

Controlling the Strongly Damping Inertia Wheel Pendulum via Nested Saturation Functions

Control del Péndulo con Rueda de Inercia Fuertemente Amortiguado mediante Funciones de Saturación Anidadas

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Abstract

In this paper we solve the stabilization problem of the strongly damping inertia wheel pendulum around its unstable equilibrium. The stabilization is accomplished by using nested saturation functions. The use of nested saturation function is possible because this system can be rewritten approximately as a chain of integrators with and nonlinear perturbation. The proposed control strategy makes the closed-loop system globally asymptotically and locally exponentially stable around the unstable inverted vertical position, even when the physical damping is presented in the model.

Keywords: Nested saturation functions, Lyapunov function, nonlinear systems.

Resumen

En este artículo resolvemos el problema de estabilización del péndulo con rueda de inercia fuertemente amortiguado alrededor de su punto de equilibrio inestable. La estabilización es lograda mediante el uso de funciones de saturación anidadas. El uso de funciones de saturación anidadas es posible porque se puede escribir una aproximación del sistema como una cadena de integradores con una perturbación no lineal. La estrategia de control que se propone hace que el sistema en lazo cerrado sea asintóticamente estable de forma global y exponencialmente estable de forma local alrededor de la posición vertical inestable, aún cuando el amortiguamiento físico está presente en el modelo.

Palabras Clave: Funciones de saturación anidadas, Función de Lyapunov, Sistemas no lineales.

1 Introduction

The inertia wheel pendulum (**IWP**) has attracted the attention of several researchers as a test bed for the effectiveness of control design techniques proposed by control theory [1, 2, 3]. The **IWP** is made up from a rotating wheel at the end, that freely spins about an axis parallel to the pendulum axis of rotation. The disk is moved by a DC-motor, while the pendulum is un-actuated. This system is controlled by the torque generated by the disk angular acceleration. Since the pendulum torque cannot be directly driven, it is an example of an under-actuated mechanical system. That is, it has fewer controllers than degrees of freedom. Basically two control maneuvers are related with this system; the first is swinging the pendulum up from the hanging position to the upright vertical position; the second consists of stabilizing the **IWP** around its unstable equilibrium point, with the two angular positions of the system at the origin. According to this issue, we mention some of the most remarkable works related to this topic. In [1] a control energy approach based on a collocated partial feedback linearization and passivity of the resulting zero dynamics is used to solve the swinging and balance problem of the **IWP**; also, it is shown that this system is feedback linearizable with respect to some suitable output, under the assumptions that the pendulum angle lies in the

upper half plane and the physical damping force is ignored. In [2], the authors transform the dynamics of the original system into a cascade nonlinear system in a strict feedback form, by using some global transformations. Based on this, a globally asymptotically stabilization around its unstable top position is presented, by means of the standard backstepping procedure. A similar idea was used in [15] to control de inverted pendulum. In [3] two nonlinear swinging-up control strategies for solving the swinging and balance of the pendulum about its unstable inverted position are used. These approaches are based on the total energy stored in the system and guarantee convergence of the pendulum to a homoclinic orbit. In [4] the interconnection and damping assignment passivity based control is used for the asymptotic stabilization of the **IWP** around its top position. The obtained closed-loop system guarantees the asymptotical convergence of all the states, for all the initial conditions, except for a set of zero measure. To do this, two necessary matching conditions have to be satisfied in order to obtain a stabilizing controller. In [5] a control strategy which combines sliding modes and generalized PI (GPI) control for the swinging up and stabilization around its unstable vertical position of the **IWP** is presented. We emphasize that to our knowledge only in [6] the undesirable effect of the damping force was considered in the control strategy, as we did it.

In this paper we deal with the asymptotic stabilization of the under-actuated and strongly damping inertia wheel pendulum (**IWP**) around its unstable top position. Our main contribution is to present a suitable set of transformations that allows us to accomplish a nested saturation based controller to bring the system to the unstable top position. That is, the obtained closed-loop system makes the strongly damping **IWP** globally asymptotically and locally exponentially stable at the origin, which coincides with the upright equilibrium point. As far as we know, the stabilization of the strongly damping **IWP** has been barely studied in the literature. In most cases, the problem has been solved designing a simple control law, made it possible by ignoring the physical damping, in the hope that this force cannot affect the closed-loop stability. However, this is not always true, because, if the physical damping is presented, it tends to destabilize the closed-loop solution, especially in the top position (see [6] and [7]). This fact can be shown by a simple linearization around the origin (for a deep study of the undesirable effect of physical damping we suggests to read [8] and [9]).

This paper is organized as follows. In Section 2 we present the dynamical model of the strongly damping **IWP** and the transformation of the original system in such a way that the obtained system looks like an integrator chain with an additional nonlinear perturbation. In Section 3 we develop the control strategy based on saturation functions. In Section 4 we present some computer simulations. Finally, we devote Section 5 to the conclusions.

2 The Inertia wheel Pendulum

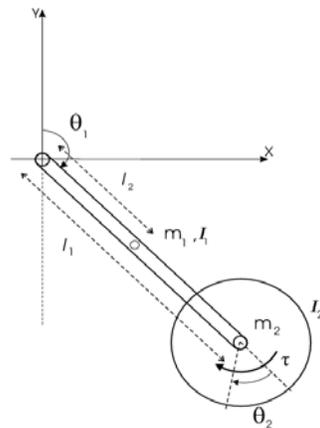


Fig. 1. The under-actuated inertia wheel pendulum (**IWP**)

The **IWP**, depicted in Figure 1, is a planar inverted pendulum with a revolving wheel at the end. The wheel pendulum is actuated while the pendulum joint at the base is unactuated. The model of this system is described by [3] as

$$\begin{aligned} (I_1 + I_2 + m_1 l_2^2 + m_2 l_1^2) \ddot{\theta}_1 + l_2 \ddot{\theta}_2 - \eta g \sin \theta_1 + \delta_1 \dot{\theta}_1 &= 0 \\ l_2 \ddot{\theta}_1 + l_2 \ddot{\theta}_2 + \delta_2 \dot{\theta}_2 &= \tau \end{aligned} \quad (1)$$

where θ_1 is the pendulum angle, θ_2 is the disk angle and τ is the torque input applied on the disk. The remaining parameters are described in the following table:

$m_{1,2}$:	pendulum and wheel masses
$l_{1,2}$:	pendulum length and distance to the center of the pendulum mass
$I_{1,2}$:	moments of pendulum and wheel inertia
$\delta_{1,2}$:	damping coefficients of the unactuated and the actuated coordinates and $\eta = m_1 l_2 + m_2 l_1$.

As can be seen, θ_1 and θ_2 are the non-actuated and the actuated system coordinates, respectively. This is because τ acts directly on the disk position. Now, to simplify the algebraic manipulations in the forthcoming developments, we divided the first equation of system (1) by I_2 and substituted

$$\tau = v l_2 + \delta_2 \dot{\theta}_2$$

in the second equation, having that system (1) can be expressed as:

$$\begin{aligned} (1 + \kappa_1) \ddot{\theta}_1 + \ddot{\theta}_2 - \kappa_2 \sin \theta_1 + \delta \ddot{\theta}_1 &= 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 &= v \end{aligned} \quad (2)$$

where

$$\kappa_1 = \frac{I_1 + m_1 l_1^2 + m_2 l_1^2}{I_2}; \kappa_2 = \frac{\eta g}{I_2}; \delta = \frac{\delta_1}{I_2} \quad (3)$$

Problem Statement

The control objective is to design a continuous feedback v to bring the pendulum to the upright position with the disk position at the origin for any arbitrary initial conditions, even if the linear dissipation force is presented in the non-actuated coordinate.

Comments 1: When damping is not physically available, several techniques can be employed to circumvent this problem (see [2, 3, 4, 10]). However, if damping is physically present in the system, then, the passivity and flatness properties are lost, i.e., the closed-loop system may become unstable in the top position or the closed-loop solution may converge to other equilibrium points [7, 11]. This fact can be shown by simple linearization of the closed-loop position around the top position. On the other hand, it is not possible to directly accomplish a model matching approach to solve the asymptotic stabilization of this system [6], due to the fact that the damping force breaks the symmetric property of the original Euler-Lagrange or Hamilton systems. However, in this case, by means of suitable transformations, it is possible to indirectly apply the model matching control energy method to locally stabilize the **IWP**, for all initial conditions except for the set equilibrium points given by, $\theta_1 = \pm k\pi, \theta_2 = 0$ with $k \in \mathbb{N} - \{0\}$.

Transforming the original structure of the system:

Now we introduce a global transformation that allows us to express system (2) as a chain of integrators with an additional nonlinear perturbation. Then, a nested-saturation controller can be used for rendering asymptotically stable the origin of the latter model.

Let us introduce the following global change of coordinates:

$$\begin{aligned} z_1 &= (1 + \kappa_1)\theta_1 + \theta_2; & \dot{z}_1 &= p_1; \\ z_2 &= \theta_1; & \dot{z}_2 &= p_2. \end{aligned} \tag{4}$$

which leads to the following nonlinear system

$$\dot{x} = A_0x + \Delta(x) + b_0u \tag{5}$$

where

$$x = \begin{bmatrix} z_1 \\ p_1 \\ z_2 \\ p_2 \end{bmatrix}; A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \kappa_1 & -\delta \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \Delta(x) = \begin{bmatrix} 0 \\ \kappa_2\phi(z_2) \\ 0 \\ 0 \end{bmatrix}; b_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The perturbation ϕ and the new controller u are defined as

$$\phi(z_2) = \sin(z_2) - z_2; u = \frac{1}{k_1}(-v - \delta\ddot{z}_2 + \kappa_2 \sin(z_2)). \tag{6}$$

Note that the structure of the above system has a similar form to the four cascade integrators with an additional nonlinear perturbation. On the other hand, the new controller u directly acts on the non-actuated coordinate θ_1 , which is the pendulum position. Contrarily, in system (2) the torque τ directly drives on the disk position. That is, we slightly change the structure of the original strongly damping IWP.

3 Control Strategy

In this section we establish the framework of our control strategy. The idea behind it consists of bringing all the states very close to the origin, where the nonlinear perturbation can be bounded by the square of the pendulum angle position. Afterwards, the stability analysis can be carried out by using a robust linear system stability analysis. In other words, we force the states of system (5) to behave as an exponentially linear system with a very small perturbation. For this purpose we use a nested saturation based controller. This technique was first introduced by Teel in the seminal works [12, 13] and used in [14] to solve the stabilization of the “Ball and Beam System”. Thereafter, this technique has been extensively used for controlling a wide class of under-actuated systems [15, 16, 17, 18, 19]

So, we proceed as follows: first, a linear transformation is used to directly propose a stabilizing controller. Secondly, it is shown that the proposed controller guarantees the boundedness of all states. Finally, we show that the closed-loop system is locally exponentially asymptotically stable after some finite time.

Before developing the control strategy, we introduce some convenient definition:

A linear saturation function ([13]):

We say that function $\sigma_m[s]:R \rightarrow R$ is a linear saturation function, if it satisfies:

$$\sigma_m[s] = \begin{cases} s & \text{if } |s| \leq m, \\ m \operatorname{sign}(s) & \text{if } |s| > m. \end{cases} \tag{7}$$

A nested based controller: based on the previous work of [13], we propose a convenient linear transformation that

allows us to propose, in a direct way, the necessary stabilizing controller u for the nonlinear system (5).

Let us first introduce a global linear transformation $q=Sx$, which is selected such that

$$SA_0S^{-1} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Sb_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

After some simple algebraic manipulations, we can propose S , as:

$$S = \begin{bmatrix} \frac{1}{\kappa_2} & \frac{\delta + 3\kappa_2}{\kappa_2^2} & 3 + \frac{\delta^2}{\kappa_2^2} & 1 \\ 0 & \frac{1}{\kappa_2} & 2 + \frac{\delta}{\kappa_2} & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{8}$$

So that system (5) can be rewritten as:

$$\begin{aligned} \dot{q}_1 &= u + q_2 + q_3 + q_4 + \left(\frac{\delta + 3\kappa_2}{\kappa_2} \right) \phi(q_3 - q_4) \\ \dot{q}_2 &= u + q_3 + q_4 + \phi(q_3 - q_4) \\ \dot{q}_3 &= u + q_4 \\ \dot{q}_4 &= u \end{aligned} \tag{9}$$

To stabilize the above system, we propose the following nested based controller u , as:

$$u = -q_4 - k\sigma_\alpha \left[\frac{1}{k} \left(q_3 + \sigma_\beta [q_2 + \sigma_\gamma [q_1]] \right) \right] \tag{10}$$

where k is a scaling positive constant.

Note that the closed-loop system, defined by equations (9) and (10), is globally Lipschitz. Consequently, all the states $\{q_i\}^1$ cannot have a finite time escape [18].

Boundedness of all states: Now, we show in four simple steps that the closed-loop solution of the proposed closed-loop system, (9) and (10), ensures that all the states are bounded. Moreover the bound of each state directly depends on the designed parameters of the controller (10).

Step 1: To show that the state q_4 is bounded, we introduce an auxiliary positive function V_1 , as:

¹ Here after, we use $\{x_i\}$ to denote $x=[x_1, x_2, x_3, x_4]^T$.

$$V_1 = \frac{1}{2}q_4^2 \quad (11)$$

Differentiating (11) and using the fourth differential equation of (9), we have:

$$\dot{V}_1 = -q_4 k \sigma_\alpha \left[\frac{q}{k} + \sigma_\beta [q_2 + \sigma_\gamma [q_1]] / k \right]$$

If $|q_4| > k\alpha$ then, from the above, we have that $\dot{V}_1 \leq 0$. Therefore, there is a finite time T_1 after which, we have:

$$|q_4(t)| < k\alpha; \forall t > T_1.$$

That is, q_4 is bounded after some finite time T_1 .

Step 2: We proceed to analyze the behavior of the state q_3 . To do this, we introduce an auxiliary positive function V_2 , as:

$$V_2 = \frac{1}{2}q_3^2 \quad (12)$$

Substituting the proposed controller (10) into the third differential equation of (9), we have:

$$\dot{q}_3 = -k\sigma_\alpha \left[\frac{1}{2}(q_3 + \sigma_\beta [q_2 + \sigma_\gamma [q_1]]) \right]. \quad (13)$$

Differentiating (12) and using (13), we obtain:

$$\dot{V}_2 = -q_3 k \sigma_\alpha \left[\frac{1}{2}(q_3 + \sigma_\beta [q_2 + \sigma_\gamma [q_1]]) \right],$$

where the control parameters, α and β , have to be selected such that $\alpha > 2\beta k$. If $|q_3| > \beta$ then, $\dot{V}_2 \leq 0$. Therefore, there is a finite time $T_2 > T_1$, after which, we have:

$$|q_3(t)| < \beta; \forall t > T_2.$$

Consequently, q_3 is also bounded after some finite time T_2 . On the other hand, defining the auxiliary variable

$$w = q_3 + \sigma_\beta [q_2 + \sigma_\gamma [q_1]],$$

we have that $|w(t)| \leq |q_3| + \beta$ for all $t > 0$, and, evidently, $|w(t)| < 2\beta$ after $t > T_2$. Since $\alpha > 2\beta/k$ clearly then

$$k\sigma_\alpha \left[\frac{1}{k} w \right] = w; t > T_2.$$

From the above, we have that control u turns out to be

$$u = -q_4 - q_3 - \sigma_\beta [q_2 + \sigma_\gamma [q_1]]; t > T_2. \tag{14}$$

Remark 1: After $t > T_2$, we have that

$$|q_3 - q_4| < \beta + k\alpha < \frac{3\alpha k}{2} \hat{=} \mu_k \tag{15}$$

Because control parameter k can be selected as we desired, we can fix it as $\mu_k < 1$. Consequently, $|q_3(t) - q_4(t)| < \mu_k < 1$, for all $t > T_2$. Then, applying the following inequality

$$|\sin(x) - x| \leq |\sin(1) - 1|x^2 = \bar{\theta}x^2; \forall |x| < 1 \tag{16}$$

into the definition of function ϕ , we clearly have

$$|\phi(q_3 - q_4)| \leq \bar{\theta}|q_3 - q_4|^2 < \theta\mu_k^2; \forall t > T_2. \tag{17}$$

Step 3: Substituting (14) into the second differential equation of (9), we obtain:

$$\dot{q}_2 = -\sigma_\beta [q_2 + \sigma_\gamma [q_1]] + \phi(q_3 - q_4); t > T_2, \tag{18}$$

Where β and γ must satisfy $\beta > 2\gamma + \bar{\theta}\mu_k^2$. In order to show that q_2 is bounded, we need to introduce the auxiliary function V_3 , as:

$$V_3 = \frac{1}{2}q_2^2. \tag{19}$$

Differentiating (19) and using (18), it produces:

$$\dot{V}_3 = -q_2(\sigma_\beta [q_2 + \sigma_\gamma [q_1]] + \phi(q_3 - q_4)).$$

Obviously, if $|q_2| > \gamma + \bar{\theta}\mu_k^2$ then $\dot{V}_3 \leq 0$ and there is a finite time $T_3 > T_2$, after which, we have:

$$|q_2(t)| < \gamma + \theta\mu_k^2; \forall t > T_3.$$

Consequently, q_2 is bounded and control u turns out to be

$$u = -q_4 - q_3 - q_2 - \sigma_\gamma [q_1]; \forall t > T_3. \tag{20}$$

Step 4: Substituting equation (20) into the first differential equation of (9), we have:

$$\dot{q}_1 = -\sigma_\gamma [q_1] + \left(\frac{\gamma}{k_2} + 3\right)\phi(q_3 - q_4); \forall t > T_3. \tag{21}$$

To show that q_1 is bounded, we define the auxiliary positive function V_4 , as:

$$V_4 = \frac{1}{2}q_1^2. \quad (22)$$

Differentiating (22) and using (21), we have:

$$\dot{V}_4 = -q_1 \left(\sigma_\gamma[q_1] + \left(\frac{\delta}{k_2} + 3 \right) \phi(q_3 - q_4) \right). \quad (23)$$

Where γ must be selected such that $\gamma > \left(\frac{\delta}{k_2} + 3 \right) \bar{\theta} \mu_k^2$. If $|q_1| > \left(\frac{\delta}{k_2} + 3 \right) \bar{\theta} \mu_k^2$ then $V_4 \leq 0$ and, there is a $T_4 > T_3$ such that

$$|q_1(t)| < \left(\frac{\delta}{k_2} + 3 \right) \bar{\theta} \mu_k^2; \forall t > T_4.$$

That is, all the states $\{q_i\}$ are bounded after $t > T_4$.

We summarize this section with the following Lemma that allows to compute the set of control parameters $\{\alpha, \beta, \gamma, \mu_k\}$, needed to guarantee the boundedness of all the states.

Lemma 1: Given the positive constants δ and k_2 and fixing $\mu_k \in (0,1)$,² the following inequalities

$$\alpha > 2\beta; \beta > 2\gamma + \bar{\theta} \mu_k^2; \gamma > \left(\frac{\gamma}{k_2} + 3 \right) \theta \mu_k^2, \quad (24)$$

are fulfilled, provided that parameters γ, β , and α are selected as:

$$\gamma = \lambda \bar{\theta} \mu_k^2 \left(\frac{\delta}{k} + 3 \right); \beta = \lambda \bar{\theta} \mu_k^2 \left(7 + \frac{2\delta}{k_2} \right); \alpha = 2\lambda \bar{\theta} \mu_k^2 \left(\frac{\delta}{k_2} + 3 \right), \quad (25)$$

Where $\lambda > 1$.

Convergence of all states to zero

We will prove that the closed-loop system given by (9) and (14) is asymptotically stable and locally exponentially stable, under the assumption of Lemma 1. That is, if the control parameters k, γ and β are selected according to Lemma 1, then the vector state q converges to zero.

² Recalling that $k = 2\mu_k/3\alpha$.

We must note that after $t > T_4$, the control law is no longer saturated, that is,

$$u = -q_1 - q_2 - q_3 - q_4,$$

and the closed-loop system turns out to be

$$\begin{aligned}\dot{q}_1 &= -q_1 + \left(\frac{\delta}{k_2} + 3\right)\phi(q_3 - q_4), \\ \dot{q}_2 &= -q_1 - q_2 + \phi(q_3 - q_4), \\ \dot{q}_3 &= -q_1 - q_2 - q_3, \\ \dot{q}_4 &= -q_1 - q_2 - q_3 - q_4.\end{aligned}\tag{26}$$

Now, in order to demonstrate the convergence of all the states to zero, we use the following Lyapunov function

$$V = \frac{1}{2} q^T q,\tag{27}$$

Differentiating (27) along the trajectories of (26), we obtain

$$\dot{V} = -q^T M q + \left(q_2 + \left(\frac{\delta}{k_2} + 3\right)q_1\right)\phi(q_3 - q_4)\tag{28}$$

where M is given by

$$M = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

Note that $\lambda_{\min}\{M\} = 1/2$ and therefore $M > 0$. Recalling that after $t > T_4$, the states $\{q_1, q_2\}$ and function ϕ satisfy the following inequalities

$$|q_1| < \bar{\theta}\mu_k^2 \left(\frac{\delta}{k_2} + 3\right); |q_2| < \bar{\theta}\mu_k^2 \left(\frac{\delta}{k_2} + 4\right); |\phi(q_3 - q_4)| < \bar{\theta}(q_3 - q_4)^2$$

Substituting the above inequalities into the second term of relation (28), we have after using the triangle inequality that

$$\left| \left(\frac{\delta}{k_2} + 3 \right) q_1 + q_2 \right| \phi(q_3 - q_4) < \bar{K} (q_3 - q_4)^2 \leq 2\bar{K} (q_3^2 + q_4^2); \tag{29}$$

where

$$\bar{K} = \bar{\theta} \mu_k^2 \left(\frac{\delta}{k_2} + 3 \right)^2 + \bar{\theta} \mu_k^2 \left(\frac{\delta}{k_2} + 4 \right). \tag{30}$$

Notice that \bar{K} can be as small as needed because $\mu_k \in (0,1)$ is selected as desired.

Therefore applying the inequality (29) into the time derivative of V (28), we evidently have

$$\dot{V} < -\frac{1}{2} [q_1^2 + q_2^2 + q_3^2 + q_4^2] + 2\bar{K} (q_3^2 + q_4^2).$$

If we force the positive constant $\bar{K} < 1/4$, then $\dot{V} < 0$, for all $q \neq 0$. That is, if \bar{K} is selected such that $\bar{K} < 1/4$, then vector state q locally exponentially converges to zero.

From the above discussion, we have:

Proposition 1: Consider the strongly damping IWP system, as described in (2), in closed-loop with

$$v = \kappa_1 q_4 + k \sigma_\alpha \left[\frac{1}{k} (q_3 + \sigma_\beta [q_2 + \sigma_\gamma [q_1]]) - \delta \dot{\theta}_1 + \kappa_2 \sin(\theta_1), \right]$$

where q is obtained via $\{q_1\} = S\{x_i\}$, where matrix S is given in (8), and the set of x_i are defined as

$$x_1 = (1 + \kappa_1)\theta_1 + \theta_2, x_2 = \dot{x}_1, x_3 = \theta_1, x_4 = \dot{\theta}_1.$$

Under the assumption that the control parameters $\{\alpha, \beta, \gamma, k\}$ are selected according to Lemma 1, then, the closed-loop system is globally asymptotically and locally exponentially stable, provided that $\bar{K} < 1/4$ where the estimated \bar{K} is given in (30).

Characteristic of the proposed control strategy

We must recall that the control strategy consists of bringing the pendulum to its upright unstable position, while the wheel spins almost freely. Once the system is close enough to its upright position, the control strategy starts to decrease the wheel’ and pendulum’ angular velocities. Under these conditions, the system turns out to be almost a linear system, because all the saturation functions are disabling. Finally, because the system is confined to move very close to the unstable equilibrium point, then it behaves as a local exponential linear system. Obviously the closed-loop system is locally robust with respect to small dynamics not considered in the model. On the other hand, an observer can be accomplished to use the estimated velocities instead of the actual ones. An exhaustive stability analysis could be carried out to assure that the proposed control strategy works well when using a high-gain observer or a reduced observer. However, this analysis is beyond the scope of this paper.

It is worth to mention that the time response when using saturation functions is, in general, very slow in comparison with other techniques.

Simulation results

In order to test the performance of the obtained control law we carried out two numerical simulations using the MATLAB™ system. The **IWP** physical parameters were set as $m_1 = 0.01\text{kg}$, $m_2 = 0.1\text{kg}$, $l_1 = 0.5\text{m}$, $l_2 = 0.35\text{m}$, $I_1 = 2.5 \times 10^{-3}\text{kgm}^2$ and $I_2 = 1.04 \times 10^{-1}\text{kgm}^2$. We include the additional linear damping term $\delta_1 = 0.1$. Consequently, the structural parameters, defined in (3), are given by $\kappa_1 = 0.16587$, $\kappa_2 = 5.4013$ and $\delta = 0.96154$. The control parameters, designed according to **Lemma 1** and **Proposition 1**, were fixed as $\alpha = 0.46$, $\beta = 0.23$, $\gamma = 0.1035$ and $\mu_k = 0.35$.

In the first experiment, we transferred the pendulum position from the lower stable equilibrium point to the upright unstable equilibrium point. That is, we fixed the initial conditions as $\theta_1(0) = \pi[\text{rad}]$, $\theta_2(0) = 0$, $\dot{\theta}_1(0) = 0$ and $\dot{\theta}_2(0) = 0$. Figure 2 shows the close-loop system response. As we can see from this figure, the state θ_1 converges to zero faster than the state θ_2 . This means that, while the wheel angular position is decreased, the pendulum angular position moves to within a very small vicinity of the origin. Once the pendulum is very close to the origin, the control action starts to regulate the wheel dynamics. In other words, firstly the control action brings the pendulum into a small vicinity of zero, while the wheel angular position decreases until it reaches its minimum; secondly the control, little by little, brings the wheel angular position to the origin. Note that this particular control maneuver cannot be carried out if we use energy based control methods, because the rest lower point is not inside of the stability domain of these kinds of control strategies (see for example, [4] and [5]).

The robustness of the proposed control law was tested in the second experiment. This experiment was set using the same parameter values and the initial conditions except for $\theta_1(0) = \pi[\text{rad}]$. Figure 3 depicts the closed-loop performance of the system when it is subject to a unknown and stochastic variation on the parameter δ , uniform distributed in $(-0.05, 0.05)$. On the other hand a significant unknown constant (*i.e. unmodeled*) perturbation in the parameters k_2 was introduced (up to 1% of its nominal values). As can be seen from Figure 3, the closed-loop response is shown to have good performance even when the system is subject to not considered perturbations. It is worth to mention that a robustness stability it out of reach of the goals of this work.

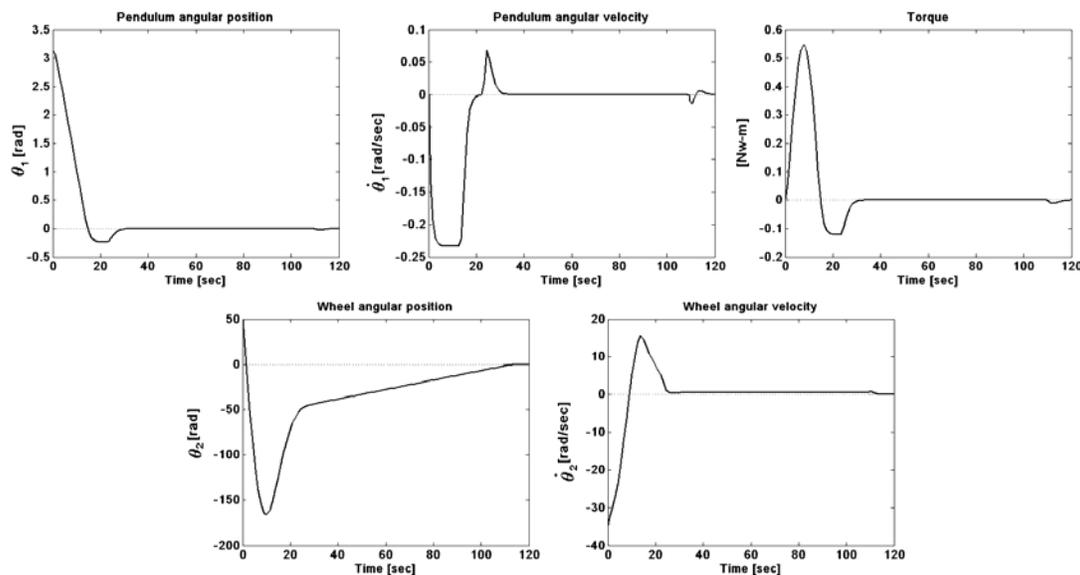


Fig. 2. Closed-loop response of all states

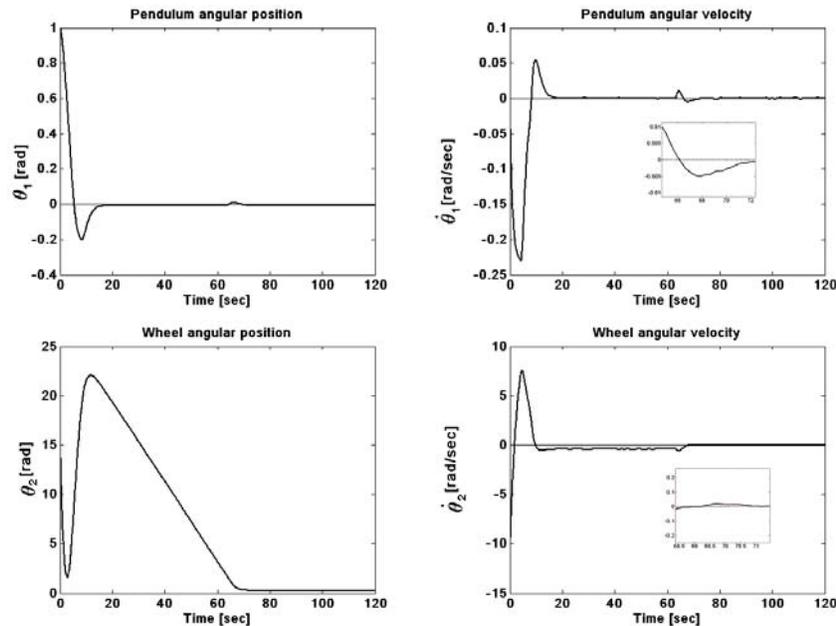


Fig. 3. Closed-loop performance for unknown and stochastic perturbations

4 Conclusions

A nested saturation based controller allows us to solve a number of interesting non-linear control stabilization problems. This powerful technique allows us to propose the necessary stabilizing controller without the necessity of having a candidate Lyapunov function for the whole system. In this case, we have applied this technique for the stabilization of the strongly damping IWP around its upright equilibrium point. Intuitively, the proposed controller consists of two stages. Firstly, we bring the pendulum close enough to the vertical unstable equilibrium point; secondly, we start to regulate the wheel angle position, until all the system states are confined inside a very small vicinity of zero, which can be estimated and contracted as desired. Afterwards, the closed-loop system behaves as an exponential linear system with a small perturbation, where it can be bounded by the square of the pendulum angular position. The latter closed-loop system, which is almost a linear system, turns out to be asymptotically stable at the origin. Convergence to zero of the closed-loop system is assured by using a simple Lyapunov method.

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