that for any ε> 0 there exists δ(ε)> 0 such that if x ∈ R_p is a vector satisfying || x - c || < δ (ε), then x ∈ D (Z) and || Z (x) - Z (c) - L (x - c) || ≤ ε || x - c ||. Therefore, it is observed that Z (s) is continuous and derivable for Re [s] ≥ 0.*

ii. *Given that* $Z(j\omega)=\frac{-\omega^2+4\omega+1}{-\omega^2+\omega+2}$ *and that* $\mathrm{Re}[j\omega]=\frac{Z(\omega)+\overline{Z}(\omega)}{2}$ *has the form* $\mathrm{Re}[Z(j\omega)]=\frac{\omega^4+\omega^2+2}{\left(2-\omega^2\right)^2+\omega^2}>0$ *for all ω ∈ $\mathbb{R}$, so Z(s) ∈ SPR0.*

3 Preservation of Structural Properties of Matrices with SPR0 Substitutions

The following statements are of particular importance in this work [11]:

Proposition 1. *Let* $M(s)\in\mathbb{R}_{pr}^{p\times m}(s)$ *be bi-proper, then* $M(f(s))\in\mathbb{R}_{pr}^{p\times m}(s)$ *is bi-proper for all f(s) ∈ SPR0.*

Lemma 1. Let $T_1(s)\in\mathbb{R}_{pr}^{p\times m}(s)$ and $T_2(s)\in\mathbb{R}_{pr}^{p\times m}(s)$ be low bi-proper equivalent transformations, and bi-proper $T_L(s)$ and $T_R(s)$ such that.

$$T_1(s)=T_L(s)\cdot T_2(s)\cdot T_R(s), \tag{21}$$

then $T_1(f(s))$ and $T_2(f(s))$ are equivalent for all f(s) ∈ SPR0.

Proof. *Given that*

$T_1(s)=T_L(s)\cdot T_2(s)\cdot T_R(s)$,

then

$T_1(f(s))=T_L(f(s))\cdot T_2(f(s))\cdot T_R(f(s))$.

and according to Proposition 1 $T_L(f(s))$ *and* $T_R(f(s))$ *are bi-proper, therefore* $T_1(s)$ *and* $T_2(s)$ *are equivalent.* ■

4 Defining the Disturbance Rejection Problem

Consider a disturbed stationary linear system (A, B, C, E) [10] described as

$$\begin{cases}(sI-A)\cdot x(s)=B\cdot u(s)+E\cdot d(s)\\ \qquad y(s)=C\cdot x(s)\end{cases}, \qquad (22)$$

where *x(s)*, *u(s)*, and *y(s)* denote the Laplace transforms of the state vectors $x(\bullet)\in\mathcal{X}\approx\mathbb{R}^n$, input $u(\bullet)\in\mathcal{U}\approx\mathbb{R}^m$, and output $y(\bullet)\in\mathcal{Y}\approx\mathbb{R}^p$, respectively, *d(s)* is the Laplace transform of the disturbance vector $d(\bullet)\in\mathcal{D}\approx\mathbb{R}^q$. $A:\mathcal{X}\to\mathcal{X}$, $B:\mathcal{U}\to X$, and $C:\mathcal{X}\to\mathcal{Y}$ are linear mappings (represented by constant real shades for freely chosen $\mathcal{X},\mathcal{U},\mathcal{Y}$, and $\mathcal{D}$).

The Problem of Interference Rejection (PRB) is defined as follows.

Definition 5. *Let* a stationary linear system (A, B, C, E) be perturbed. Find (if possible) a linear mapping F: $\mathcal{X}\to\mathcal{U}$ such that [3]

i. $$C\cdot\left(sI_n-\left(A+B\cdot F\right)\right)^{-1}\cdot E\equiv 0, \qquad (23)$$

ii. $$.\sigma\left(A+B\cdot F\right)\subset\mathbb{C}^- \qquad (24)$$

Note 2. The above definition states that the feedback u (s) = Fx(s) rejects the disturbance d(s) in the retrofitted system:

$$\begin{cases}sI_n-\left(A+B\cdot F\right)\cdot x(s)=E\cdot d(s)\\ \qquad\qquad y(s)=C\cdot x(s)\end{cases} \qquad (25)$$

ensuring internal stability.

5 Geometric Solution

Theorem 1. Let (A, B, C) be a fictitious system and (A, [BE], C) be a combined system:

$$\begin{cases}\dot{x}(t):=A\cdot x(t)+\left[B\ E\right]\begin{pmatrix}u(t)\\ d(t)\end{pmatrix},\\ y(s)=C\cdot x(t)\end{cases} \qquad (26)$$

i.e. (A, B, C) corresponds to the fictitious system (A, B, C, E) when d(t) = 0, while (A, [BE], C) results when d(t) is notionally regarded as an input control. Furthermore, from (23) we have

$$y(s)=T(s)\cdot u(s)+T_d(s)\cdot d(s) \qquad (27)$$

with $T(s):=C\cdot(sI-A)^{-1}\cdot B$ and $T_d(s):=C\cdot(sI-A)^{-1}\cdot E$.

Now consider the control law

$$u(t)=Fx(t), \qquad (28)$$

where *F: X → U*. From the system in (28) we obtain

$$\begin{cases}x(s)=(sI-A)^{-1}Bu(s)+(sI-A)^{-1}Ed(s)\\ u(s)=Fx(t)\end{cases} \qquad (29)$$

and

$$u(s)=C(s)\cdot d(s), \qquad (30)$$

where

$$C(s)=:\left(I-F(sI-A)^{-1}B\right)^{-1}F(sI-A)^{-1}E. \qquad (31)$$

Note 3. Compensator resulting control law in (28) with C(s) defined in (29) is strictly proper.

Then the Disturbance Rejection with Internal Stability (PRPEI) problem is defined in algebraic terms as follows.

Definition 6. The problem is to find an application F represented by a constant matrix selected for the involved base such that

i. $$T_d(s) := T(s)\cdot C(s) + T_d(s) = \\ = C\cdot\left(sI_n - (A + B\cdot F)\right)^{-1}\cdot E = 0, \quad (32)$$

ii. $$\sigma(A + B\cdot F) \subset \mathbb{C}^-. \quad (33)$$

The geometric solution to the problem is as follows.

Remark 5. In agreement with [4], the Disturbance Rejection problem has a solution if and only if there is the largest sub-space (A, B)-invariant contained in kerC.

6 Structural Solution

Here we present a structural solution to the Disturbance Rejection problem without measuring the disturbance [9]. It is necessary to introduce two fictitious systems built from a perturbed system. The first modified system is

$$\begin{cases} \dot{x}(t) := \widetilde{A}\cdot x(t) + \widetilde{B}\cdot x(t) + \widetilde{E}\cdot d(s) \\ y(s) = \widetilde{C}\cdot x(t) \end{cases}, \quad (34)$$

where

$$\widetilde{A} = \begin{bmatrix} A & I_n \\ 0 & 0 \end{bmatrix}; \widetilde{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}, \\ \widetilde{C} = \begin{bmatrix} C & 0 \end{bmatrix}; \widetilde{E} = \begin{bmatrix} E \\ 0 \end{bmatrix} \quad (35)$$

which are associated fictitious systems $\left(\widetilde{A}, \widetilde{B}, \widetilde{C},\right)$ and $\left(\widetilde{A}, \left[\widetilde{B}\ \widetilde{E}\right], \widetilde{C}\right)$, whose respective transfer functions are

$$\widetilde{T}(s) = \widetilde{C}\cdot\left(sI - \widetilde{A}\right)^{-1}\cdot\widetilde{B} \quad (36)$$

and

$$\left[\widetilde{T}(s)\ \widetilde{T}_d(s)\right] = \widetilde{C}\cdot\left(sI - \widetilde{A}\right)^{-1}\cdot\left[\widetilde{B}\ \widetilde{E}\right]. \quad (37)$$

The following theorem provides necessary and sufficient conditions for solving PRPEI.

Theorem 2. In agreement with [5], PRPEI has a solution if and only if

$$rank\left\{\widetilde{T}(s)\right\} = rank\left\{\left[\widetilde{T}(s) \quad \widetilde{T}_d(s)\right]\right\} \quad (38)$$

and

$$C_\infty^+\left\{\widetilde{T}(s)\right\} = C_\infty^+\left\{\left[\widetilde{T}(s) \quad \widetilde{T}_d(s)\right]\right\}. \quad (39)$$

7 Numerical Example

Consider the following stationary linear single input, single output system described by

$$\sum = \begin{cases} \dot{x}(t) := A\cdot x(t) + B\cdot u(t) \\ y(s) = C\cdot x(t) \end{cases} \quad t \geq 0 \quad (40)$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 2 & -3 & 1 \end{bmatrix}. \quad (41)$$

Let us denote the inverse Laplace transform as $\mathfrak{I}^{-1}\{\bullet\}$. Then the trajectory has

$$x(t) = \mathfrak{I}^{-1}\{x(s)\} \\ = \mathfrak{I}^{-1}\left\{(s\cdot I - A)^{-1}\cdot\left(x(0) + B\cdot u(s)\right)\right\}. \quad (42)$$

For an input of the form $u(t) = g\cdot e^{\alpha\cdot t}$, $t \geq 0$ with *g* and *α* finite constants, whose Laplace transform is

$$u(t) = \frac{g}{s - \alpha} \ . \quad (43)$$

Then it is transformed into

$$x(t)=\mathfrak{I}^{-1}\left\{\left(\frac{1}{s^3-3s-2}\cdot\begin{bmatrix} s^2-3 & s & 1 \\ 2 & s^2 & s \\ 2s & 3s^2 & s^2 \end{bmatrix}\right)\cdot\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0)+\dfrac{g}{s-\alpha} \end{pmatrix}\right\}. \quad (44)$$

Now, we consider the case when *α=1* and *g=1*. It can be verified that for the vector of initial conditions $x(0)=\left(-\frac{1}{4}\ \ -\frac{1}{4}\ \ -\frac{1}{4}\right)^T$ the system trajectory is

$$\begin{aligned} x(t) &= \mathfrak{I}^{-1}\left\{\left(\frac{1}{s^3-3s-2}\cdot\begin{bmatrix} s^2-3 & s & 1 \\ 2 & s^2 & s \\ 2s & 3s^2 & s^2 \end{bmatrix}\right)\cdot\begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4}+\frac{1}{s-1} \end{pmatrix}\right\} \\ &=^{-1}\left\{\begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix}\cdot\left(\frac{1}{s-1}\right)\right\}=\begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4}+\frac{1}{(s-1)} \end{pmatrix}\cdot\mathfrak{I}^{-1}\left\{\frac{1}{s-1}\right\} \\ &= x(0)\cdot e^t \end{aligned} \quad (45)$$

and therefore the output is

$$\begin{aligned} y(t) &= C\cdot x(t) \\ &= \begin{bmatrix} 2 & -3 & 1 \end{bmatrix}\cdot\begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix}\cdot e^t. \\ &= 0, \forall t \ge 0 \end{aligned} \quad (46)$$

Similarly, for the case of α=2, it can be verified that for *g=0* and the vector of initial conditions $x(0)=(1\ 2\ 4)^T$, we have

$$\begin{aligned} x(t) &= \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}\cdot e^{2\cdot t}, \forall t \ge 0, \\ y(t) &= 0, \forall t \ge 0. \end{aligned} \quad (47)$$

It is can therefore be seen that for α =1 and α=2, there is a combination of the values of *g* and *x(0)* for which the trajectory of the system is equal to $x(0)\cdot e^{\alpha\cdot t}$ for all $t \ge 0$ and the output of the system is equal to zero for all $t \ge 0$, for an input of the form $u(t)=g\cdot e^{\alpha\cdot t}$. In fact, the values of α are the only ones for which this situation occurs.

8 Conclusions

In this work, the structural and geometric conditions that ensure the existence of at least one controller for a feedback static state which decouples the disturbance of a perturbed output system are presented. This guarantees that the (A, B, C, D, E) system in closed loop has stability under the controller action.

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References

1. **Wonham, W.M. (1985).** *Linear Multivariable Control. A Geometric Approach*. Springer-Verlag.
2. **Goodwin, G.C. & Sink, K.S.** *Adaptive Filtering, Prediction and Control*. Englewood Clifis, NJ, Prentice-Hall.
3. **Narenda, K.S. & Taylor, J.H. (1973).** *Frequency Domain Criteria for Absolute Stability*. New York, Academic Press.
4. **Tovar, J.C. (2004).** *SPR0 substitutions in the disturbance rejection.* Master Thesis, CINVESTAV-IPN, 55 p.
5. **Martinez-García, J.C., Malabre, M., & Dion, J.M. (1999).** Commault, Condensed Structural Solutions

ISSN 2007-9737

to the Disturbance Rejection and Decoupling Problems with Stability. *International Journal of Control*, Vol. 72, No. 15, pp. 1392–1401. DOI:10.1080/002071799220191

6. **Martínez-García, J.C. & Malabre, M. (1995).** Simultaneous Rejection and Regular Row by Row Decoupling with Stability: A Geometric Approach. *IEEE Transaction on Automatic Control*, Vol. 40, No. 2, pp. 365–369. DOI:10.1109/9.341812
7. **Martínez, J.C. (1994).** *Contribution a l'Étude des Propiétés Structurelles des Systemes Linéaires en veu de leur commande*. Thése de Doctorat, 14 Université de Nantes, pp. 26–31.
8. **Vidyasagar, M., (1985).** *Control System Synthesis: A Factorization Approach.* MIT Press, Cambrige, Massachusetts, USA.
9. **Morse, A.S. (1973).** Structural Invariants of Linear Multivariable Systems. *SIAM J. of Control and Optimization,* Vol. 24, No. 1, pp. 446–465. DOI:10.1137/0311037
10. **Fernández-Anaya, Guillermo, Martínez-García, J.C., Kučera, V., & Aguilar George, D. (2003).** MIMO Systems Properties Preservation under SPR0 Substitutions. *IEEE Transactions on Circuits and Systems. DOI:10.1109/TCSII.2004.824050*
11. **Fernandez, G. (1999).** Preservation of SPR functions and Stabilization by Substitutions in SISO plants. *IEEE Transactions on Automatic Control,* Vol. 44, pp. 2171–2172. DOI:10.1109/9.802939

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